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On the Representation of D.G. Seminear-rings

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Abstract. Some results about the representation of d.g. seminear-rings are given. While not all d.g. seminear-rings are faithful, it seems that there are conditions under which the d.g. seminear-rings are. To every (non-faithful) d.g. seminear-ring (S,T), we associate a faithful d.g. seminear-ring $(\breve{S},\breve{T})$ and prove the existence of such a d.g. seminear-ring. Finally, we show how adjoining an identity to a given d.g. seminear-ring will give the faithfulness we desire.

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1 Introduction

The theory of seminear-rings can be developed in many directions. One way is to study an important class of seminear-rings known as distributively generated (d.g.) seminear-rings. It seems that some concepts can be established and a lot of results can be obtained. The theory of near-rings have been studied intensively by Pilz [9], Meldrum [5], Clay [1], and others. The class of d.g. near-rings (see for example [2] and [3]) is an important part of this subject. In this context, it seems that some ideas and results concerning d.g. near-rings can be extended to the case of d.g. seminear-rings. While some fundamental ideas are defined in a way analogous to the case of d.g. near-rings, some concepts are not because semigroups are involved rather than groups. In this paper, we intend to develop an important aspect of seminear-rings by studying the representation of the class of d.g. seminear-rings. In this direction, we will show in Section 2 that while not all d.g. seminear-rings are faithful [6], there are some conditions under which the d.g. seminear-rings are. In Section 3, we prove that for every non-faithful d.g. seminear-ring, we can associate a faithful d.g. seminear-ring. Finally, in Section 4, we obtain some results about faithfulness and adjoining identities to d.g. seminear-rings.

In order to start, we need the following basic definitions and preliminaries.

A (left) seminear-ring is a set S with two operations $+$ and $\cdot$ such that both $(S,+)$ and $(S,\cdot)$ are semigroups, and the left distributive law is satisfied, that is, $a(b+c) = ab + ac$ for all $a,b,c \in S$. An element $d \in S$ is called distributive if

$(a+b)d = ad + bd$ for all $a, b \in S$. The set of all distributive elements of S forms a subsemigroup of $(S, \cdot)$. Let H be an additive semigroup. The set $\mathrm{M}(H)$ of all mappings of H into itself with pointwise addition and multiplication as composition of maps forms a seminear-ring. A seminear-ring S is called a d.g. seminear-ring if S contains a multiplicative subsemigroup $(T, \cdot)$ of distributive elements which generates $(S, +)$. T need not be the whole set of distributive elements, and such a d.g. seminear-ring is denoted by (S, T). The concept of d.g. seminear-rings was first studied in [6] as a generalization for the case of d.g. near-rings which was used earlier by Neumann [7, 8].

Consider the set $\mathrm{M}(H)$ as above. Then $\mathrm{End}(H)$, the set of all endomorphisms of H, forms a subsemigroup of $\mathrm{M}(H)$ and generates a d.g. seminear-ring denoted by $(\mathrm{E}(H), \mathrm{End}(H))$. A mapping $\theta : S \to D$ is called a seminear-ring homomorphism if θ is both a semigroup homomorphism from $(S, +)$ to $(D, +)$ and also from $(S, \cdot)$ to $(D, \cdot)$. A d.g. seminear-ring homomorphism $\theta : (S, T) \to (D, U)$ is a seminear-ring homomorphism which maps T into U. It is well known that a semigroup homomorphism $\theta : (S, +) \to (D, +)$ is a d.g. seminear-ring homomorphism from (S, T) to (D, U) if and only if θ is a semigroup homomorphism from $(T, \cdot)$ to $(U, \cdot)$. Let S be a seminear-ring. A semigroup H is called an S-module if there is a seminear-ring homomorphism $\theta : S \to \mathrm{M}(H)$, and such a homomorphism is called a representation of S. A representation θ is called faithful if $\mathrm{Ker}\,\theta$ is trivial. In this case, S is called a faithful seminear-ring. An S-module H is called monogenic if $H = hS$ for some $h \in H$. Let (S, T) be a d.g. seminear-ring. A representation θ of S is a d.g. representation if there is an S-module H associated with the representation θ such that $T\theta \subseteq \mathrm{End}(H)$. Note that a d.g. representation of (S, T) on H is a d.g. seminear-ring homomorphism from (S, T) to $(\mathrm{E}(H), \mathrm{End}(H))$.

Let Ω be a variety of semigroups. Given a set X, $F_\Omega(X)$ denotes the free additive semigroup in Ω on X. Let T be a multiplicative semigroup and define the semigroup $\mathrm{Frs}_\Omega(X, T)$ as the free additive semigroup in the variety Ω on the set of symbols $\{x, t_x : x \in X,\ t \in T\} = T_x$. For each $t \in T$, define a map $t^* : T_x \to \mathrm{Frs}_\Omega(X, T)$ by $x \cdot t^* := t_x$ and $(m_x)t^* := (mt)_x$ for all $x \in X$ and $m \in T$, which we extend to an endomorphism of $\mathrm{Frs}_\Omega(X, T)$. Let $T^* = \{t^* : t \in T\}$, then T^* is a semigroup of endomorphisms of $\mathrm{Frs}_\Omega(X, T)$. It can be easily seen that $T^* \cong T$ and hence we can assume that T is a semigroup of endomorphisms of $\mathrm{Frs}_\Omega(X, T)$ which will generate a d.g. seminear-ring, denoted by $(\mathrm{Frs}_\Omega(T), T)$ and called the free d.g. seminear-ring on T in Ω. We refer to [6] for the basic concept and results related to $\mathrm{Frs}_\Omega(X, T)$.

2 D.G. Representation and Faithfulness

We start with the following lemma which deals with monogenic S-modules.

Lemma 2.1. *A representation of a d.g. seminear-ring (S, T) on a monogenic S-module H is a d.g. representation.*

Proof. For a generator $h \in H$, consider the map $\theta_h : S \to H$ given by $(s)\theta_h = hs$ for all $s \in S$. Then θ_h is an S-homomorphism, and the fact that H is monogenic forces θ to be an epimorphism. Let $\sigma_h = \mathrm{Ker}\,\theta_h$, then $H \cong S/\sigma_h$ as an S-module. Now every element $t \in T$ is distributive, and consequently, t induces an endomorphism of $S/\sigma_h \cong H$. That is, $T \subseteq \mathrm{End}(H)$, as desired. □

The following example shows that, in general, not every representation of a d.g. seminear-ring is a d.g. representation.

Example 2.2. Let (S,T) be a d.g. seminear-ring and let H be a semigroup which properly contains a copy of $(S,+)$. For $s \in S$, define a map $\theta_s : S \to \mathrm{M}(H)$ by

$$(h)\theta_s = \begin{cases} s & \text{if } h \notin S, \\ hs & \text{if } h \in S. \end{cases}$$

Then $\theta_s \in \mathrm{M}(H)$. Now choose $h_1, h_2 \in H \setminus S$ such that $h_1 + h_2 \notin S$. So for a non-idempotent element $t \in T$, we have $(h_1+h_2)t = t$, while $h_1 t + h_2 t = t + t$. Hence, T cannot act as an endomorphism of H, which shows that such a representation is not a d.g. representation.

Now we give some attention to the d.g. representations that are faithful.

Lemma 2.3. *Let (S,T) in Ω have a faithful representation. Let $H = \mathrm{Frs}_\Omega(x,S,T)$ be the free (S,T)-semigroup on one generator x. Then the representation of (S,T) on H is faithful.*

Proof. Suppose that (S,T) has a faithful representation on M in Ω. Then for any pair $s_1 \neq s_2$ in S, there exists $m \in M$ such that $ms_1 \neq ms_2$. Now we map x to m and extend this mapping to an (S,T)-homomorphism $\theta : H \to M$. Let us suppose that the representation of (S,T) on H is not faithful. Then there exist $s_1, s_2 \in S$ such that $s_1 \neq s_2$ and $xs_1 = xs_2$. It follows that $ms_1 = (x\theta)s_1 = (xs_1)\theta = (xs_2)\theta = (x\theta)s_2 = ms_2$, which is a contradiction. Hence, (S,T) has a faithful representation on H. □

Theorem 2.4. *Let (S,T) be a d.g. seminear-ring. If T has a left identity, then (S,T) is faithful.*

Proof. Clearly, (S,T) has a d.g. representation θ on $(S,+)$. If e is a left identity for T, then it is also a left identity for S, since for $s = \sum_{i=1}^n t_i \in S$, we have

$$es = e\Big(\sum_{i=1}^n t_i\Big) = \sum_{i=1}^n et_i = \sum_{i=1}^n t_i = s.$$

Consider $\mathrm{Ker}\,\theta = \{(a,b) \in S \times S : xa = xb \;\forall x \in S\} = \{(a,a) : a \in S\}$. Hence, θ is a faithful representation of (S,T). □

Theorem 2.5. *Let (S,T) be a d.g. seminear-ring. If T is a set of free generators for S in Ω, then (S,T) has a faithful representation.*

Proof. It follows from [6, Theorem 2]. □

3 Lower Faithful D.G. Seminear-rings

Although a d.g. seminear-ring (S,T) may not have a faithful representation, it is the homomorphic image of a faithful d.g. seminear-ring, namely, $(\mathrm{Frs}(T),T)$. This idea will lead to the following work.

Definition 3.1. Let (S,T) be a d.g. seminear-ring. The lower faithful d.g. seminear-ring for (S,T) is a faithful d.g. seminear-ring $(\breve{S},\breve{T})$ with a d.g. seminear-ring homomorphism $\theta : (S,T) \to (\breve{S},\breve{T})$ such that $T\theta = \breve{T}$, and if (D,U) is a faithful d.g. seminear-ring and $\phi : (S,T) \to (D,U)$ is a d.g. seminear-ring homomorphism, then there exists a unique d.g. seminear-ring homomorphism $\psi : (\breve{S},\breve{T}) \to (D,U)$ such that $\phi = \theta\psi$.

Our first aim is to prove the existence of the lower faithful d.g. seminear-ring for a given d.g. seminear-ring. To this aim, we need the following.

Lemma 3.2. *Let $\theta : (S,T) \to (D,U)$ be a d.g. seminear-ring homomorphism. Let H be a (D,U)-semigroup with representation ϕ. Then H can be defined as an (S,T)-semigroup.*

Proof. This is easily seen if we define $\psi : (S,T) \to (\mathrm{E}(H), \mathrm{End}(H))$ by $\psi = \theta\phi$. □

The following result which deals with relationship between representations can be easily checked; so we omit its proof.

Lemma 3.3. *Let (S,T) and (D,U) be d.g. seminear-rings. Let $\theta : (S,T) \to (D,U)$ be a d.g. seminear-ring homomorphism. Let H be a (D,U)-semigroup with representation ϕ. Then (S,T) has a d.g. representation on H given by $\theta\phi$, and the kernel of this representation is $\theta^{-1}(\mathrm{Ker}\,\phi)$.*

Lemma 3.4. *Let (S,T) be a d.g. seminear-ring and let $H = \mathrm{Frs}(x,S,T)$ be the free (S,T)-semigroup on one element x. Let $\sigma = \{(s_1,s_2) \subseteq S\times S : hs_1 = hs_2 \ \forall\, h \in H\}$. If θ is a representation of (S,T) on a semigroup K, then $\mathrm{Ker}\,\theta \supseteq \sigma$.*

Proof. Assume to the contrary that σ is not contained in $\mathrm{Ker}\,\theta$. Let $(s_1,s_2) \in \sigma \setminus \mathrm{Ker}\,\theta$. Then there exists $k \in K$ such that $k(s_1\theta) \neq k(s_2\theta)$. Now we map x to k and extend this mapping to an (S,T)-homomorphism $\phi : H \to K$. Thus, we have $k(s_1\theta) = x\phi s_1 = (xs_1)\phi = (xs_2)\phi = (x\phi)s_2 = k(s_2\theta)$, which contradicts our assumption. Hence, $\mathrm{Ker}\,\theta \supseteq \sigma$. □

Now we prove the existence of the lower faithful d.g. seminear-rings.

Theorem 3.5. *Let (S,T) be a d.g. seminear-ring. Let σ be as defined in the above lemma. Then the lower faithful d.g. seminear-ring for (S,T) is $(S,T)/\sigma$.*

Proof. Let $H = \mathrm{Frs}_\Omega(x,S,T)$. Then $(S,T)/\sigma$ has a faithful d.g. representation on H. Consider the canonical homomorphism $\theta : (S,T) \to (S,T)/\sigma$. Then it is clear that $T\theta = T\sigma/\sigma$. So it only remains to verify the last part of Definition 3.1. Let (D,U) be a faithful d.g. seminear-ring with a d.g. seminear-ring homomorphism $\phi : (S,T) \to (D,U)$. Let $K = \mathrm{Frs}_\Omega(x,D,U)$ be the free (D,U)-semigroup on the element x, and let η be the representation of (D,U) on K. By Lemma 2.3, $\mathrm{Ker}\,\eta$ is trivial. Applying Lemma 3.2, we deduce that K is an (S,T)-semigroup and the kernel of the d.g. representation $\phi\eta$ is $\mathrm{Ker}\,\phi$. This means that $\mathrm{Ker}\,\phi \supseteq \sigma$ by Lemma 3.4. Hence, there is a unique homomorphism $\psi : (S,T)/\sigma \to (D,U)$ such that $\theta\psi = \phi$. This completes the proof. □

Theorem 3.6. *Let (S,T) be a faithful d.g. seminear-ring in a variety Ω, and let $H = \mathrm{Frs}_\Omega(x,S,T)$. Then $H = H_1 * H_2$ is the free product of H_1 and H_2, where H_1 is the free semigroup on one generator in Ω and H_2 is a semigroup isomorphic to $(S,+)$.*

Proof. Let $(F,T) = (\mathrm{Frs}(T),T)$ be the free d.g. seminear-ring on the semigroup T, and let K be the free (F,T)-semigroup on one element x. By [6, Theorem 2], K is the free semigroup in Ω on the set $T_x = \{x, t_x : t \in T\}$. Let ρ be the kernel of the canonical map from (F,T) to (S,T). Then $(S,T) = (F,T)/\rho$ and $H = K/(K\rho)^K$, where $(K\rho)^K$ is the least congruence on K which contains $K\rho$. We can write $K = M_1 * M_2$, where M_1 is the semigroup generated by x, M_2 is the semigroup generated by the set $\{t_x : t \in T\}$, both M_1 and M_2 are free semigroups in Ω, and $*$ indicates the free product. By the definition of the action of (F,T) on K, we know that $KF = M_2$. Hence, $K\rho \subseteq M_2$, and by a standard result from universal algebra, we can deduce that $H \cong M_1 * (M_2/(K\rho)^{M_2})$, where $(K\rho)^{M_2}$ is the least congruence on M_2 containing $K\rho$. Again, $HS = M_2/(K\rho)^{M_2}$, identifying H and $M_1 * (M_2/(K\rho)^{M_2})$. Note that $M_2/(K\rho)^{M_2} = xS \cong (S,+)$. Hence, taking $H_1 = M_1$ and $H_2 = M_2/(K\rho)^{M_2}$, we get $H \cong H_1 * H_2$, as desired. □

4 Adjoining an Identity

It is known that to every seminear-ring we can adjoin an identity. If we consider a d.g. seminear-ring (S,T) and form a d.g. seminear-ring by adjoining an identity to T, then, in general, the elements which were distributive in the seminear-ring (S,T) are no longer distributive in the new one. Our aim is to adjoin identities to d.g. seminear-rings (S,T) without losing the distributivity of the elements of T.

Lemma 4.1. *Let (S,T) be a faithful d.g. seminear-ring. Let $H = \mathrm{Frs}_\Omega(x,S,T)$ be the free (S,T)-semigroup on the generator x. Let $U = T \cup \{1\}$, where 1 is the identity map on H, and consider the d.g. seminear-ring (L,U) contained in $\mathrm{E}(H)$. Then $(L,+)$ is isomorphic to H.*

Proof. Since $T \subseteq U$, we have $(S,T) \subseteq (L,U)$, and by Lemma 2.3, (S,T) has a faithful d.g. representation on H. Hence, we can assume that $T \subseteq \mathrm{End}(H)$ and that T generates $S \subseteq \mathrm{E}(H)$. Observe that the semigroup $(L,+)$ is a faithful (S,T)-module since (L,U) has an identity. Furthermore, $(L,+)$ is generated by $U = T \cup \{1\}$. Thus, $(L,+)$ is generated by $\{1\}$ as an (S,T)-semigroup. Since H is a free (S,T)-semigroup on $\{x\}$, the map $\theta : x \mapsto 1$ can be extended to an (S,T)-epimorphism from H to $(L,+)$, which we again denote by θ. Recall that (L,U) has a faithful representation on H. Consider the map $\phi : (L,+) \to (H,+)$ defined by $l\phi = xl$. Clearly, ϕ is an (L,U)-homomorphism from $(L,+)$ to $(H,+)$ such that $1\phi = x1 = x$ and $t\phi = xt = t_x$ for all $t \in T \subseteq S$. Moreover, $x\theta\phi = 1\phi = x$ and $t_x\theta\phi = t\phi = t_x$, showing that $\theta\phi$ is the identity map on the generating set T_x for H, and hence is the identity map on H. Also, $1\phi\theta = x\theta = 1$ and $t\phi\theta = t_x\theta = t$, showing that $\phi\theta$ is the identity map on U, and hence is the identity map on $(L,+)$ being generated by U. This shows that θ and ϕ are both isomorphisms, which completes the proof. □

The following theorem gives a connection between faithfulness and adjoining an identity to a given d.g. seminear-ring.

Theorem 4.2. *A d.g. seminear-ring* (S, T) *can be embedded by a d.g. monomorphism in a d.g. seminear-ring* (L, U) *with identity if and only if it is faithful.*

Proof. If (S, T) is embeddable by a d.g. monomorphism in a d.g. seminear-ring (L, U) with identity, then it is easy to see that $(L, +)$ is a faithful (S, T)-module and so (S, T) is faithful. Conversely, suppose that (S, T) is a faithful d.g. seminear-ring with a faithful (S, T)-module H. Let $L = \mathrm{E}(H)$ and $U = \mathrm{End}(H)$. Then (S, T) can be embedded in (L, U) by a d.g. monomorphism. □

Theorem 4.3. *Let* (S, T) *be a faithful d.g. seminear-ring in a variety* Ω*. Then we can adjoin an identity to* (S, T) *to obtain a d.g. seminear-ring* (L, U) *in* Ω *with* $U = T \cup \{1\}$ *by the following construction:* $(L, +) = \mathrm{sg}\langle 1 \rangle * (S, +)$*, that is,* $(L, +)$ *is the free product in* Ω *of the free semigroup in* Ω *on the identity element* 1 *and a copy of* $(S, +)$*.*

Proof. The d.g. seminear-ring we need is (L, U) given in Lemma 4.1, while the semigroup $(L, +)$ is described in Theorem 3.6. Note that the product in L can be easily defined since we know about the product in T; and $T \cup \{1\} = U$ generates L. □

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References

[1] J.R. Clay, *Near-rings: Geneses and Applications*, Oxford University Press, Oxford, 1992.

[2] A. Fröhlich, Distributively generated near rings (I): Ideal theory, *Proc. London Math. Soc.* (3) **8** (1958) 74–94.

[3] A. Fröhlich, Distributively generated near rings (II): Representation theory, *Proc. London Math. Soc.* (Ser. 3) **8** (1958) 95–108.

[4] J.D.P. Meldrum, The representation of d.g. near-rings, *Austral. Math. Soc.* **16** (1973) 467–480.

[5] J.D.P. Meldrum, *Near-rings and Their Links with Groups*, Pitman Research Notes in Mathematics, 134, London, 1985.

[6] J.D.P. Meldrum, M. Samman, On free d.g. seminear-rings, *Riv. Mat. Univ. Parma* (Ser. V) **6** (1997) 93–102.

[7] H. Neumann, Near-rings connected with free groups, in: *Proc. International Congress of Mathematics* (Amsterdam, 1954), II, pp. 46–47.

[8] H. Neumann, On varieties of groups and their associated near-rings, *Math. Z.* **65** (1956) 36–69.

[9] G. Pilz, *Near-rings*, 2nd ed., North-Holland, Amsterdam, 1983.