Robust global synchronization of two complex dynamical networks

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We investigate the synchronization of two coupled complex dynamical networks, a problem that has been termed outer synchronization in the literature. Our approach relies on (a) a basic lemma on the eigendecomposition of matrices resulting from Kronecker products and (b) a suitable choice of Lyapunov function related to the synchronization error dynamics. Starting from these two ingredients, a theorem that provides a sufficient condition for outer synchronization of the networks is proved. The condition in the theorem is expressed as a linear matrix inequality. When satisfied, synchronization is guaranteed to occur globally, i.e., independently of the initial conditions of the networks. The argument of the proof includes the design of the gain of the synchronizer, which is a constant square matrix with dimension dependent on the number of dynamic variables in a single network node, but independent of the size of the overall network, which can be much larger. This basic result is subsequently elaborated to simplify the design of the synchronizer, to avoid unnecessarily restrictive assumptions (e.g., diffusivity) on the coupling matrix that defines the topology of the networks and, finally, to obtain synchronizers that are robust to model errors in the parameters of the coupled networks. An illustrative numerical example for the outer synchronization of two networks of classical Lorenz nodes with perturbed parameters is presented. © 2013 AIP Publishing LLC.

The seminal work in Ref. 1 sparked the interest in the so-called complex network models, which have become a major tool in the analysis of many physical, biological, and social phenomena. One particular topic that has attracted the researchers’ attention is the analysis of how synchronization occurs in this class of models, with the expectation of gaining new insights of the interactions taking place in real-world complex systems. Most previous research is focused on the synchronization of a collection of interconnected nodes, forming a single network, where each node is a dynamical system governed by a set of nonlinear differential equations, possibly displaying chaotic dynamics. In this paper, we study an extended version of this problem. In particular, we consider a setup consisting of two complex networks which are coupled unidirectionally, in such a way that a set of signals from the master network are injected into the response network, and then investigate how synchronization is attained. Our analysis is fairly general and provides easy-to-check sufficient conditions for global synchronization, i.e., conditions that ensure that the networks synchronize independently of their initial conditions. The proposed approach is based on a simple definition of the synchronization error and the use of a suitable Lyapunov function. From these ingredients, we build up several new theorems that ensure the attainment of synchronization in various scenarios, including, e.g., cases in which the coupling matrix of the networks is non-diffusive (hence we cannot avoid this assumption, which is almost invariably made in the literature) or in which the network parameters are known only up to a bounded perturbation. In all the cases of interest, we show that the scheme for coupling the networks is very simple, as it reduces to the computation of a single gain matrix whose dimension is independent of the number of network nodes. Although the main aim of this work is to provide analytical insights, some numerical illustrations (obtained by computer simulations) are also presented.

I. INTRODUCTION

In recent years, the study of synchronization phenomena in complex dynamical networks has attracted the interest of many researchers. Indeed, complex networks have become a mainstream area of research for over a decade, as they have been identified as powerful tools for modeling a variety of real-world systems [1,2] that otherwise appear intractable. The synchronization between two such networks (coupled in some suitable manner) was termed “outer” synchronization in Ref. 3, in order to distinguish it from the more thoroughly studied problem of “inner” synchronization, i.e., the synchronization of the nodes in a single network. The spread of an infectious disease across different groups of individuals is an example of a real-world phenomenon that can be modeled by way of outer synchronization. [3] Other examples can be found in Refs. 3 and 5.

The amount of analytical results related to outer synchronization that can be found in the literature is still limited. In Refs. 3 and 6, synchronization between two continuous-time [3] and discrete-time [6] complex networks in a master-slave configuration is investigated using similar approaches. In both cases, the coupling matrix that determines the topology of the networks is assumed to be diffusive and the design of

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the synchronizers is based on the calculation of Jordan canonical forms. Only local synchronization is guaranteed (as a result of using a linearization scheme in the analysis) and the master and slave systems have to be fully deterministic, i.e., no unknown perturbations of the network variables or parameters are considered. Another example of using linearization techniques to attain local outer synchronization is Ref. 4. Similar to Refs. 3 and 6, the networks are assumed to be fully deterministic and the coupling matrix diffusive, and additionally, balanced.

The work in Ref. 8 is concerned with achieving outer synchronization in finite time between two networks whose state variables are perturbed by an additive Brownian motion process. The networks are assumed to have the same node dynamics but possibly different topological structure. The same as in the previous references, the coupling matrices are assumed to be diffusive. An important feature of the scheme in Ref. 8 is that the necessity to gather signals from all the nodes in the master network in order to compute the synchronizer for each individual node in the slave network. For large networks, this feature may arise obvious difficulties with the practical complexity of the scheme. The model parameters are also assumed to be deterministic and known, the same as in Refs. 3, 4, and 6.

An example of a contribution where synchronization is robust to perturbations in the model parameters, which can be different and unknown across different network nodes and across different networks, is Ref. 9. Some of the schemes proposed in the latter work are also proved to be valid for networks with non-diffusive coupling matrices. However, similar to Refs. 3, 4, and 6, the results in Ref. 9 rely on the use of linearization techniques for the analysis, and hence, they can only guarantee local outer synchronization. The problem of inner synchronization has been studied for stochastic networks, where the dynamics of the nodes is contaminated by additive noise (modeled as Brownian motion) and random failures of signal transmissions among the nodes are also modeled. In this case, it is possible to obtain probabilistic criteria for inner synchronization. However, this approach does not account for uncertainties in the network parameters and has not been extended to the case of outer synchronization yet.

In this paper, we investigate robust schemes for global outer synchronization of two diffusively coupled complex dynamical networks in which the model parameters are not known, i.e., they are subject to an unknown perturbation with respect to their nominal values. Our approach relies on

(a) a basic lemma on the eigendecomposition of matrices resulting from Kronecker products and
(b) a suitable choice of a Lyapunov function related to the synchronization error dynamics.

Starting from these two ingredients, a theorem that provides a sufficient condition for the global outer synchronization of two networks with known parameters and a diffusive coupling matrix is proved. The sufficient condition in the latter theorem is formally given as a linear matrix inequality (LMI) that has to be satisfied by the system of coupled networks. The argument of the proof includes the design of the gain of the synchronizer, which is a constant square matrix with dimension given by the number of dynamic variables in a single network node. Therefore, the complexity of the scheme is independent of the size of the overall network, which can be much larger.

The basic result is subsequently elaborated, first in order to simplify the design of the synchronizer while holding the assumption of the coupling matrix being diffusive. Then, the latter assumption is relaxed and a sufficient condition for global outer synchronization is given. The corresponding LMI involves the maximum eigenvalue of the coupling matrix but avoids any other assumptions on it (in particular, the coupling matrix is not assumed to be diffusive anymore). Next, we investigate schemes that reduce the dimension of the synchronizer signals, which can be made lesser than the dimension of the state in a single node. Finally, we obtain synchronizers that are robust to model errors in the parameters of the networks. As before, sufficient conditions for global synchronization are given in the form of a LMI with only mild assumptions on the coupling matrix. An illustrative numerical example for the outer synchronization of two networks of classical Lorenz nodes with perturbed parameters is presented.

The rest of the paper is organized as follows. In Sec. II, we present a formal description of the network model and a statement of the synchronization problem to be addressed. A set of auxiliary results, including the key lemma on the eigendecomposition of matrices resulting from Kronecker products, is presented in Sec. III. In Sec. IV, we introduce the main analytical results. A numerical example is presented in Sec. V, and finally, Sec. VI is devoted to the conclusions.

II. NETWORK MODEL AND PROBLEM STATEMENT

A. Network model

Consider a network that consists of \( N \) identical nodes, each one being a dynamical subsystem described by an \( n \)-dimensional system of differential equations. The network is formally described as

\[
\dot{x}_i = Ax_i + f(x_i) + \sum_{j=1}^{N} c_{ij}Lx_j, \quad i = 1, \ldots, N, \tag{1}
\]

where \( x_i(t) \in \mathbb{R}^n \) is the state vector of node \( i \) at time \( t \), \( f \) comprising \( n \) real variables, \( A \in \mathbb{R}^{n \times n} \) is an \( n \times n \) matrix with real entries that describes the linear component of the node dynamics, while \( f : \mathbb{R}^n \to \mathbb{R}^n \) is a continuous function which describes the nonlinear component of the dynamics of the individual node. Function \( f \) is assumed to be Lipschitz with constant \( \eta \), i.e.,

\[
|f(x) - f(y)| \leq \eta \|x - y\| \quad \forall x, y \in \mathbb{R}^n, \tag{2}
\]

where \( \| \cdot \| \) denotes the Euclidean norm of a vector. The entries of the matrix \( L \in \{0, 1\}^{N \times n} \) are binary, linking scalar state variables in \( x_i(t) \), while the \( N \times N \) coupling matrix \( C \in \mathbb{R}^{N \times N} \) indicates the network topology, i.e., the connections existing among the \( N \) nodes. We write \( c_{ij} \) to denote the (real)
entry in the $i$-th row and $j$-th column of $C$. If $c_{ij} > 0$ then there exists a link from node $i$ to node $j$ ($i \neq j$), while $c_{ij} = 0$ when the nodes are not connected directly. If positive, the entry $c_{ij}$ indicates also the strength of the connection between the nodes $i$ and $j$.

In order to investigate outer synchronization between two identical networks, we consider Eq. (1) as the master network and assume the response system to be coupled with the master through the scheme

$$\dot{y}_i = Ay_i + f(y_i) + \sum_{j=1}^{N} c_{ij} L y_j + u_i, \quad i = 1, \ldots, N$$

where $y_i(t) \in \mathbb{R}^n$ is the $n \times 1$ state vector of the $i$-th node in the response network, $u_i(t) \in \mathbb{R}^d$ is the synchronizer signal, defined as $u_i = K(y_i - x_i)$, and $K$ is a constant matrix which will be later designed in such a way that outer synchronization can be guaranteed. To be specific, we define global outer synchronization schemes can be found without assuming diffusivity, symmetry, balance, and irreducibility.

Definition 1. Networks (1) and (3) synchronize globally if, and only if,

$$\lim_{t \to \infty} \|y_i(t) - x_i(t)\| = 0, \quad \text{for } i = 1, 2, \ldots, N.$$  \hspace{1cm} (4)

Matrix $C$ is a key element for characterizing the dynamics of both Eqs. (1) and (3). In the literature, various assumptions are commonly made in order to simplify the analysis of the class of systems described by Eq. (1). Specifically, most authors assume the following properties.

- Irreducibility: The network is connected in such a way that there are no isolated clusters of nodes.
- Diffusivity: The matrix $C$ satisfies $\sum_{j=1}^{N} c_{ij} = 0$, $i = 1, 2, \ldots, N$. As a consequence, its diagonal elements can be written as $c_{ii} = -\sum_{j=1, j \neq i}^{N} c_{ij}$ (hence $c_{ii} < 0$ if the network is connected).
- Balance: The matrix $C$ satisfies $\sum_{j=1}^{N} c_{ij} = 0$, $i = 1, 2, \ldots, N$.
- Symmetry: for all $i, j$, $c_{ij} = c_{ji}$. A network with nondirected edges has a symmetric coupling matrix.

Obviously, if $C$ is symmetric, then it is balanced, but the opposite is not necessarily true. In this paper, we relax all of these assumptions. Indeed, we show that appropriate synchronization schemes can be found without assuming diffusivity, symmetry, balance, and irreducibility.

In the sequel, we assume that the matrix $L$ in Eqs. (1) and (3) is an $n \times n$ identity matrix, $L = I_n$. This is done for the sake of clarity in the presentation of the analytical results, but they can be extended for the other values of $L$.

### B. Problem statement

Let us introduce the $n \times 1$ error signal $e_i(t) = y_i(t) - x_i(t) \in \mathbb{R}^n$. The error dynamics are described by the differential equation

$$\dot{e}_i = \dot{y}_i - \dot{x}_i$$

$$= Ae_i + f(y_i) - f(x_i) + \sum_{j=1}^{N} c_{ij} e_j + u_i,$$

where the synchronizer signal can be rewritten as $u_i = Ke_i$.

Stacking together the error signals for the $n$ pairs of nodes in the overall system, we can define the $nN \times 1$ global error vector

$$e(t) = [e_1^T(t), e_2^T(t), \ldots, e_N^T(t)] \in \mathbb{R}^{nN},$$

where the superscript $\top$ denotes transposition, and the resulting error dynamics can be compactly written as

$$\dot{e} = \left( I_N \otimes (A + K) + C \otimes I_n \right) e + f,$$

where

$$f = \begin{bmatrix} f(y_1(t)) - f(x_1(t)) \end{bmatrix}$$

is the $nN \times 1$ vector collecting the synchronization error in the nonlinear components of the nodes.

The goal of this paper is to find sufficient conditions for the two networks to synchronize globally, i.e., to ensure that $\lim_{t \to \infty} \|e(t)\| = 0$ irrespective of the initial conditions. Note that, since the matrices $A$ and $C$ and the nonlinear function $f$ are given, synchronization has to be achieved by a proper design of the gain matrix $K$ alone.

### III. ANCILLARY RESULTS

Studying the global synchronization of the networks (1) and (3) amounts to analyzing the global asymptotic stability of the $nN$-dimensional error signal $e(t)$, whose dynamics are determined by Eq. (7). Since the number of nodes in the network, $N$, can be very large in practical applications, matrix-vector calculations involving $e(t)$ (and the coupling matrix $C$ as well) may turn out prohibitive, and hence, there is a need to find analytical methods which are both rigorous and computationally efficient. In this section, we review a number of auxiliary results that will be later used to alleviate these difficulties.

Definition 2. (Adapted from Ref. 15) Let $z = [z_1, \ldots, z_m]^T \in \mathbb{R}^m$ be an $m \times 1$ variable vector and let $F_i = F_i^T \in \mathbb{R}^{m \times m}$, $i = 1, \ldots, m$, be a sequence of symmetric matrices. An LMI has the form

$$F(z) \triangleq F_0 + \sum_{i=1}^{m} z_i F_i < 0,$$  \hspace{1cm} (9)

where the notation “$F(z) < 0$” indicates that $F(z)$ is negative definite.

The inequality (9) is a convex constraint on the space of $z$, i.e., the set $\{z \in \mathbb{R}^m : F(z) < 0\}$ is convex. An analogous definition can be given with $F(z) > 0$, i.e., when $F(z)$ is a positive definite matrix.

The next lemma plays a main role in the stability analysis of Sec. IV and we need to introduce some notation for its proper statement. Consider an $n \times n$ matrix $A$ and an $m \times m$ matrix $B$ with eigenvalues and eigenvectors $(\lambda_i, \nu_i) \in \mathbb{R}^n \times \mathbb{R}$ and $(\mu_i, \eta_i) \in \mathbb{R}^m \times \mathbb{R}$, respectively, i.e.,
\[ Ax_i = \nu_i y_i, \quad i = 1, \ldots, n \quad \text{and} \]
\[ By_j = \mu_j y_j, \quad j = 1, \ldots, m, \]
where \( x_i = [x_{i1}, x_{i2}, \ldots, x_{in}]^T \) and \( y_j = [y_{j1}, y_{j2}, \ldots, y_{jm}]^T \).

Let us also introduce the \( nm \times nm \) matrix
\[ T = A \otimes I_n + I_n \otimes B, \quad (10) \]
where \( \otimes \) denotes the Kronecker product. The eigenvalues and eigenvectors of \( T \) are denoted as \( z_i \) and \( \tau_i \), respectively, i.e.,
\[ T z_i = \tau_i z_i, \quad i = 1, \ldots, nm, \quad (11) \]
where \( z_i = [z_{i1}, z_{i2}, \ldots, z_{im}]^T \in \mathbb{R}^{nm} \) is the \( i \)-th eigenvector of \( T \).

The following lemma makes a connection between the eigenvalues and eigenvectors of \( T \) and those of \( A \) and \( B \).

**Lemma 1.** Let \( X = [x_1, \ldots, x_n] \in \mathbb{R}^{n \times m} \) and \( Y = [y_1, \ldots, y_m] \in \mathbb{R}^{m \times m} \) be the matrices whose columns are the eigenvectors of \( A \) and \( B \), respectively. The eigenvectors of \( T \) have the form \( z_i = \psi_i (X \otimes Y) \), \( i = 1, \ldots, nm \), where \( \psi_i (M) \) is the operator that selects the \( i \)-th column of matrix \( M \), and the eigenvalues of \( T \) have the form:
\[ \tau_i = \nu_k + \mu_j, \quad (12) \]
where \( i = 1, 2, \ldots, mn \), \( k = [i/n] + 1 \) and \( j = i + n - kn \). (For a real number \( r \in \mathbb{R} \), \( [r] \) is the floor operator, \( [r] = \max\{a \in \mathbb{Z} : a \leq r\} \).

**Proof.** See Lemma 2 in Ref. 9.

The following lemma states that the eigenvalues of a matrix \( A \) are shifted by a constant \( k \) when we perform the operation \( A + kI \).

**Lemma 2.** Let \( \eta_i, i = 1, \ldots, n \) be the eigenvalues of the \( n \times n \) matrix \( A \). The eigenvalues of matrix \( A + kI_n \), where \( k \) is an arbitrary real constant, are
\[ \xi_i = \eta_i + k, \quad i = 1, \ldots, n. \quad (13) \]

**Proof.** See Ref. 16.

The next lemma states the well-known Lyapunov stability condition and makes a connection among the eigenvalues of a matrix, asymptotic system stability, and the satisfaction of an associated LMI.

**Lemma 3.** Let \( A \) be an \( n \times n \) matrix and let \( x(t) \in \mathbb{R}^n \) be a dynamic vector. The following statements are equivalent:

- The linear system \( \dot{x} = Ax \) is asymptotically stable around \( x = 0 \), i.e., \( \lim_{t \to \infty} x(t) = 0 \).
- All the eigenvalues of matrix \( A \) have negative real part.
- There exists a symmetric and positive definite matrix \( P \) such that the LMI
\[ PA + A^T P < 0 \]
is satisfied.

**Proof.** See Ref. 17.

As mentioned in Sec. I, most of the earlier work on the outer synchronization of complex dynamical networks builds upon the assumption that the coupling matrix \( C \) that describes the intra-network topology is diffusive. The reason is that this property entails a number of other useful results involving the eigenvalues of \( C \), which are stated by the next lemma.

**Lemma 4.** All the eigenvalues of diffusive matrix \( C \) have nonpositive real parts. Moreover, 0 is an eigenvalue of \( C \) in general and, if \( C \) is irreducible, then 0 is an eigenvalue with multiplicity one.

**Proof.** See Ref. 18.

Finally, the inequality below will be used in the stability analysis of the error dynamics (7) using Lyapunov functions.

**Lemma 5.** Choose arbitrary matrices \( A \) and \( B \) with compatible dimensions. The inequality
\[ A^T B + B^T A \leq A^T Q A + B^T Q^{-1} B \]
is satisfied for any positive definite square matrix \( Q \) with suitable dimensions.

**Proof.** See Ref. 19.

### IV. GLOBAL SYNCHRONIZATION

In this section, we introduce sets of sufficient conditions, expressed as LMIs that involve the gain matrix \( K \), for global synchronization of the networks (1) and (3). We start, in Sec. IV A, with basic results for systems where the model parameters are exactly known. We analyze the cases in which the coupling matrix is diffusive (the same as in the existing literature) and then relax this assumption. We also seek schemes where the dimension of the synchronizer signals can be reduced (namely, where it can be made smaller than the state space dimension \( n \)) in Sec. IV B. In Sec. IV C, we extend our analysis to systems of coupled networks where the model parameters are subject to an unknown (albeit bounded) perturbation.

#### A. Main results

Recall that the local synchronizer signals at the network nodes have the form \( u_i(t) = Ke_i(t) \), where \( e_i(t) = y_i(t) - x_i(t) \) is the synchronization error at the \( i \)-th node and \( K \) is a constant (network wide) gain matrix of dimension \( n \times n \), while \( \eta > 0 \) is the Lipschitz constant of the nonlinear function \( f \). The following theorem provides a sufficient condition for the gain matrix \( K \) to guarantee the global synchronization of networks (1) and (3) when the coupling matrix \( C \) is diffusive. This is a fundamental result that allows several extensions of the analysis as the assumptions on the model are changed.

**Theorem 1.** Assume that the coupling matrix \( C \) is diffusive. If there exist a symmetric and positive definite matrix \( X \in \mathbb{R}^{n \times n} \) and a positive definite matrix \( W \in \mathbb{R}^{n \times n} \) such that the LMI
\[ \left( A + \frac{1 + \eta^2}{2} I_n \right) X + X \left( A + \frac{1 + \eta^2}{2} I_n \right)^T + W + W^T < 0 \quad (14) \]
is satisfied, then the gain matrix \( K = WX^{-1} \) guarantees that the networks (1) and (3) synchronize globally.
Proof. Consider the radially unbounded Lyapunov function
\[ V = e^\top \tilde{P} e, \]  
where \( \tilde{P} \) is some symmetric positive definite matrix of dimension \( nN \times nN \). If \( \tilde{V} < 0 \), \( \forall e \neq 0 \), then the system of differential equations (7) is asymptotically stable around \( e(t) = 0 \) (irrespective of its initial condition). Therefore, it is enough to show that Eq. (14) implies \( \tilde{V} < 0 \) when the gain matrix is chosen as \( K = X^{-1}W \), and we proceed to prove the latter result.

The derivative \( \tilde{V} \) can be easily obtained as
\[ \dot{V} = e^\top \tilde{P} \dot{e} + \dot{e}^\top \tilde{P} e. \]  
If we denote, for conciseness,
\[ Y = I_N \otimes (A + K) \]  
and
\[ Z = C \otimes I_N, \]
then substituting Eq. (7) into Eq. (16) yields
\[ \dot{V} = e^\top (\tilde{P}(Y + Z) + (Y + Z)^\top \tilde{P}) e + e^\top \tilde{P}\dot{f} + \dot{f}^\top \tilde{P} e. \]  
Moreover, according to Lemma 5, we may choose an arbitrary positive definite matrix \( Q \) of dimension \( nN \times nN \) to obtain the inequality
\[ \dot{V} \leq e^\top (\tilde{P}(Y + Z) + (Y + Z)^\top \tilde{P}) e + e^\top \tilde{P}Q\tilde{P}e + \dot{f}^\top Q^{-1} \dot{f} \]  
and, using the fact that \( \| \dot{f} \| < \eta \| e \| \) (that follows from \( f \) being Lipschitz with constant \( \eta > 0 \)), we arrive at
\[ \dot{V} \leq e^\top (\tilde{P}(Y + Z) + (Y + Z)^\top \tilde{P} + \tilde{P}Q\tilde{P} + \eta^2 Q^{-1})e. \]  
Since \( Q \) is positive definite but otherwise arbitrary, we can select \( Q = \tilde{P}^{-1} \) which, when substituted into Eq. (21), results in
\[ \dot{V} \leq e^\top \left( \tilde{P} \left( Y + Z + \frac{1 + \eta^2}{2} I_{nN} \right) + \left( Y + Z + \frac{1 + \eta^2}{2} I_{nN} \right)^\top \tilde{P} \right) e. \]  
Let us write
- \( \{ \mu_i; i = 1, \ldots, n \} \) for the eigenvalues of matrix \( A + K \),
- \( \{ \zeta_j; j = 1, \ldots, N \} \) for the eigenvalues of the coupling matrix \( C \), and
- \( \{ \xi_\ell; \ell = 1, \ldots, nN \} \) for the eigenvalues of \( Y + Z + \frac{1 + \eta^2}{2} I_{nN} \).

On one hand, Lemma 1 enables us to obtain the eigenvalues of \( Y + Z \) in terms of the eigenvalues of \( A + K \) and \( C \). On the other hand, Lemma 2 yields the eigenvalues of \( Y + Z + \frac{1 + \eta^2}{2} I_{nN} \) given those of \( Y + Z \). Combining both results leads to
\[ \zeta_j = \mu_i + \zeta_j + \frac{1 + \eta^2}{2}, \quad \ell \in \{1, \ldots, nN\}, \]  
where the subscript \( \ell \) is a function of \( i \) and \( j \), namely,
\[ \ell = Ni + j, \quad \text{with} \quad i = 0, \ldots, n - 1, \quad \text{and} \quad j = 1, \ldots, N. \]

The following argument brings the proof to a conclusion. Assume that the inequality (14) holds true. If we pre-and post-multiply by \( X^{-1} \) then we obtain the inequality
\[ X^{-1} \left( A + \frac{1 + \eta^2}{2} I_n \right) + \left( A + \frac{1 + \eta^2}{2} I_n \right)^\top X^{-1} + X^{-1} W X^{-1} < 0. \]  
If we let \( K = WX^{-1} \) and denote \( P = X^{-1} \) (hence, \( P \) is symmetric and positive definite), Eq. (24) can be rewritten as
\[ P \left( A + \frac{1 + \eta^2}{2} I_n + K \right) + \left( A + \frac{1 + \eta^2}{2} I_n + K \right)^\top P < 0. \]  
Using Lemma 2, the eigenvalues of the matrix \( A + \frac{1 + \eta^2}{2} I_n + K \) are shown to have the form \( \{ \mu_i + \frac{1 + \eta^2}{2}; i = 1, \ldots, n \} \).

Moreover, from Lemma 3, the inequality (25) implies that the eigenvalues of \( A + \frac{1 + \eta^2}{2} I_n + K \) must have negative real parts, hence
\[ \Re \{ \mu_i + \frac{1 + \eta^2}{2} \} < 0, \quad i = 1, \ldots, n. \]  
Since the coupling matrix \( C \) is diffusive, it follows from Lemma 4 that
\[ \Re \{ \zeta_j \} \leq 0, \quad j = 1, \ldots, N, \]  
and Eqs. (23), (26), and (27) together imply that
\[ \Re \{ \xi_\ell \} < 0, \quad \ell = 1, \ldots, nN. \]

However, Lemma 3 and Eq. (28) show that
\[ \tilde{P} \left( Y + Z + \frac{1 + \eta^2}{2} I_{nN} \right) + \left( Y + Z + \frac{1 + \eta^2}{2} I_{nN} \right)^\top \tilde{P} < 0 \]  
for any symmetric \( \tilde{P} > 0 \). Combining Eqs. (22) and (29) yields \( \tilde{V} < 0, \forall e \neq 0 \), which concludes the proof. \( \square \)

Let \( \{ \xi_i; i = 1, \ldots, n \} \) be the eigenvalues of matrix \( A \) and let \( a^+ = \max_{1 \leq i \leq n} \Re \{ \xi_i \} \) be the maximum over the real parts of these eigenvalues. The following theorem shows that, in practice, it is possible to compute very simple gain matrices that lead to global outer synchronization of the networks.

**Theorem 2.** Assume that the coupling matrix \( C \) is diffusive and choose a real constant \( k \) satisfying the inequality
\[ k > \frac{1 + \eta^2}{2} + a^+. \]  

If the gain matrix is selected as \( K = -k I_n \) then the networks (1) and (3) synchronize globally.

Proof. From Lemma 2, the eigenvalues of \( A + K = A - k I_n \) have the form \( \mu_i = a_i - k \), \( i = 1, \ldots, n \). Therefore, from Eq. (30), we easily obtain that

\[
\Re \{ \mu_i + \frac{1 + \eta^2}{2} \} = \Re \{ a_i \} + \frac{1 + \eta^2}{2} - k \\
\leq \alpha^+ + \frac{1 + \eta^2}{2} - k < 0. \tag{31}
\]

Combining Eqs. (31) and (27), we arrive at Eq. (28) which, in turn, leads to the inequality (29) and, as a consequence, \( \dot{V} < 0 \) for all \( \forall e \neq 0 \). \( \Box \)

Theorems 1 and 2 still place the (common) assumption of diffusivity on the coupling matrix \( C \). However, a simple combination of the latter results yields a sufficient condition for synchronization that is free of any assumption on matrix \( C \). Specifically, let \( c^+ = \max_{1 \leq j \leq n} \Re \{ \lambda_j \} \) be the maximum over the real parts of the eigenvalues of the coupling matrix. The following result establishes a sufficient condition for synchronization with possibly non-diffusive matrix \( C \).

**Theorem 3.** If there exist a symmetric positive definite matrix \( X \) and positive definite matrix \( W \) such that the LMI

\[
(A + \left( \frac{1 + \eta^2}{2} + c^+ \right) I_n) X + X (A + \left( \frac{1 + \eta^2}{2} + c^+ \right) I_n)^T + W + W^T < 0 \tag{32}
\]

is satisfied, then the gain matrix \( K = WX^{-1} \) yields global synchronization of the networks (1) and (3).

Proof. Let \( \zeta_j, 1 \leq j \leq n \), be the eigenvalues of the matrix \( A + K + \left( c^+ + \frac{1 + \eta^2}{2} \right) I_n \). From Eq. (23), Lemma 2 and the definition of \( c^+ \), it follows that

\[
\max_{1 \leq j \leq n} \Re \{ \zeta_j \} \leq \max_{1 \leq j \leq n} \Re \{ \lambda_j \}.
\]

Therefore, if \( \max_{1 \leq j \leq n} \Re \{ \zeta_j \} < 0 \) then \( \max_{1 \leq j \leq n} \Re \{ \lambda_j \} < 0 \) which, in turn, implies \( \dot{V} < 0 \) for all \( \forall e \neq 0 \), and hence the global synchronization of the networks.

Lemma 3 shows that \( \max_{1 \leq j \leq n} \Re \{ \zeta_j \} < 0 \) if, and only if,

\[
P \left( A + K + \left( c^+ + \frac{1 + \eta^2}{2} \right) I_n \right) + \left( A + K + \left( c^+ + \frac{1 + \eta^2}{2} \right) I_n \right)^T P < 0 \tag{33}
\]

for some symmetric positive definite matrix \( P \). However, if the LMI (32) holds true and we choose \( P = X^{-1} \) and \( W = KX \) (hence, \( K = WX^{-1} \)), then Eq. (33) follows immediately, which concludes the proof. \( \Box \)

Remarkably, it is possible to apply Theorem 3 without explicitly calculating the eigenvalues of matrix \( C \). Note that the latter may be a computationally heavy task in practice, since it can be expected that \( N \gg n \). To avoid such a calculation, we resort to the following auxiliary lemma.

**Lemma 6 (Gershgorin circle theorem).** Let \( A \) be a complex \( n \times n \) matrix with entries \( a_{ij} \), and let \( \mathcal{D}(a_{ii}, \zeta_i) \) be the closed disc centered at \( a_{ii} \) with radius \( \zeta_i = \sum_{j \neq i} |a_{ij}| \). All eigenvalues of \( A \) lie in at least one of the Gershgorin discs \( \mathcal{D}(a_{ii}, \zeta_i), 1 \leq i \leq n \).

Proof. See Ref. 20.

Let \( c = \max_{1 \leq j \leq n} |a_{ij} + \sum_{j \neq i} |c_{ij}| \). The following corollary provides a simpler way of applying Theorem 3 in practical setups.

**Corollary 1.** If there exist a symmetric positive definite matrix \( X \) and positive definite matrix \( W \) such that the LMI

\[
(A + \left( \frac{1 + \eta^2}{2} + c \right) I_n) X + X (A + \left( \frac{1 + \eta^2}{2} + c \right) I_n)^T + W + W^T < 0 \tag{34}
\]

is satisfied, then the gain matrix \( K = WX^{-1} \) yields global synchronization of the networks (1) and (3).

Proof. From Lemma 6, it is seen that \( c^+ \leq c \). The result is then straightforward from Theorem 3. \( \Box \)

**B. Lower dimensional gain matrix**

In many cases, the networks may not be coupled through the full dimensional state signals \( x_i(t) \in \mathbb{R}^n \), \( i = 1, \ldots, N \). In control problems, in which the slave system modeled by network (3) has to be steered by the master system modeled by Eq. (1), this may be due to physical limitations of the actuators that should apply the control signals. In other scenarios, there may be real-world constraints in the manner the two systems modeled by the networks (1) and (3) can interact.

In order to account for such limitations, we substitute the synchronizer signals \( u_i = K(y_i - x_i) \) in network (3) by \( \bar{u}_i = B \bar{K}(y_i - x_i) \), \( i = 1, \ldots, N \), where \( B \in \mathbb{R}^{n \times m} \) is a given tall matrix, i.e., \( m < n \), and \( \bar{K} \in \mathbb{R}^{m \times m} \) is a short gain matrix, whose dimension is reduced with respect to the original matrix \( K \in \mathbb{R}^{m \times m} \). As a result of this substitution, the error dynamics of Eq. (7) becomes

\[
\dot{e} = (K_l \otimes (A + B \bar{K}) + C \otimes I_n) e + \bar{f}. \tag{35}
\]

It turns out that it is possible to carry out the same kind of global stability analysis that led to Theorem 1 when the networks are coupled through the lower-dimensional gain matrix \( \bar{K} \).

**Theorem 4.** Assume that the coupling matrix \( C \) is diffusive and \( B \in \mathbb{R}^{n \times m} \). If there exist \( W \in \mathbb{R}^{m \times m} \) and a symmetric and positive definite matrix \( X \in \mathbb{R}^{n \times n} \) such that the LMI

\[
(A + \left( \frac{1 + \eta^2}{2} \right) I_n) X + X (A + \left( \frac{1 + \eta^2}{2} \right) I_n)^T + BW + W^T < 0 \tag{36}
\]

is satisfied, then the gain matrix \( \bar{K} = WX^{-1} \) guarantees that the networks (1) and (3) synchronize globally.

Proof. The proof is essentially the same as for Theorem 1. In particular, we can work with exactly the same radially unbounded Lyapunov function \( V \) and arrive at the inequality (22). Then we write \( \{ \xi_{ij}; i = 1, \ldots, n \} \) for the eigenvalues of \( C \), \( \{ \xi_{ij}; i = 1, \ldots, n \} \) for the eigenvalues of \( Y + Z + \frac{1 + \eta^2}{2} I_n \).
but introduce \( \{ \tilde{\mu}_i \mid i = 1, \ldots, n \} \) for the eigenvalues of \( A + \mathbf{BK} \).
Then, from Lemmas 1 and 2, we obtain
\[
\zeta_\ell = \tilde{\mu}_j + \frac{1 + \eta^2}{2}, \quad \ell \in \{1, \ldots, nN\},
\]
where, \( \ell = Ni + j \), with \( i = 0, \ldots, n - 1 \), and \( j = 1, \ldots, N \), which is the straightforward counterpart of Eq. (23).

Finally, assume that Eq. (36) holds true. If we pre- and post-multiply by \( X^{-1} \) in Eq. (36) and then let \( \bar{K} = WX^{-1} \) and \( P = X^{-1} \) we obtain
\[
P\left( A + \frac{1 + \eta^2}{2} I_n + \mathbf{BK} \right) + \left( A + \frac{1 + \eta^2}{2} I_n + \mathbf{BK} \right)^\top P < 0.
\]

From Eq. (38), we can use Lemmas 2 and 3 in exactly the same way as in the proof of Theorem 1 to show that \( \Re(\zeta_\ell) < 0 \), \( \ell = 1, \ldots, nN \). The latter inequality combined with Lemma 3 brings us to Eq. (29) again which, combined with Eq. (22), yields \( V < 0 \) if \( \epsilon \neq 0 \).

Theorem 4 guarantees that, if the matrices \( W \in \mathbb{R}^{m \times n} \) and \( X > 0 \) exist, then global synchronization is attained with the reduced-dimension gain matrix \( \bar{K} = WX^{-1} \). However, depending on the pair of matrices \( (A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \), it may happen that no \( X > 0 \) and \( W \) exist that satisfy the LMI (36). This difficulty can be removed if we assume the pair of matrices \( (A, B) \) to be controllable.

Definition 3. The pair \( (A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \), \( m \leq n \), is controllable if the \( n \times nm \) real matrix \( [B, AB, \ldots, A^{(n-1)}B] \) has rank \( n \).

This definition refers to the classical notion of controllability. Recently, controllability of networks has been introduced in some references\(^{22-24} \) as the problem of applying appropriate control signals to a group of nodes of a network in such a way that a certain goal is achieved (e.g., attain inner synchronization) without actuating on other nodes directly.

If the pair \( (A, B) \) is controllable according to Definition 3 then we can select the eigenvalues of the sum \( A + \mathbf{BK} \) by adequately choosing \( \bar{K} \).

Lemma 7. A pair \( (A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \) is controllable if, and only if, for any valid\(^{25} \) choice \( \mathcal{U} = \{ \nu_i \mid i = 1, \ldots, n \} \) there exists \( \mathcal{K}_U \in \mathbb{R}^{m \times n} \) such that \( \mathcal{U} \) is the set of eigenvalues of the sum matrix \( A + \mathbf{BK}_U \).

Proof. See Ref.\(^{21} \) pp. 829–832.

Therefore, the pair \( (A, B) \) being controllable is actually a sufficient condition for global synchronization of the networks (1) and (3).

**Theorem 5.** Assume that the coupling matrix \( C \) is diffusive and \( B \in \mathbb{R}^{n \times m} \), \( m \leq n \), is such that the pair \( (A, B) \) is controllable. Then, there exists \( \tilde{K} \in \mathbb{R}^{m \times n} \) such that the networks (1) and (3) synchronize globally.

Proof. Recall that \( \{ \tilde{\mu}_i \mid i = 1, \ldots, n \} \) denotes the set of eigenvalues of \( A + \mathbf{BK} \). From Lemma 2, the eigenvalues of \( A + \mathbf{BK} + \frac{1 + \eta^2}{2} I_n \) have the form \( \nu_i = \tilde{\mu}_i + \frac{1 + \eta^2}{2} \), \( i = 1, \ldots, n \). However, if \( (A, B) \) is controllable, then, from Lemma 7, there exists \( \tilde{K} \) such that \( \Re(\nu_i) < -\frac{1 + \eta^2}{2} \), and hence \( \bar{K} \) is controllable.

From Lemma 3, \( \Re(\nu_i) < 0 \) for \( i = 1, \ldots, n \) implies that there exists some positive definite and symmetric \( P \in \mathbb{R}^{n \times n} \) such that
\[
P\left( A + \frac{1 + \eta^2}{2} I_n + \mathbf{BK} \right) + \left( A + \frac{1 + \eta^2}{2} I_n + \mathbf{BK} \right)^\top P < 0.
\]

If we choose \( X = P^{-1} \) and \( W = \tilde{K}P^{-1} \) then it is seen that Eq. (39) is equivalent to the LMI (36) in the statement of Theorem 4.

Remark 1. A gain matrix \( \tilde{K} \in \mathbb{R}^{m \times n} \) that guarantees synchronization can be found by computing \( X > 0 \) and \( W \) that satisfy the LMI (36). Theorem 5 simply guarantees that \( X > 0 \) and \( W \) exist.

Remark 2. It is straightforward to extend Theorem 4, in the same way as we have done with Theorem 1, to obtain results analogous to Theorems 2 and 3 and Corollary 1 when the networks are linked through the signals \( u_i(t) = \tilde{K}(y_i(t) - x_i(t)) \in \mathbb{R}^m \), \( m < n \). In particular, if \( \tilde{c} = \max_{1 \leq i \leq n} |c_{ii}| + \sum_{i \neq j} |c_{ij}| \) and there exist \( X > 0 \) symmetric and \( W \in \mathbb{R}^{m \times n} \) such that the LMI
\[
\begin{align*}
\left( A + \frac{1 + \eta^2}{2} I_n + \tilde{c} I_n \right) X + X \left( A + \frac{1 + \eta^2}{2} I_n + \tilde{c} I_n \right)^\top + BW + W^\top B^\top & < 0
\end{align*}
\]
is satisfied, then the gain matrix \( \tilde{K} = WX^{-1} \) yields global synchronization of the networks (1) and (3).

C. Robust synchronization

Very often, the fixed parameters of the networks, including the gain matrix \( K \), cannot be known exactly and, additionally, the system dynamics may be “contaminated” by external unknown perturbations. Such difficulties may typically arise from modeling errors or, simply, from the impossibility to characterize a physical, or otherwise real-world, system faithfully enough. For this reason, it is highly desirable to determine whether a synchronization scheme can be robust, i.e., whether global synchronization can be guaranteed despite such model mismatches and/or perturbations.

In this section, we assume that the matrices \( A \) and \( K \) appearing in Eqs. (1) and (3) are only available up to an unknown, bounded but possibly time-varying, mismatch, and additionally, we further introduce unknown additive perturbations that affect the slave network. The latter disturbance can be arbitrary, but bounded as well.

To be precise, the master and response networks are modelled as
\[
\begin{align*}
\dot{x}_i &= (A + \Delta A)x_i + f(x_i) + \sum_{j=1}^{N} c_{ij}x_j \\
\dot{y}_i &= (A + \Delta A)y_i + f(y_i) + \sum_{j=1}^{N} c_{ij}y_j + B(K + \Delta K)(y_i - x_i) + d_i(y_i - x_i),
\end{align*}
\]
respectively, for \( i = 1, \ldots, N \), where \( A \in \mathbb{R}^{n \times n} \) and \( K \in \mathbb{R}^{n \times m} \) are nominal (known) values, while \( \Delta A(t) \in \mathbb{R}^{n \times n} \) and \( \Delta K(t) \in \mathbb{R}^{m \times n} \) are unknown perturbations of the nominal values, and \( d_i(t)(y_i(t) - x_i(t)) \) is an additional disturbance of
Combining Eqs. (41) and (42), the dynamics of the overall synchronization error \( e(t) \in \mathbb{R}^{nN} \) is shown to be governed by the differential equation
\[
\dot{e} = (I_N \otimes ((A + \Delta A) + B(K + \Delta K)) + C \otimes I_n) e + \bar{f} + De,
\]
where
\[
D = \begin{pmatrix}
d_1I_n & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & d_NI_n
\end{pmatrix} \in \mathbb{R}^{mN \times nN}
\]
is a “disturbance” matrix, unknown but with diagonal form.

In order to study whether the perturbed networks (41) and (42) synchronize globally, i.e., whether \( \lim_{t \to \infty} \|e(t)\| = 0 \), we constrain the perturbations \( \Delta A(t), \Delta B(t) \) and \( d(t) \) to be uniformly bounded from above over time. In particular, we assume that there exist finite constants \( \gamma > 0, \kappa > 0 \) and \( \tau > 0 \) such that
\[
\| \Delta A(t) \| < \gamma, \quad (45)
\]
\[
\| B \Delta K(t) \| < \kappa, \quad \text{and} \quad (46)
\]
\[
\| D \| < \tau, \quad (47)
\]
where, for a square matrix \( M \), we use notation \( \| M \| \) to indicate the absolute value of the maximum eigenvalue of \( M \). If these upper bounds hold, it is straightforward to obtain an analog of Theorems 1 and 4 for the perturbed system of Eq. (43).

**Theorem 6.** Assume that the coupling matrix \( C \) is diffusive, \( B \in \mathbb{R}^{n \times m} \) is given and the inequalities (45)–(47) hold. If there exist \( X \in \mathbb{R}^{n \times n} \), positive definite and symmetric, and \( W \in \mathbb{R}^{m \times n} \) such that the LMI
\[
\begin{align*}
\left( A + \frac{4 + \gamma^2 + \kappa^2 + \eta^2 + \tau^2}{2} I_n \right) X + X \left( A + \frac{4 + \gamma^2 + \kappa^2 + \eta^2 + \tau^2}{2} I_n \right)^\top & + BW + W^\top B^\top < 0
\end{align*}
\]
is satisfied, then the gain matrix \( \tilde{K} = WX^{-1} \) guarantees that the networks (41) and (42) synchronize globally.

**Proof.** Let us consider again the radially unbounded Lyapunov function of Eq. (15). If we substitute Eq. (43) into Eq. (16) and then apply the bounds (45)–(47), we readily obtain (by way of Lemma 5, exactly the same as in the proof of Theorem 1)
\[
\dot{V} < e^\top \left( \tilde{P} \left( Y + Z + \frac{4 + \eta^2 + \gamma^2 + \kappa^2 + \tau^2}{2} I_{NN} \right) \right) e,
\]
which is the counterpart of the inequality (22), the only difference being the (larger) constant \( \frac{4 + \eta^2 + \gamma^2 + \kappa^2 + \tau^2}{2} \) instead of \( \frac{1 + \eta^2}{2} \) obtained in the absence of perturbations. It is now straightforward to follow the argument in the proofs of Theorems 1 and 4 to complete the proof.

**Remark 3.** Theorem 6 can be extended in the same manner as Theorems 1 and 4 to account for networks with non-diffusive coupling matrix \( C \). In particular, if \( \tilde{c} = \max_{1 \leq j \leq n} |c_{ij} + \sum_{k \neq j} |c_{ij}| | \) and there exist \( X > 0 \) symmetric and \( W \in \mathbb{R}^{m \times n} \) such that the LMI
\[
\begin{align*}
(A + \frac{4 + \gamma^2 + \kappa^2 + \eta^2 + \tau^2}{2} + \tilde{c} I_n) X + X (A + \frac{4 + \gamma^2 + \kappa^2 + \eta^2 + \tau^2}{2} + \tilde{c} I_n)^\top & + BW + W^\top B^\top < 0
\end{align*}
\]
is satisfied, then the gain matrix \( \tilde{K} = WX^{-1} \) yields global synchronization of the networks (41) and (42).

Moreover, the existence of \( W \) and \( X \) that satisfy Eq. (50) (and, hence, the existence of the gain matrix \( \tilde{K} = WX^{-1} \)) is guaranteed whenever the pair \( (A, B) \) is controllable.

**V. NUMERICAL SIMULATION**

In this section, we present computer simulation results that illustrate the application of Theorem 6 in Sec. IV C. In particular, we have considered the coupling of two scale-free networks, with \( N = 10 \) nodes each and the topology depicted in Figure 1. The \( 10 \times 10 \) coupling matrix that determines the connectivity of the network is
\[
C = \begin{pmatrix}
-4 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

![FIG. 1. Graphical representation of the topology determined by the coupling matrix C in Eq. (51). The links represented with thin solid lines correspond to coefficients of the form \( c_{ij} = 1 \); those with dashed lines correspond to coefficients of the form \( c_{ij} = 2 \); and the links with thick solid lines correspond to coupling coefficients of the form \( c_{ij} = 3 \).](image-url)
which can be readily shown to be diffusive. Each node in the master network corresponds to a classical 3-dimensional Lorenz system, i.e.,

\[
\begin{pmatrix}
\dot{x}_{1,1}(t) \\
\dot{x}_{1,2}(t) \\
\dot{x}_{1,3}(t)
\end{pmatrix} =
\begin{pmatrix}
\sigma(x_{1,1}(t)) - \sigma(t) & \sigma(t) & 0 \\
\rho \beta(t) - 1 & 0 & 0 \\
0 & 0 & -\beta(t)
\end{pmatrix}\begin{pmatrix}
x_{1,1}(t) \\
x_{1,2}(t) \\
x_{1,3}(t)
\end{pmatrix} +
\begin{pmatrix}
0 \\
-\lambda_{1,1}(t) x_{1,3}(t) \\
\lambda_{1,1}(t) x_{1,2}(t)
\end{pmatrix},
\]  

(52)

for \(i = 1, \ldots, N\), where \(\sigma(t) = \sigma + \Delta_\sigma(t)\), \(\beta(t) = \beta + \Delta_\beta(t)\) are perturbed parameters, with nominal values \((\sigma, \beta) = (10, 28, 8/3)\) and bounded unknown perturbations \(-1 \leq \Delta_\sigma(t), \Delta_\beta(t) \leq 1\). Therefore, the nominal system matrix \(A\) and perturbation matrix \(\Delta A(t)\) can be written as

\[
A = \begin{pmatrix}
-\sigma & \sigma & 0 \\
\rho & 0 & 0 \\
0 & 0 & \beta
\end{pmatrix}
\quad \text{and} \quad
\Delta A(t) = \begin{pmatrix}
-\Delta_\sigma(t) & \Delta_\sigma(t) & 0 \\
\Delta_\beta(t) & 0 & 0 \\
0 & 0 & \Delta_\beta(t)
\end{pmatrix},
\]

respectively. It is relatively straightforward to compute an upper bound for \(\|\Delta A(t)\|\). Indeed, the maximum eigenvalues of \(\Delta A(t)\) can be calculated directly to yield (at time \(t\))

\[
\Delta_\sigma(t)^2 + \frac{1}{2} \Delta_\beta(t)^2 + \frac{1}{2} \sqrt{4\Delta_\sigma(t)^4 + 4\Delta_\beta(t)^4}.
\]

Hence, from the assumption \(-1 \leq \Delta_\sigma(t), \Delta_\beta(t) \leq 1\) and Lemma 2, we readily obtain that \(\|\Delta A(t)\| \leq \gamma^2 = (\sqrt{3^2 + 3^2})^2\). In general, looser bounds can be obtained (in a simpler way, without explicitly computing eigenvalues) via the Gershgorin circle theorem (Lemma 6).

The additive disturbance in Eq. (42) is modeled by assuming that the \(d_i(t)\)'s, \(i = 1, \ldots, N\) are unknown random numbers in the interval \([-1, 1]\).

The Lipschitz constant of the nonlinear function \(f\) in Eq. (52) can be computed by taking the numerical \(L^2\) norm of the Jacobian \(J = \frac{\partial f}{\partial x}\) over the interval \([0, 1000]\), which yielded \(\eta = 53\) in our simulation.

In order to interconnect the nodes in the master and slave networks, we have selected the matrix \(B = [0, 1, 1]^T\). It is easy to check that this choice makes the pair \((A, B)\) controllable and, hence, it guarantees that the networks can be synchronized (see Theorem 5 and Remark 1). Also, since \(B\) has dimensions \(3 \times 1\), we establish only one signal channel between each pair of nodes and there is no actuation on \(y_i, i = 1, \ldots, N\), since the first entry of \(B\) is null. This should be compared with existing schemes in the literature. In Ref. 4, for instance, the same example is addressed but three signal channels are used and the parameters have to be exactly known (there are no perturbations).

We have used the LMI toolbox of Matlab to solve the LMI of Theorem 6. For \(B = [0, 1, 1]^T\), this yields a nominal gain matrix \(K = [-2.6, -0.027, 0.016] \times 10^6\). We note that the magnitude of the gain coefficients is large. However, in cases where this may lead to implementation difficulties, it is possible to trade off between the number of signal channels and the gain amplitudes. For example, if we take

\[
B = \begin{pmatrix}
0 & 1 & 1 \\
1 & 1 & 0
\end{pmatrix}
\]

then the nominal gain matrix becomes \(\tilde{K} = \begin{pmatrix}
-3.5 & 3.6 & -3.7 \\
8.5 & -8.8 & 8.8
\end{pmatrix} \times 10^3\). The results shown in this section correspond all of them to the choice \(B_1 = [0, 1, 1]^T\) and \(\tilde{K} = [-2.6, -0.027, 0.016] \times 10^6\).

The unknown perturbation of the gain matrix is assumed to have the form:

![FIG. 2. Time-varying perturbations of the network parameters.](image-url)
where $-1 \leq \Delta K_i(t), \Delta K_e(t) \leq 1$. As a result, the eigenvalues of $\mathcal{B} \Delta K(t)$ are $\{0, 0, \Delta K_i(t) + \Delta K_e(t)\}$, hence we can safely choose $\gamma_0^2 = 4$ as the upper bound in Eq. (46).

Figure 2 displays the values of the perturbations $\Delta c(t)$, $\Delta \beta(t)$, $\Delta K_i(t)$, and $\Delta K_e(t)$ over time for our simulation (recall that $d_i(t)$, $i = 1, \cdots, N$, are random, with $|d_i(t)| < 1$ as well).

Finally, Figure 3 plots the norm of the synchronization error, $|e(t)|$, versus time. We observe how $e(t)$ converges toward zero very quickly.

**VI. CONCLUSION**

We have addressed the problem of outer synchronization between two networks of nonlinear (possibly chaotic) dynamical oscillators. Our approach is based on a simple definition of the synchronization error and a proper choice of a radially unbounded Lyapunov function. Starting from these two ingredients, we have provided sets of sufficient conditions for the global synchronization of the networks, irrespective of their initial condition. Although the first such result, Theorem 1, relies on relatively restrictive assumptions (diffusive coupling matrix, perfectly known network parameters), we have subsequently relaxed them to obtain sufficient conditions that ensure global synchronization when the coupling matrix is non-diffusive, when the number of connections between the two networks is reduced and when the network parameters are only known up to a bounded perturbation. In all cases, the conditions for synchronization are expressed in terms of the feasibility of an LMI whose dimension is independent of the number of nodes in the networks. Such LMIs are simple to solve using standard software routines and the solutions can be used explicitly to design inter-network connections that guarantee synchronization. To summarize, the key contributions of the proposed approach compared to previous work are: (a) to avoid linearizations and other approximations in the analysis, hence ensuring that synchronization is attained independently of the networks initial conditions, (b) to avoid computations whose complexity depends on the network size (i.e., the number of nodes), and (c) to derive synchronization schemes that work with reduced-dimensional (even one-dimensional) signal channels between pairs of nodes. We have also provided computer simulation results that illustrate the application of the selected theoretical findings in the paper.

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25 The elements of $\mathcal{U}$ must either be real or appear in conjugate pairs in order to be valid eigenvalues of a real matrix.