Some matrix transformations and generalized core of double sequences

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\textbf{A R T I C L E I N F O}

Article history:
Received 8 May 2008
Received in revised form 24 December 2008
Accepted 31 December 2008

Keywords:
Double sequence
Double series
Matrix transformation
Core of a double sequence

\textbf{A B S T R A C T}

A four-dimensional matrix transformation is said to be regular if it maps every bounded-convergent double sequence into a convergent sequence with the same limit. Firstly, Robison [G.M. Robison, Divergent double sequences and series, Trans. Amer. Math. Soc. 28 (1926) 50–73] presented the necessary and sufficient conditions for regular matrix transformations of double sequences. In this paper, the conditions of Robison are extended to the class of regular matrix transformations between the double sequence spaces \(c_2^{PB}(p)\) and \(c_2^{PB}(q)\). We also characterize the matrix classes \((\alpha c_2^{PB}(p), \alpha c_2^{PB}(q)), (\alpha c_2^{PB}(p), \alpha c_2^{PB}(q)), (c_2^{PB}(p), c_2^{PB}(q)), (c_2^{PB}(p), c_2^{PB}(q))\) and \((\ell_2^{PB}(p), \ell_2^{PB}(q))\). Furthermore, we define the core of a real sequence belonging to the more general class \(c_2^{PB}(p)\) and establish some results related to this new type of core by using our matrix classes \((c_2^{PB}(p), c_2^{PB}(q)), (\ell_2^{PB}(p), \ell_2^{PB}(q))\).

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1. Introduction

Let \(w^2\) denote the space of all complex double sequences. It is obvious that \(w^2\) is a linear space under the coordinatewise addition and scalar multiplication. It is well known that \(\ell_2^{\infty}\) is a Banach space of all bounded sequences \(x = (x_{kl})\) with the norm \(\|x\|_\infty = \sup_{k,l \in \mathbb{N}} |x_{kl}|\). The concept of the convergence of double sequences was presented by Pringsheim [1] as follows:

A double sequence \((x_{kl})\) converges to \(L\) if and only if for every \(\varepsilon > 0\) there exists a positive integer \(N\) such that

\[ |x_{kl} - L| < \varepsilon \]

for all \((k, l)\) such that \(k, l > N\). The limit \(L\) is called Pringsheim limit (or double limit) of the double sequence. The set of all convergent (in Pringsheim’s sense) double sequences and null double sequences are denoted by \(c_2^r\) and \(c_2^n\), respectively.

Recall that if a single sequence is convergent, then it is also bounded. But, this case does not hold for a double sequence, i.e., the convergence in Pringsheim’s sense of a double sequence does not imply the boundedness of the double sequence. Hence, we define the spaces \(c_2^{PB} = c_2^r \cap \ell_2^{\infty}\) and \(c_2^{PB} = c_2^n \cap \ell_2^{\infty}\). Móricz [2] proved that the space \(c_2^{PB}\) of all convergent (in Pringsheim’s sense) double sequences is complete under the pseudonorm \(\|x\|_p = \lim_{k,l \to \infty} \sup_{k,l \leq n} |x_{kl}|\) and the set \(c_2^{PB}\) of all bounded-convergent double sequences, the set \(c_2^{PB}\) of all bounded-null double sequences are Banach spaces under the norm \(\|\cdot\|_\infty\). Gökhan and Çolak [3–5] have extended these spaces to the following double sequence spaces and proved that these sets are complete paranormed spaces under some conditions:

\[ \ell_2^{PB}(p) = \left\{ x = (x_{kl}) \in w^2 : \sup_{k,l \leq n} |x_{kl}|^{1/p} < \infty \right\} \]
Let \( p = (p_{kl}) \) be a double sequence of strictly positive real numbers \( p_{kl} \).

In the case \( p_{kl} = a \) for all \( k, l \in \mathbb{N} \), where \( a > 0 \) is constant, we obtain \( \ell^\infty_2(p) = \ell^\infty_2, c^p_2(p) = c^p_2, o_c^p_2(p) = o_c^p_2, c^p_2(p) = c^p_2 \) and \( o_c^p_2(p) = o_c^p_2 \). Furthermore, when all terms of \( (p_{kl}) \), excluding the first finite number of \( k \) and \( l \), are constant and all are equal to \( a > 0 \), then also we obtain \( c^p_2(p) = c^p_2, o_c^p_2(p) = o_c^p_2, c^p_2(p) = c^p_2 \) and \( o_c^p_2(p) = o_c^p_2 \).

Given a double series \( \sum_{k,l=1}^\infty x_{kl} \) we define its partial sum by the formula \( \sum_{m,n}^\infty = \sum_{k,l=1}^\infty x_{kl} \). The sum of the double series \( \sum_{k,l=1}^\infty x_{kl} \) is denoted as \( \text{lim}_{m,n \to \infty} S_{mn} \).

Let \( X \) and \( Y \) be two nonempty subsets of \( w^2 \) and let \( A = (a_{mn}) \) be a four-dimensional matrix. We say that the matrix \( A \) defines a matrix transformation from \( X \) to \( Y \) and we denote it by \( A : X \to Y \) if for every double sequence \( x = (x_{kl}) \in X \) the sequence \( Ax = (y_{mn}) = \sum_{n=1}^\infty a_{kl}x_{kl} \) is in \( Y \), where \( y_{mn} = \sum_{k,l=1}^\infty a_{kl}x_{kl} \) and the series converge for each \( m, n \). By \( (X; Y) \), we denote the class of matrices \( A \) such that \( A : X \to Y \). If, \( (y_{mn}) \) converges to the same limit \( L \), whenever \( (x_{kl}) \) is a bounded convergent sequence with limit \( L \), then the transformation is said to be regular. A regular matrix transformation from \( X \) to \( Y \) is denoted by \( (X; Y)_\text{reg} \).

The characterizations of some four-dimensional matrix transformations between the double sequence spaces \( \ell^\infty_2, c^p_2 \) and \( c^p_2 \) have already been given by Robison [6]. In that work, Robison presented the necessary and sufficient conditions of regular matrix transformation for double sequences.

Then, the matrix characterization of strong regularity for double sequences was studied by Móricz and Rhoades [7]. Mursaleen and Savaş [8] defined and characterized the almost regular matrices for double sequences. Recently, Mursaleen [9] characterized the four-dimensional almost strongly regular matrices of double sequences. Gökhan [10] extended the conditions of the regular matrix transformations of Robison to the matrix class \( (c^p_2, c^p_2)_\text{reg} \).

### 2. Matrix transformations

In this section, we characterize the matrix classes \( (c^p_2, c^p_2), (c^p_2, c^p_2)_\text{reg}, (o_c^p_2, o_c^p_2), (o_c^p_2, o_c^p_2)_\text{reg} \) and \( (c^p_2, c^p_2)_\text{reg} \).

**Theorem 2.1.** Let \( 0 < p_{kl} \leq \sup_{k,l=1}^\infty p_{kl} = H < \infty \) and \( M = \max(1, H) \). Then \( A \in (c^p_2, c^p_2)_\text{reg} \) if and only if

1. \( \lim_{m,n \to \infty} a_{mn}^m = 0 \) for every \( k, l \in \mathbb{N} \),
2. \( \lim_{m,n \to \infty} \sum_{k,l=1}^\infty a_{mn}^m = a \)
3. \( \sum_{k,l=1}^\infty a_{mn}^m B^{1/p_{kl}} \) converges for each \( m \) and \( n \) and for some \( B > 1 \),
4. \( \lim_{m,n \to \infty} \sum_{k,l=1}^\infty a_{mn}^m B^{1/p_{kl}} = 0 \), for \( k = 1, 2, \ldots \text{ and for some } B > 1 \),
5. \( \lim_{m,n \to \infty} \sum_{k,l=1}^\infty a_{mn}^m B^{1/p_{kl}} = 0 \), for \( l = 1, 2, \ldots \text{ and for some } B > 1 \),
6. \( \sup_{m,n} \sum_{k,l=1}^\infty a_{mn}^m B^{1/p_{kl}} = C < \infty \) for every \( B > 1 \).

When these conditions are satisfied, we have \( \lim_{m,n \to \infty} y_{mn} = ax_l \), where \( \lim_{m,n \to \infty} |x_{kl} - \ell|^{p_{kl}} = 0 \).

**Proof.** **Sufficiency.** Let \( x \in c^p_2 \). Then given any \( \varepsilon > 0 \) there exist \( K, N \in \mathbb{N} \) such that

\[ |x_{kl} - \ell|^{p_{kl}/M} < \frac{\min(1, \varepsilon)}{C + 1} < 1 \]

for every \( k > K, l > N \). Therefore, we have

\[ B^{-1/p_{kl}} |x_{kl} - \ell| < |x_{kl} - \ell| < \left( \frac{\min(1, \varepsilon)}{C + 1} \right)^{M/p_{kl}} < \frac{\varepsilon}{C + 1} \]

for every \( k > K, l > N \). We thus have

\[ \sum_{k=K+1}^\infty \sum_{l=N+1}^\infty |a_{kl}^m| B^{1/p_{kl}} B^{-1/p_{kl}} |x_{kl} - \ell| < \frac{\varepsilon C}{C + 1} < \varepsilon \]

for every \( k > K, l > N \) and every \( m, n \in \mathbb{N} \). Hence we obtain that

\[ \lim_{m,n \to \infty} \sum_{k=K+1}^\infty \sum_{l=N+1}^\infty |a_{kl}^m| |x_{kl} - \ell| = 0. \]
Furthermore, there is a real number $R > 1$ such that $|x_k|^{p_{kl}} \leq R$ for all $k, l \in \mathbb{N}$ since $x \in \ell^\infty_2(p)$. Hence we obtain a real number $R_1 > 1$ such that

$$|x_k - \ell| \leq R_1^{1/p_{kl}}$$

for all $k, l \in \mathbb{N}$, where $R_1 = \left[ R^{1/M} + \max(1, |\ell|) \right]^M$. From condition (ii), we have

$$\sum_{k,l=1}^{\infty} a_{kl}^{mn} + r_{mn} = a$$

where $\lim_{m,n \to \infty} r_{mn} = 0$. Whence

$$|y_{mn} - a\ell| \leq \sum_{k,l=1}^{\infty} |a_{kl}^{mn}| |x_k - \ell| + |\ell r_{mn}|$$

$$\leq \sum_{k,l=1}^{K} |a_{kl}^{mn}| R_1^{1/p_{kl}} + \sum_{k=K+1}^{\infty} |a_{kl}^{mn}| R_1^{1/p_{kl}} + \sum_{l=1}^{L} \sum_{k=K+1}^{\infty} |a_{kl}^{mn}| R_1^{1/p_{kl}} + \sum_{k=K+1}^{\infty} \sum_{l=L+1}^{\infty} |a_{kl}^{mn}| |x_k - \ell| + |\ell r_{mn}|$$

and therefore $\lim_{m,n \to \infty} y_{mn} = a\ell$ and

$$\sum_{k,l=1}^{\infty} a_{kl}^{mn} x_k \leq \sum_{k,l=1}^{\infty} |a_{kl}^{mn}| R_1^{1/p_{kl}} \leq C < \infty$$

for all $m, n \in \mathbb{N}$, i.e., $(y_{mn}) \in \ell^p_2$.

**Necessity.** The proof of necessity of the conditions (i), (ii), (iii) is easily obtained from the proof of Theorem 1 in [10].

(iv) We assume that conditions (i) and (iii) are satisfied but that (iv) does not hold. Then there exists a real number $Q$ such that

$$\lim_{m,n \to \infty} \sum_{l=1}^{\infty} |a_{k_0,l}^{m,n}| B_1^{1/p_{k_0,l}} = Q > 0$$

for some $k_0 \in \mathbb{N}$ and every $B > 1$. Hence we may choose strictly increasing sequences $(m(q)), (n(q))$ and $(r(q))$ such that

$$\left| \sum_{l=1}^{\infty} |a_{k_0,l}^{m,q,n(q)}| B_1^{1/p_{k_0,l}} - Q \right| < \frac{Q}{10} \quad (1)$$

and

$$\sum_{l=1}^{r(q)} |a_{k_0,l}^{m,q,n(q)}| B_1^{1/p_{k_0,l}} < \frac{Q}{10} \quad (2)$$

and

$$\sum_{l=r(q)+1}^{\infty} |a_{k_0,l}^{m,q,n(q)}| B_1^{1/p_{k_0,l}} < \frac{Q}{10} \quad (3)$$

hence, we have

$$\left| \sum_{l=r(q)+1}^{\infty} a_{k_0,l}^{m,q,n(q)} B_1^{1/p_{k_0,l}} - Q \right| \leq \sum_{l=1}^{\infty} |a_{k_0,l}^{m,q,n(q)}| B_1^{1/p_{k_0,l}} - Q + \sum_{l=1}^{r(q)} |a_{k_0,l}^{m,q,n(q)}| B_1^{1/p_{k_0,l}}$$

$$+ \sum_{l=r(q)+1}^{\infty} |a_{k_0,l}^{m,q,n(q)}| B_1^{1/p_{k_0,l}}$$

$$< \frac{3Q}{10} \quad (4)$$

using (1)–(3). Now, we define a sequence $x = (x_k)$ as follows:

$$x_k = \begin{cases} 0, & \text{if } k \neq k_0 \\ (-1)^i B_1^{1/p_{k_0,i}} \text{sgn}(a_{k_0,i}^{m,q,n(q)}), & \text{if } r(q) < i \leq r(q + 1) \\ B_1^{1/p_{k_0,i}}, & \text{if } k = k_0 \\ \text{otherwise and } k = k_0 \end{cases}$$
for any \( B > 1 \). It is trivial that \( x \in c_2^B(p) \). Then we obtain that

\[
|y_{m(q), n(q)} - (-1)^q Q| \leq \frac{Q}{2}
\]

from (2)–(4). Thus \( (y_{mn}) \not\in c_2^B(p) \), hence (iv) is necessary.

(v) The necessity of (v) is established in a similar manner.

(vi) Suppose that conditions (i) and (iii) are satisfied and (vi) does not hold. Then we may choose strictly increasing sequences \((m(p)), (n(p))\) and \((r(p)), (s(p))\) such that

\[
\sum_{k=1}^{r(p-1)} \sum_{l=1}^{s(p-1)} |a_{kl}^{m(p), n(p)}| B^{1/p_{kl}} \leq 2^{2p-2} \text{ (from (i))}
\]

(5)

\[
\sum_{k,l=1}^{\infty} |a_{kl}^{m(p), n(p)}| B^{1/p_{kl}} \geq 2^{2p}
\]

(6)

and from (iii) we have

\[
\sum_{k=r(p-1)+1}^{\infty} \sum_{l=s(p-1)+1}^{\infty} |a_{kl}^{m(p), n(p)}| B^{1/p_{kl}} + \sum_{k=1}^{r(p-1)} \sum_{l=s(p-1)+1}^{\infty} |a_{kl}^{m(p), n(p)}| B^{1/p_{kl}} + \sum_{k=1}^{r(p-1)} \sum_{l=1}^{s(p-1)} |a_{kl}^{m(p), n(p)}| B^{1/p_{kl}} + \sum_{k=l=s(p-1)+1}^{\infty} \sum_{l=1}^{r(p-1)} |a_{kl}^{m(p), n(p)}| B^{1/p_{kl}}
\]

(7)

Since

\[
\sum_{k,l=1}^{\infty} \leq \sum_{k=1}^{r(p-1)} \sum_{l=1}^{s(p-1)} + \sum_{k=1}^{r(p-1)} \sum_{l=s(p-1)+1}^{\infty} + \sum_{k=1}^{r(p-1)} \sum_{l=1}^{s(p-1)} + \sum_{k=r(p-1)+1}^{\infty} \sum_{l=1}^{s(p-1)+1}
\]

we have, from (5), (6), (7),

\[
\sum_{k=1}^{r(p-1)} \sum_{l=s(p-1)+1}^{\infty} |a_{kl}^{m(p), n(p)}| B^{1/p_{kl}} + \sum_{k=1}^{r(p-1)} \sum_{l=1}^{s(p-1)} |a_{kl}^{m(p), n(p)}| B^{1/p_{kl}} + \sum_{k=1}^{r(p-1)} \sum_{l=1}^{s(p-1)} \sum_{l=1}^{s(p-1)} |a_{kl}^{m(p), n(p)}| B^{1/p_{kl}} + \sum_{k=1}^{r(p-1)} \sum_{l=1}^{s(p-1)} \sum_{l=1}^{s(p-1)} |a_{kl}^{m(p), n(p)}| B^{1/p_{kl}}
\]

\[
\geq 2^{2p} - 2^{2p-2} - 2^{2p-2} = 2^{2p-1}.
\]

(8)

Now, we define a sequence \((x_{kl})\) as follows:

\[
x_{kl} = \begin{cases} B^{1/p_{kl}} \text{sgn} a_{kl}^{m(p), n(p)}, & \text{if } 1 \leq k \leq r(p-1) \\
B^{1/p_{kl}} \text{sgn} a_{kl}^{m(p), n(p)}, & \text{if } 1 \leq l \leq s(p-1), \\
0, & \text{otherwise.}
\end{cases}
\]
It is easy to see that \((x_{kl}) \in c^\mathbb{B}_2(p)\). But, we obtain that
\[
\left| y_{m(n),p} \right| = \left| \sum_{k,l=1}^{\infty} a_{kl}^{m,n} x_{kl} \right| \\
\geq \left( \sum_{k=1}^{\infty} \sum_{l=(p-1)+1}^{\infty} \left| a_{kl}^{m,n} \right| B_{1/p}^{1/p} \right) + \left( \sum_{k=(p-1)+1}^{\infty} \sum_{l=1}^{\infty} \left| a_{kl}^{m,n} \right| B_{1/p}^{1/p} \right) \\
+ \left( \sum_{k=1}^{\infty} \sum_{l=(p-1)+1}^{\infty} \left| a_{kl}^{m,n} \right| B_{1/p}^{1/p} \right) - \left( \sum_{k=(p-1)+1}^{\infty} \sum_{l=1}^{\infty} \left| a_{kl}^{m,n} \right| B_{1/p}^{1/p} \right) \\
\geq 2^{2p-1} - 2^{2p-2} \quad \text{(from (5) and (8))} \\
= 2^{2p} (2^{-1} - 2^{-2}) = 2^{2p-2}.
\]
Hence \(\lim_{p \to \infty} \left| y_{m(n),p} \right| = +\infty\). This contradicts our hypothesis. This completes the proof. □

**Example.** Let \(p_{kl} = \frac{1}{k+l}\) for all \(k, l \in \mathbb{N}\). Now, let us define the matrix \(A = (a_{kl}^{mn})\) as follows:
\[
a_{kl}^{mn} = \begin{cases} \frac{1}{mn(2B)^{k+l}}, & 1 \leq k \leq m \text{ and } 1 \leq l \leq n \\ 0, & \text{otherwise} \end{cases}
\]

where \(B\) is any real number such that \(B > 1\). Then \(A \in (c^\mathbb{B}_2(p), c^\mathbb{B}_2)\) since
\begin{itemize}
  \item[(i)] \(\lim_{m,n \to \infty} a_{kl}^{mn} = 0\) for each \(k, l \in \mathbb{N}\),
  \item[(ii)] \(\lim_{m,n \to \infty} \sum_{k,l=1}^{m,n} a_{kl}^{mn} = \lim_{k,l=1}^{m,n} \sum_{k,l=1}^{m,n} \frac{1}{mn(2B)^{k+l}} = \frac{1}{B^{-1}B},\)
  \item[(iii)] \(\lim_{k,l=1}^{m,n} \left| a_{kl}^{mn} \right| B_{1/p}^{1/p} = \sum_{k,l=1}^{m,n} \frac{1}{mn(2B)^{k+l}} \text{ converges for each } m \text{ and } n,\)
  \item[(iv)] \(\lim_{m,n \to \infty} \sum_{k=1}^{\infty} a_{kl}^{mn} B_{1/p}^{1/p} = \lim_{m,n \to \infty} \frac{1}{B^{-1}B} \sum_{k=1}^{\infty} \frac{1}{2B^{k}} = 0\) for \(k = 1, 2, \ldots,\)
  \item[(v)] \(\lim_{m,n \to \infty} \sum_{l=1}^{\infty} a_{kl}^{mn} B_{1/p}^{1/p} = \lim_{m,n \to \infty} \frac{1}{B^{-1}B} \sum_{l=1}^{\infty} \frac{1}{2B^{l}} = 0\) for \(l = 1, 2, \ldots,\)
  \item[(vi)] \(\sup_{m,n} \sum_{k,l=1}^{m,n} \left| a_{kl}^{mn} \right| B_{1/p}^{1/p} = \sup_{m,n} \sum_{k,l=1}^{m,n} \frac{1}{mn(2B)^{k+l}} = 1.\)
\end{itemize}

**Theorem 2.2.** Let \(0 < p_{kl} \leq \sup_{k,l=1}^{p_{kl}} = H < \infty \) and \(M = \max(1, H)\). Then \(A \in (c^\mathbb{B}_2(p), c^\mathbb{B}_2)\) if and only if
\begin{itemize}
  \item[(i)] \(\lim_{m,n \to \infty} a_{kl}^{mn} = 0\) for each \(k, l \in \mathbb{N}\),
  \item[(ii)] \(\lim_{m,n \to \infty} \sum_{k,l=1}^{m,n} a_{kl}^{mn} = 1,\)
  \item[(iii)] \(\lim_{k,l=1}^{m,n} \left| a_{kl}^{mn} \right| B_{1/p}^{1/p} \text{ converges for each } m \text{ and } n \text{ and for some } B > 1,\)
  \item[(iv)] \(\lim_{m,n \to \infty} \sum_{k=1}^{\infty} a_{kl}^{mn} B_{1/p}^{1/p} = 0\) for \(k = 1, 2, \ldots\) and for some \(B > 1,\)
  \item[(v)] \(\lim_{m,n \to \infty} \sum_{l=1}^{\infty} a_{kl}^{mn} B_{1/p}^{1/p} = 0\) for \(l = 1, 2, \ldots\) and for some \(B > 1,\)
  \item[(vi)] \(\sup_{m,n} \sum_{k,l=1}^{m,n} \left| a_{kl}^{mn} \right| B_{1/p}^{1/p} = C < \infty \) for some \(B > 1.\)
\end{itemize}

**Proof.** The proof follows from Theorem 2.1. □

The proofs of Theorems 2.3 and 2.4 are easily obtained from the proof of Theorem 1 in [10]. Therefore we omit it.

**Theorem 2.3.** Let \(p = (p_{kl})\) be any sequence of strictly positive real numbers and \(q = (q_{kl})\) be any bounded sequence of strictly positive real numbers. Then \(A \in (c^\mathbb{B}_2(p), a^\mathbb{B}_2)\) if and only if
\begin{itemize}
  \item[(i)] \(\lim_{k,l=1}^{m,n} \left| a_{kl}^{mn} \right| B_{1/p}^{1/p} \text{ converges for each } m \text{ and } n \text{ and for some } B > 1,\)
  \item[(ii)] \(\lim_{m,n \to \infty} a_{kl}^{mn} = 0\) for each \(k, l \in \mathbb{N}\),
  \item[(iii)] \(\lim_{k,l=1}^{m,n} \left| a_{kl}^{mn} \right| B_{1/p}^{1/p} = 0\) for \(k = 1, 2, \ldots\) and for some \(B > 1,\)
  \item[(iv)] \(\lim_{m,n \to \infty} \sum_{k=1}^{\infty} a_{kl}^{mn} B_{1/p}^{1/p} = 0\) for \(l = 1, 2, \ldots\) and for some \(B > 1,\)
  \item[(v)] \(\lim_{m,n \to \infty} \sum_{k,l=1}^{m,n} a_{kl}^{mn} B_{1/p}^{1/p} = 0\) for every real number \(B > 1.\)
\end{itemize}

Notice that if we also consider the condition \(\sum_{k,l=1}^{m,n} \left| a_{kl}^{mn} \right| B_{1/p}^{1/p} \leq C < \infty \) for all \(m, n \in \mathbb{N}\), we shall find the following inequality:
\[
\left| y_{m,n} \right|^{p_{mn}} \leq \left( \sum_{k,l=1}^{m,n} \left| a_{kl}^{mn} x_{kl} \right| \right)^{p_{mn}} \leq \left( \sum_{k,l=1}^{m,n} \left| a_{kl}^{mn} \right| B_{1/p}^{1/p} \right)^{p_{mn}} \leq C^{p_{mn}} \leq \max(1, C^M),
\]
where $|x_{kl}|^{p_{kl}} \leq B$ and $M = \max(1, \sup_{m,n \geq 1} q_{mn})$. Hence, we may give the following corollary:

**Corollary 2.1.** Let $p = (p_{kl})$ be any sequence of strictly positive real numbers and $q = (q_{kl})$ be any bounded sequence of strictly positive real numbers. Then $A \in (c_{2}^{p}(p), o_{2}^{p}(q))$ if and only if

$$\sum_{k,l=1}^{\infty} a_{kl}^{mn} B^{1/p_{kl}} \leq C,$$

for some $C > 0$ and for all $n, m \in \mathbb{N}$, holds and the conditions (i), (ii), (iii), (iv) and (v) in Theorem 2.3 are satisfied.

**Theorem 2.4.** Let $p = (p_{kl})$ and $q = (q_{kl})$ be any bounded sequences of strictly positive real numbers. Then $A \in (c_{2}^{p}(p), o_{2}^{p}(q))$ if and only if

(i) $\lim_{m,n \to \infty} a_{kl}^{mn} \leq C$ for each $m$ and $n$ and for some integer $B > 1$,

(ii) $\lim_{m,n \to \infty} a_{kl}^{mn} = 0$ for each $k, l \in \mathbb{N}$,

(iii) $\lim_{m,n \to \infty} \sum_{k,l=1}^{\infty} a_{kl}^{mn} = 0$,

(iv) $\lim_{m,n \to \infty} \sum_{k,l=1}^{\infty} a_{kl}^{mn} B^{1/p_{kl}} \leq C$ for some $C > 0$,

(v) $\lim_{m,n \to \infty} \sum_{k,l=1}^{\infty} a_{kl}^{mn} B^{1/p_{kl}} = 0$ for $l = 1, 2, \ldots$ and for some $B > 1$,

(vi) $\lim_{m,n \to \infty} \sum_{k,l=1}^{\infty} a_{kl}^{mn} B^{1/p_{kl}} = 0$ for every $B > 1$.

**Corollary 2.2.** Let $p = (p_{kl})$ and $q = (q_{kl})$ be any bounded sequences of strictly positive real numbers. Then $A \in (c_{2}^{p}(p), o_{2}^{p}(q))$ if and only if

(i) $\sum_{k,l=1}^{\infty} |a_{kl}^{mn}|^{q_{mn}} = 0$ for each $k, l \in \mathbb{N}$,

(ii) $\lim_{l=1}^{\infty} a_{kl}^{mn} = 0$ for all $m, n \in \mathbb{N}$, and for some integer $B > 1$, then for the double sequence $x = (B^{1/p_{kl}} \text{sgn} a_{kl}^{mn}) \in \ell_{2}^{p}(p)$, we have $(y_{mn}) \not\in o_{2}^{p}(q)$. □

Now we give some examples for the classes $(c_{2}^{p}(p), o_{2}^{p}(q))$, $(c_{2}^{p}(p), o_{2}^{p}(q))$, $(c_{2}^{p}(p), o_{2}^{p}(q))$, $(c_{2}^{p}(p), o_{2}^{p}(q))$, $(c_{2}^{p}(p), o_{2}^{p}(q))$.

**Examples.** (i) Let $p_{kl} = \frac{k}{l+1}$ and $q_{kl} = 1$ for all $k, l \in \mathbb{N}$. Now, let us define the matrix $A = (a_{kl}^{mn})$ as follows:

$$a_{kl}^{mn} = \frac{1}{mn(k+1)(l+1)B^{k+l}}$$

where $B$ is any real number such that $B > 1$. Then, it is easy to see that $A \in (c_{2}^{p}(p), o_{2}^{p}(q))$ and $A \in (c_{2}^{p}(p), o_{2}^{p}(q))$.

Note that $A \in (c_{2}^{p}(p), o_{2}^{p}(q))$ and $A \in (c_{2}^{p}(p), o_{2}^{p}(q))$ since

$$\sup_{m,n} \sum_{k,l=1}^{\infty} \frac{1}{mn(k+1)(l+1)B^{k+l}} \leq 1.$$

(ii) Let $p_{kl} = \frac{1}{l(k!)(l)!} B^{k+l}$, $q_{kl} = 1 + \frac{1}{m}$ for all $m, n \in \mathbb{N}$ and

$$a_{kl}^{mn} = \frac{1}{m (k!) (l!) B^{k+l}}$$

where $B > 1$ is any real number. Then $A \in (c_{2}^{p}(p), o_{2}^{p}(q))$ and $A \in (c_{2}^{p}(p), o_{2}^{p}(q))$.

**Theorem 2.5.** Let $p = (p_{kl})$ and $q = (q_{kl})$ be any sequences of strictly positive real numbers. Then $A \in (\ell_{2}^{p}(p), o_{2}^{p}(q))$ if and only if

(i) $\lim_{m,n \to \infty} \sum_{k,l=1}^{\infty} a_{kl}^{mn} = 0$ for some $B > 1$,

(ii) $\sum_{k,l=1}^{\infty} a_{kl}^{mn} B^{1/p_{kl}}$ converges for each $m$ and $n$, and for some $B > 1$.

**Proof.** (\Rightarrow) Let $x \in \ell_{2}^{p}(p)$. Then, there is a real number $B > 1$ such that

$$\max(1, \sup_{k,l \in \mathbb{N}} |x_{kl}|^{p_{kl}}) < B$$

for all $k, l \in \mathbb{N}$. Using this and condition (ii), we obtain that

$$\left| \sum_{k,l=1}^{\infty} a_{kl}^{mn} x_{kl} \right| \leq \sum_{k,l=1}^{\infty} |a_{kl}^{mn}| B^{1/p_{kl}} < \infty,$$

i.e. the series $\sum_{k,l=1}^{\infty} a_{kl}^{mn} x_{kl}$ exists for each $m, n \in \mathbb{N}$. Then, we have

$$\lim_{m,n \to \infty} \sum_{k,l=1}^{\infty} a_{kl}^{mn} x_{kl} \leq \lim_{m,n \to \infty} \sum_{k,l=1}^{\infty} |a_{kl}^{mn}| B^{1/p_{kl}} = 0$$

for some integer $B > 1$, i.e. $(y_{mn}) \not\in o_{2}^{p}(q)$.

(\Rightarrow) Now, if (i) fails then $\lim_{m,n \to \infty} \sum_{k,l=1}^{\infty} a_{kl}^{mn} B^{1/p_{kl}} \neq 0$ for all $B > 1$, then for the double sequence $x = (B^{1/p_{kl}} \text{sgn} a_{kl}^{mn}) \in \ell_{2}^{p}(p)$, we have $(y_{mn}) \not\in o_{2}^{p}(q)$. □
The necessity of (ii) similarly follows.

**Example.** Let $a_{kl}^{mn} = \left(\frac{1}{mn(2B)^{k+l}}\right)$ for some integer $B > 1$, $p_{kl} = \frac{1}{k+l}$ and $q_{kl} = 1$ for all $k, l \in \mathbb{N}$. Since

(i) $\lim_{m,n \to \infty} \sum_{k,l=1}^{\infty} \frac{1}{mn(2B)^{k+l}} = \lim_{m,n \to \infty} \frac{1}{mn} = 0$,

(ii) It is easily seen that $\sum_{k,l=1}^{\infty} \frac{1}{mn(2B)^{k+l}}$ converge for each $m$ and $n$, so we have $\left(\frac{1}{mn(2B)^{k+l}}\right) \in (\ell^2_\infty(p), \ell^2(p))$.

Note that, in addition to conditions of Theorem 2.5, if $q = (q_{kl})$ is any bounded sequence of strictly positive real numbers and $\sup_{m,n \geq 1} \sum_{k,l=1}^{\infty} a_{kl}^{mn} \left| A_1^{1/p}\right| B^{1/p} = C$, for some $C > 0$, is satisfied then we obtain that $A \in (\ell^2_\infty(p), \ell^2(p))$.

### 3. An application: Generalized core

The well-known Knopp core of a bounded real single sequence $x = (x_n)$ is defined to be the closed interval $[\liminf x_n, \limsup x_n]$. In [11] Maddox determined the necessary and sufficient conditions to hold the inequality $\limsup x \leq \limsup x$, where $x \in \ell^\infty$. The concept of Pringsheim’s core of a real bounded double sequence $x = (x_{kl})$ was given by Patterson [12] as the closed interval $[\liminf x_{kl}, \limsup x_{kl}]$. In this section we generalize the Pringsheim’s core for $x \in \ell^\infty_\infty(p)$ and establish core theorems by using our matrix classes $(c^p_2(p), c^p_2)$ and $(c^p_2(p), c^p_2)$.

**Definition 3.1.** Let $x = (x_{kl})$ be a real double sequence and $x \in \ell^\infty_\infty(p)$. Then we define the generalized core of $x$ as

$$ (P) - \text{core}(x) = \left[\liminf |x_{kl}|^{p_{kl}}, \limsup |x_{kl}|^{p_{kl}}\right]. $$

First we give the following lemmas which will be useful to prove our main theorems related to this generalized core.

**Lemma 3.1.** Let $1 \leq \inf_{k,l \geq 1} p_{kl} = h < \sup_{k,l \geq 1} p_{kl} = H < \infty$. If $A$ is a $4$-dimensional matrix of real or complex numbers such that

(i) $\lim_{m,n \to \infty} a_{kl}^{mn} = 0$ for each $k, l \in \mathbb{N}$,

(ii) $\lim_{m,n \to \infty} \sum_{k,l=1}^{\infty} \left| a_{kl}^{mn}\right| B^{1/p}= 0$, for $k = 1, 2, \ldots$ and for some $B > 1$,

(iii) $\lim_{m,n \to \infty} \sum_{k,l=1}^{\infty} \left| a_{kl}^{mn}\right| B^{1/p} = 0$, for $l = 1, 2, \ldots$ and for some $B > 1$,

(iv) $\limsup_{m,n \to \infty} \sum_{k,l=1}^{\infty} \left| a_{kl}^{mn}\right| = M$,

hold then for any $x \in \ell^2_\omega(p)$, we have

$$ \limsup |y_{mn}| \leq M \max\left(\limsup |x_{kl}|^{p_{kl}}h, (\limsup |x_{kl}|^{p_{kl}})^H\right) $$

where $y_{mn} = A_{mn}(x)$.

Moreover, if $(a_{kl}^{mn})$ is a matrix of real numbers then there is a double sequence $(x_{kl})$ of real numbers, such that $0 < \limsup_{k,l \to \infty} |x_{kl}|^{p_{kl}} < \infty$ and

$$ \limsup |y_{mn}| = M \max\left(\limsup |x_{kl}|^{p_{kl}}h, (\limsup |x_{kl}|^{p_{kl}})^H\right). $$

**Proof.** Let $x \in \ell^2_\omega(p)$ and define $\ell = \limsup |x_{kl}|^{p_{kl}} < \infty$ so that there is a real number $B > 1$ such that $|x_{kl}|^{p_{kl}} \leq B$ for all $k, l \in \mathbb{N}$. Then given $\varepsilon > 0$, we can choose a positive integer $N$ such that $|x_{kl}|^{p_{kl}} < (\ell + \varepsilon)^{1/H}$ for each $k, l > N$. Thus,

$$ |y_{mn}| \leq \sum_{k,l=0}^{N} \left| a_{kl}^{mn}\right| |x_{kl}| + \sum_{k=0}^{\infty} \sum_{l=k+1}^{\infty} \left| a_{kl}^{mn}\right| B^{1/p} + \sum_{l=0}^{\infty} \sum_{k=l+1}^{\infty} \left| a_{kl}^{mn}\right| B^{1/p} + \sum_{k,l=0}^{\infty} \left| a_{kl}^{mn}\right| (\ell + \varepsilon)^{1/H} $$

$$ = \sum_{k,l=0}^{N} \left| a_{kl}^{mn}\right| |x_{kl}| + \sum_{k=0}^{\infty} \sum_{l=k+1}^{\infty} \left| a_{kl}^{mn}\right| B^{1/p} + \sum_{l=0}^{\infty} \sum_{k=l+1}^{\infty} \left| a_{kl}^{mn}\right| B^{1/p} $$

$$ + \max\left(\left((\ell + \varepsilon)^{1/H}\right)^h, ((\ell + \varepsilon)^{1/H})^H\right) \sum_{k,l=N+1}^{\infty} \left| a_{kl}^{mn}\right|. $$

Using the conditions (i), (ii), (iii) and (iv), we get $\limsup |y_{mn}| \leq M. \max(\ell^H, \ell^h)$, since $\varepsilon > 0$ is arbitrary.
To prove the second part, suppose that $M > 0$, since the case $M = 0$ is trivial. Using the conditions (i), (ii), (iii) and (iv) we may choose strictly increasing sequences $(m(i))$, $(n(j))$, $(k(i))$ and $(l(j))$, $i, j = 1, 2, \ldots$, such that

$$
\sum_{k,l=1}^{\infty} |a_{kl}^{m(i),n(j)}| > M - \frac{1}{i+j} \quad \left( \Rightarrow \sum_{k,l=1}^{\infty} |a_{kl}^{m(i),n(j)}| B^{1/p_{kl}} > M - \frac{1}{i+j} \right),
$$

$$
\sum_{k=1}^{k(i)-1} \sum_{l=1}^{l(j)-1} |a_{kl}^{m(i),n(j)}| B^{1/p_{kl}} < \frac{1}{i+j},
$$

and

$$
\sum_{k=1}^{k(i)-1} \sum_{l=1}^{l(j)-1} |a_{kl}^{m(i),n(j)}| B^{1/p_{kl}} < \frac{1}{i+j}.
$$

and

$$
\sum_{k=1}^{l(j)-1} \sum_{l=1}^{l(j)-1} |a_{kl}^{m(i),n(j)}| B^{1/p_{kl}} < \frac{1}{i+j}.
$$

where $B > 1$. Hence we obtain that

$$
\sum_{k=1}^{k(i)-1} \sum_{l=1}^{l(j)-1} |a_{kl}^{m(i),n(j)}| B^{1/p_{kl}} = \sum_{k,l=1}^{\infty} |a_{kl}^{m(i),n(j)}| B^{1/p_{kl}} - \sum_{k=1}^{k(i)-1} \sum_{l=1}^{l(j)-1} |a_{kl}^{m(i),n(j)}| B^{1/p_{kl}}
$$

$$
\quad - \sum_{k=1}^{k(i)-1} \sum_{l=1}^{l(j)-1} |a_{kl}^{m(i),n(j)}| B^{1/p_{kl}} - \sum_{k=1}^{k(i)-1} \sum_{l=1}^{l(j)-1} |a_{kl}^{m(i),n(j)}| B^{1/p_{kl}}< M - \frac{5}{i+j}.
$$

Now, we define a double sequence $(x_{kl})$ as follows

$$
x_{kl} = \begin{cases} 
B^{1/p_{kl}} \text{sgn}(a_{kl}^{m(i),n(j)}), & \text{if } k(i-1) + 1 \leq k \leq k(i) - 1 \\
1, & \text{if } k \geq k(i) \text{ and } l \geq l(j) - 1 \\
B^{1/p_{kl}}, & \text{otherwise},
\end{cases}
$$

where $B > 1$. It is trivial that $||x|| \leq B$ and lim sup $|x_{kl}|^{p_{kl}} = 1$. Then the matrix transform $(y_{mn})$ of this sequence is such that

$$
|y_{m(i),n(j)}| \geq \sum_{k=1}^{k(i)-1} \sum_{l=1}^{l(j)-1} |a_{kl}^{m(i),n(j)}| B^{1/p_{kl}} - \sum_{k=1}^{k(i)-1} \sum_{l=1}^{l(j)-1} |a_{kl}^{m(i),n(j)}| B^{1/p_{kl}} \quad \geq \quad M - \frac{5}{i+j} - \frac{4}{i+j} = M - \frac{9}{i+j}.
$$

Now it follows that

$$
\limsup |y_{mn}| \geq M = M \max \left[ (\limsup |x_{kl}|^{p_{kl}})^{\frac{1}{h}}, (\limsup |x_{kl}|^{p_{kl}})^{\frac{1}{h}} \right].
$$

From the first part of the lemma and (10), we must have the equality in (9). \(\Box\)

**Lemma 3.2.** Let $(p_{kl})$ be a convergent double sequence of strictly positive real numbers. Then
The limit \( \lim_{k,l \to \infty} a^{kl} \) exists for any real number \( \alpha > 0 \).

If \( x = (x_{kl}) \in c^{PB}_2(p) \) and \( (p_{kl}) \) is bounded, then \( \lim_{k,l \to \infty} |x_{kl}|^{p_{kl}} \) exists.

**Proof.** (i) is trivial.

(ii) It is well known that for any \( a, b \in \mathbb{C} \) and \( 0 < p \leq 1 \), the inequalities
\[
|a^p - b^p| \leq |a - b|^p \leq |a|^p + |b|^p,
\]
hold. Let \((p_{kl})\) be a convergent and bounded double sequence of strictly positive real numbers and \( x = (x_{kl}) \in c^{PB}_2(p) \). Then \( |x_{kl} - L_{p_{kl}}| \to 0 \) as \( k, l \to \infty \) for some \( L \). From (i) there exists an \( \ell > 0 \) such that \( |L_{p_{kl}}| \to \ell \) as \( k, l \to \infty \). Now from the preceding inequality we have
\[
|x_{kl} - L_{p_{kl}}| / |L_{p_{kl}}| \leq |x_{kl} - L_{p_{kl}}| / |L_{p_{kl}}| + |L_{p_{kl}}| / |L_{p_{kl}}| \to 0
\]
and from this
\[
\liminf_{k,l \to \infty} |x_{kl}|^{p_{kl}/M} \geq |x_{kl}|^{p_{kl} - |L|^{1/M} / |L|^{1/M}} \to 0,
\]
as \( k, l \to \infty \) whence \( |x_{kl}|^{p_{kl}} \to |\ell| \) as \( k, l \to \infty \), so \( \lim_{k,l \to \infty} |x_{kl}|^{p_{kl}} \) exists, where \( M = \max \{1, \sup_{k,l} p_{kl}\} \). This completes the proof. \( \Box \)

**Theorem 3.1.** Let \((p_{kl})\) be a convergent double sequence of strictly positive real numbers such that \( 1 \leq \inf_{k,l \geq 1} p_{kl} = h < \sup_{k,l \geq 1} p_{kl} = H \to \infty \) and \( A = (a^{mn}) \) be a four-dimensional matrix. If \( x = (x_{kl}) \in \ell^\infty_2(p) \) and
\[
\limsup_{m,n \to \infty} A_{mn}(x) \leq \limsup_{m,n \to \infty} |x_{kl}|^{p_{kl}} \tag{11}
\]
then

(i) \( A \in (c^{PB}_2(p), c^{PB}_2(p)) \).

(ii) \( \lim_{m,n \to \infty} \sum_{k,l=1}^{\infty} |a^{mn}_{kl}| \leq 1 \).

**Proof.** Let \( x \in c^{PB}_2(p) \) and (11) be hold. Then from Lemma 3.2(ii)
\[
\liminf_{k,l \to \infty} |x_{kl}|^{p_{kl}} = \limsup_{k,l \to \infty} |x_{kl}|^{p_{kl}} = \lim |x_{kl}|^{p_{kl}}
\]
exists and also
\[
\limsup_{k,l \to \infty} A_{mn}(-x) \leq \limsup_{k,l \to \infty} (-|x_{kl}|^{p_{kl}}).
\]
Hence, since \( \liminf_{k,l \to \infty} |x_{kl}|^{p_{kl}} = - \limsup_{k,l \to \infty} (-|x_{kl}|^{p_{kl}}) \) we have
\[
\liminf_{k,l \to \infty} |x_{kl}|^{p_{kl}} \leq \liminf_{k,l \to \infty} A_{mn}(x),
\]
and thus by (11), we have
\[
\liminf_{k,l \to \infty} |x_{kl}|^{p_{kl}} \leq \liminf_{k,l \to \infty} A_{mn}(x) \leq \limsup_{k,l \to \infty} A_{mn}(x) \leq \limsup_{k,l \to \infty} |x_{kl}|^{p_{kl}}.
\]
Hence \( Ax \in c^{PB}_2 \) and \( \limsup_{m,n \to \infty} A_{mn}(x) = \lim |x_{kl}|^{p_{kl}} \), i.e. \( A \in (c^{PB}_2(p), c^{PB}_2(p)) \). Thus the conditions of Lemma 3.1 are satisfied. By Lemma 3.1, there exists \( x \in \ell^\infty_2(p) \) such that \( \lim sup_{m,n} |x_{kl}|^{p_{kl}} = 1 \) and \( \lim sup_{m,n} |y_{mn}| = M \) which gives
\[
\limsup_{m,n \to \infty} \sum_{k,l=1}^{\infty} |a^{mn}_{kl}| \leq 1.
\]
Note that
\[
\limsup_{m,n \to \infty} \sum_{k,l=1}^{\infty} |a^{mn}_{kl}| > 1 \tag{12}
\]
does not hold. Indeed, we assume that (12) holds. Then we have
\[
\limsup_{m,n \to \infty} \sum_{k,l=1}^{\infty} |a^{mn}_{kl}| = 1 + \alpha \text{ for some } \alpha > 0.
\]
Using (12) and the conditions of \( A \in (c^{PB}_2(p), c^{PB}_2(p)) \), we can choose strictly increasing sequences of positive integers \((m(i)), (n(j)) \), \((k(i)) \) and \((l(j)) \), \( i, j = 1, 2, \ldots \) such that
\[
\sum_{k=k(i-1)+1}^{k(i)-1} \sum_{l=l(j-1)+1}^{l(j)-1} |a^{m(i),n(j)}_{kl}| > 1 + \frac{\alpha}{2}.
\]
Lemma 3.1

Proof. bounded. Now this, let \( H \). Hence, we have
\[
\lim_{x} \sum_{k} \left| a_{kl} \right| B^{1/p_{kl}} < \frac{\alpha}{8},
\]
\[
\lim_{x} \sum_{k} \sum_{l} \left| a_{kl} \right| B^{1/p_{kl}} < \frac{\alpha}{8},
\]
\[
\lim_{x} \sum_{k} \sum_{l} \left| a_{kl} \right| B^{1/p_{kl}} < \frac{\alpha}{8}.
\]

Therefore \( \left| y_{m(i),n(j)} \right| > 1 + \frac{\alpha}{2} - \frac{\alpha}{2} = 1 \) for \( x = (x_{kl}) \) as defined in Lemma 3.1. So we have
\[
\lim \sup |x_{kl}|^{p_{kl}} = 1 < \lim \sup |y_{mn}|
\]
which contradicts (11). Hence (ii) must hold. \( \square \)

Remark. Note that \( x \in \ell_{2}^{p_{kl}}(p) \) does not imply the existence of \( \lim_{k,l \to \infty} |x_{kl}|^{p_{kl}} \) in general. To show this, let \( x_{kl} = L + \frac{1}{k} \) and \( p_{kl} = 2 + \frac{(-1)^{k+l}}{k+l+1} \) for all \( k, l \in \mathbb{N} \), where \( L > 0 \) is a fixed number. Then \( p_{kl} > 0 \) for all \( k, l \in \mathbb{N} \) and \((p_{kl})\) is bounded. Therefore \( |x_{kl} - L|^{p_{kl}} = \frac{1}{k^{2} + \frac{(-1)^{k+l}}{k+l+1}} \). Now
\[
|x_{kl} - L|^{p_{kl}} = \begin{cases} 
1 & \text{if } k + l \text{ is even} \\
\frac{1}{k^{2} + \frac{(-1)^{k+l}}{k+l+1}} & \text{if } k + l \text{ is odd}.
\end{cases}
\]

Hence, we have \( \lim |x_{kl} - L|^{p_{kl}} = 0 \). But \( \lim \sup |x_{kl}|^{p_{kl}} = L^{3} \) and \( \lim \inf |x_{kl}|^{p_{kl}} = L \), therefore \( |x_{kl}|^{p_{kl}} \) does not exist.

However, in the case of the sequence \((p_{kl})\) is convergent and bounded double sequence of strictly positive real numbers (see Lemma 3.2), \( \lim_{k,l \to \infty} |x_{kl} - L|^{p_{kl}} = 0 \) and \( \lim_{k,l \to \infty} |x_{kl}|^{p_{kl}} = L_{1} \), but limits \( L \) and \( L_{1} \) may not be equal in general. To see this, let \( x_{kl} = L + \frac{1}{k} \) and \( p_{kl} = 3 + \frac{1}{k+l+1} \) for all \( k, l \in \mathbb{N} \), where \( L > 0 \) is a fixed number. As is seen, \((p_{kl})\) is convergent and bounded. Now \( |x_{kl}|^{p_{kl}} \to L^{3} = L_{1} \) and \( L \neq L_{1} \), for \( L \neq 1 \).

Theorem 3.2. Let \((p_{kl})\) be a double sequence of strictly positive real numbers such that \( 0 < p_{kl} < \sup_{k,l \geq 1} p_{kl} = H < \infty \) and \( A = (a_{kl})^{m} \) be a four-dimensional matrix. If \( A \) satisfies the conditions

(i) \( A \in \ell_{2}^{p_{kl}}(p), \ell_{2}^{p_{kl}} \text{ reg.} \)

(ii) \( \lim_{m,n \to \infty} \sum_{k,l=1}^{\infty} \left| a_{kl}^{mn} \right| = 1 \) then for all \( x = (x_{kl}) \in \ell_{2}^{p_{kl}}(p) \), we have
\[
\lim \sup A_{mn}(x) \leq \lim \sup |x_{kl}|^{p_{kl}}.
\]

Proof. For \( K, L > 1 \) we have
\[
|A_{mn}(x)| = \left| \sum_{k,l=1}^{\infty} a_{kl}^{mn} x_{kl} \right|
\leq \sum_{k,l=1}^{\infty} x_{kl} \left( \left| a_{kl}^{mn} \right| + \left| a_{kl}^{mn} \right| - \left| a_{kl}^{mn} \right| \right)
\leq B^{1/H} \sum_{k} \sum_{l=1}^{L} \left| a_{kl}^{mn} \right| + \sum_{k} \sum_{l=1}^{L} \left| a_{kl}^{mn} \right| B^{1/p_{kl}}
\leq B^{1/H} \sum_{k} \sum_{l=1}^{L} \left| a_{kl}^{mn} \right| B^{1/p_{kl}} + \sup_{k > K, l > L} \left| x_{kl} \right|^{p_{kl}} \sum_{k} \sum_{l=1}^{L} \sum_{k=K+1}^{\infty} \sum_{l=K+1}^{\infty} \left| a_{kl}^{mn} \right| B^{1/p_{kl}} + \sum_{k} \sum_{l=1}^{L} \left| a_{kl}^{mn} \right| B^{1/p_{kl}} + \sup_{k > K, l > L} \left| x_{kl} \right|^{p_{kl}} \sum_{k} \sum_{l=1}^{L} \sum_{k=K+1}^{\infty} \sum_{l=K+1}^{\infty} \left| a_{kl}^{mn} \right| B^{1/p_{kl}}.
\]

where \( \sup_{k,l \geq 1} |x_{kl}|^{p_{kl}} = B \). Using condition (ii) and the conditions of Theorem 2.1, we have \( \lim \sup A_{mn}(x) \leq \lim \sup |x_{kl}|^{p_{kl}} \).

This completes the proof of the theorem. \( \square \)
Acknowledgements

This research was supported by FUBAP (The Management Union of the Scientific Research Projects of Firat University) under the Project Number 1179 when Professor Mursaleen visited Firat University (May–June, 2007). He is very much grateful to the Firat University for providing hospitality.

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