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ON SKEW ARMENDARIZ OF LAURENT SERIES TYPE RINGS

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Let α be an automorphism of a ring R. We study the skew Armendariz of Laurent series type rings (α-LA rings), as a generalization of the standard Armendariz condition from polynomials to skew Laurent series. We study on the relationship between the Baerness and p.p. property of a ring R and these of the skew Laurent series ring $R[[x, x^{-1}; \alpha]]$, in case R is an α-LA ring. Moreover, we prove that for an α-weakly rigid ring R, $R[[x, x^{-1}; \alpha]]$ is a left p.q.-Baer ring if and only if R is left p.q.-Baer and every countable subset of $S_\ell(R)$ has a generalized countable join in R. Various types of examples of α-LA rings are provided.

Key Words: Armendariz rings; Skew Laurent series rings; Baer rings; Weakly rigid rings.

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1. INTRODUCTION

Throughout this article, R denotes an associative ring with unity, and α is an automorphism of R. Denote $R[[x, x^{-1}; \alpha]]$, the ring of formal skew Laurent series, whose elements are of the form $\sum_{i=-m}^{\infty} a_i x^i$, with usual addition and multiplication subject to the rule $x^i a = \alpha^i(a)x^i$, for each i.

A ring R is called *Baer* (resp., *quasi-Baer*) if the left annihilator of every nonempty subset (left ideal) of R is generated, as a left ideal, by an idempotent of R. Kaplansky [16] introduced the Baer rings to abstract various properties of rings of operators on a Hilbert space. Clark [8] introduced the quasi-Baer rings and used them to characterize a finite dimensional twisted matrix units semigroup algebra over an algebraically closed field. The definitions of Baer and quasi-Baer rings are left-right symmetric. A ring R is called a *left* (resp., *right*) *p.p.*-ring if the left (resp., right) annihilator of each element of R is generated by an idempotent.

In [5], Birkenmeier, Kim, and Park, defined a ring to be called *left* (resp., *right*) *principally quasi-Baer* (or simply *left* [resp., *right*] *p.q.-Baer*) if the left (resp., right)

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annihilator of every principal left (resp., right) ideal of R is generated by an idempotent, as a left (resp., right) ideal of R. Equivalently, R is left p.q.-Baer if R modulo the left annihilator of any principal left ideal is projective. Note that in a reduced ring R (which has no nonzero nilpotent elements), R is a p.q.-Baer ring if and only if R is a p.p.-ring. The class of p.q.-Baer rings includes all biregular rings, all quasi-Baer rings, all *abelian* (i.e., every idempotent is central) p.p.-rings and is closed under direct products and Morita invariance.

Recall from [2], an idempotent $e \in R$ is *left* (resp., *right*) *semicentral* in R if $ere = re$ (resp., $ere = er$), for all $r \in R$. The set of all left semicentral idempotents of R is denoted by $S_\ell(R)$. Since the left annihilator of a left ideal is an ideal, we see that the left annihilator of a principal left ideal is generated by a right semicentral idempotent in a left p.q.-Baer ring. The set of all idempotent elements of R is denoted by $I(R)$.

In [4], Birkenmeier, Kim, and Park showed that the quasi-Baer condition is preserved by $R[[x, x^{-1}; \alpha]]$. In [3], they showed that R is left p.q.-Baer if and only if $R[x]$ is left p.q.-Baer. But it is not equivalent to that $R[[x]]$ is left p.q.-Baer. In fact, there exists a commutative von Neumann regular ring R (hence p.q.-Baer) such that the ring $R[[x]]$ is not p.q.-Baer [3, Example 3.6]. In [9, Theorem 3] Fraser and Nicholson proved that the ring $R[[x]]$ is reduced and left p.p. if and only if R is reduced and left p.p. and any countable family of idempotents in R has a join in $I(R)$. Liu in [21, Theorem 3] showed that $R[[x]]$ is right p.q.-Baer if and only if R is right p.q.-Baer and any countable family of idempotents in R has a generalized join when all left semicentral idempotents are central. Indeed, for a right p.q.-Baer ring, the condition "left semicentral idempotents are central" is equivalent to assume R is semiprime [9, Proposition 1.17]. Huang [15] showed that, in Liu's result, the condition requiring all left semicentral idempotents being central, is redundant. In [22, Theorem 3.5], the authors extended Huang's result and showed that for any α-compatible ring R, $R[[x; \alpha]]$ is right p.q.-Baer if and only if $R[[x, x^{-1}; \alpha]]$ is right p.q.-Baer if and only if R is right p.q.-Baer and every countable subset of right semicentral idempotents has a generalized countable join.

A ring R is said to be *Armendariz* if the product of two polynomials in $R[x]$ is zero if and only if the product of their coefficients is zero. This definition was given by Rege and Chhawchharia in [25] using the name Armendariz since Armendariz had proved in [1] that reduced rings satisfied this condition. Kim et al. in [17] called a ring R power-serieswise Armendariz, if $a_i b_j = 0$, for all i, j, whenever power series $f(x) = \sum_{i=0}^{\infty} a_i x^i$, $g(x) = \sum_{j=0}^{\infty} b_j x^j$ in $R[[x]]$ satisfy $f(x)g(x) = 0$. In [14] Hong, Kim, and Kwak extended the Armendariz property of rings to skew polynomial rings $R[x; \alpha]$: For an endomorphism α of a ring R, R is called an α-*skew Armendariz* ring if for polynomials $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j$ in $R[x; \alpha]$, $f(x)g(x) = 0$ implies that $a_i \alpha^i(b_j) = 0$, for each i, j.

In this note, we apply the concept of Armendariz ring to skew Laurent series rings over general noncommutative rings. We shall call a ring R is an α-*Armendariz of Laurent series type* ring (or simply, α-LA ring), if for each $f(x) = \sum_{i=-m}^{\infty} a_i x^i$ and $g(x) = \sum_{j=-n}^{\infty} b_j x^j \in R[[x, x^{-1}; \alpha]]$, $f(x)g(x) = 0$ implies that $a_i \alpha^i(b_j) = 0$, for each $i \geq -m$ and $j \geq -n$. We show that a number of interesting properties of an α-LA ring R such as the Baer and the α-quasi Baer property transfer to $R[[x, x^{-1}; \alpha]]$ and vice versa. Also, we study various types of examples of α-LA rings, extending the class of α-LA rings to non-semiprime rings.

2. SKEW ARMENDARIZ OF LAURENT SERIES TYPE RINGS

In this section, we resolve the structure of α-LA rings and obtain various necessary or sufficient conditions for a ring to be α-LA, unifying and generalizing a number of known Armendariz-like conditions.

We start by the following example which shows that there exists an Armendariz ring R with an automorphism α that is not an α-LA ring.

Example 2.1. Let $R = R_1 \oplus R_2$, where R_i is a reduced ring, for $i = 1, 2$. Then R is reduced and hence Armendariz. Let $\alpha : R \to R$ be an automorphism defined by $\alpha((a, b)) = (b, a)$. Then for $f(x) = (1, 0) - (1, 0)x$ and $g(x) = (0, 1) + (1, 0)x + (0, 1)x^2 + \cdots$ in $R[[x, x^{-1}; \alpha]]$, $f(x)g(x) = 0$ but $(1, 0)^2 \neq 0$. Thus R is not an α-LA ring.

In the following, we prove that R is an α-LA ring, if it satisfies the equivalent conditions of the following theorem.

Theorem 2.2. *Let R be a ring and α an automorphism of R. Then the following statements are equivalent*:

(i) *R is an α-LA ring*;
(ii) *For each $f(x) = \sum_{i=-m}^{\infty} a_i x^i$ and $g(x) = \sum_{j=-n}^{\infty} b_j x^j \in R[[x, x^{-1}; \alpha]]$, if $f(x)g(x) = 0$, then $a_0 b_j = 0$, for each $j \geq -n$*;
(iii) *For each $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x, x^{-1}; \alpha]]$, if $f(x)g(x) = 0$, then $a_i \alpha^i(b_j) = 0$, for each $i, j \geq 0$*;
(iv) *For each $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x, x^{-1}; \alpha]]$, if $f(x)g(x) = 0$, then $a_0 b_j = 0$, for each $j \geq 0$.*

Proof. We only need to prove (iv) $\Rightarrow$ (i). Let $f(x)g(x) = 0$, where $f(x) = \sum_{i=-m}^{\infty} a_i x^i$ and $g(x) = \sum_{j=-n}^{\infty} b_j x^j \in R[[x, x^{-1}; \alpha]]$. We have

$$\begin{aligned} f(x)g(x) &= \left(\sum_{i=-m}^{\infty} a_i x^i \right) \left(\sum_{j=-n}^{\infty} b_j x^j \right) \\ &= \left(\sum_{i=-m}^{\infty} a_i x^{i+m} \right) x^{-m} \left(\sum_{j=-n}^{\infty} b_j x^j \right) \\ &= \left(\sum_{i=-m}^{\infty} a_i x^{i+m} \right) \left(\sum_{j=-n}^{\infty} \alpha^{-m}(b_j) x^{j-m} \right) = 0. \end{aligned}$$

By multiplying x^{m+n} from the right-hand side of above equation, we get $(\sum_{i=-m}^{\infty} a_i x^{i+m})(\sum_{j=-n}^{\infty} \alpha^{-m}(b_j) x^{j+n}) = 0$. Thus $a_{-m}\alpha^{-m}(b_j) = 0$, for each $j \geq -n$, by (iv). Therefore,

$$
\begin{aligned}
f(x)g(x) &= \Big(\sum_{i=-m+1}^{\infty} a_i x^i\Big)\Big(\sum_{j=-n}^{\infty} b_j x^j\Big)\\
&= \Big(\sum_{i=-m+1}^{\infty} a_i x^{i+m-1}\Big)x^{-m+1}\Big(\sum_{j=-n}^{\infty} b_j x^j\Big)\\
&= \Big(\sum_{i=-m+1}^{\infty} a_i x^{i+m-1}\Big)\Big(\sum_{j=-n}^{\infty} \alpha^{-m+1}(b_j) x^{j-m+1}\Big) = 0.
\end{aligned}
$$

By multiplying x^{m+n-1} from the right-hand side of above equation, we get $(\sum_{i=-m+1}^{\infty} a_i x^{i+m-1})(\sum_{j=-n}^{\infty} \alpha^{-m+1}(b_j)x^{j+n}) = 0$. Thus $a_{-m+1}\alpha^{-m+1}(b_j) = 0$, for each $j \geq -n$, by (iv). By continuing in this way, we get $a_i\alpha^i(b_j) = 0$, for each $i \geq -m$ and $j \geq -n$ and the proof is complete.

In [18, Lemmas 7, 8] Kim and Lee proved that Armendariz rings are abelian and when R is an abelian ring, then every idempotent of $R[[x]]$ is in R and $R[[x]]$ is abelian. Now, we state similar results for the skew Laurent series ring $R[[x, x^{-1}; \alpha]]$.

Proposition 2.3. *Let R be an α-LA ring. Then we have the following statements*:

(i) $\alpha(e) = e$, *for each* $e^2 = e \in R$;
(ii) *If* $e^2 = e \in R[[x, x^{-1}; \alpha]]$, *then* $e \in R$;
(iii) *R is an abelian ring*;
(iv) *$R[[x, x^{-1}; \alpha]]$ is an abelian ring.*

Proof. (i). Let $e^2 = e \in R$. Suppose that $f(x) = ex^{-1} - e$ and $g(x) = (1 - \alpha(e)) + (1 - \alpha^2(e))x + (1 - \alpha^3(e))x^2 + \cdots \in R[[x, x^{-1}; \alpha]]$. We have $f(x)g(x) = 0$. Since R is an α-LA ring, $e(1 - \alpha(e)) = 0$ and, consequently, $e = e\alpha(e)$. On the other hand, $h(x)k(x) = 0$, where $h(x) = (1 - e)x^{-1} - (1 - e)$ and $k(x) = \alpha(e) + \alpha^2(e)x + \alpha^3(e)x^2 + \cdots \in R[[x, x^{-1}; \alpha]]$. Thus$(1 - e)\alpha(e) = 0$, since R is an α-LA ring and consequently $\alpha(e) = e\alpha(e)$. Hence $e = \alpha(e)$ and the result follows.

(ii) Let $e = \sum_{i=-m}^{\infty} e_i x^i$ be an idempotent of $R[[x, x^{-1}; \alpha]]$. Since $(1 - e)e = 0$, we have $(1 - e_0)e_i = 0$, for each i. Thus $e_i = e_0e_i$, for each i. On the other hand, since $e(1 - e) = 0$, we have $e_0(1 - e_0) = 0$ and $e_0e_i = 0$, for each $i \neq 0$. Thus $e_i = 0$, for each $i \neq 0$. Hence $e = e_0 \in R$ and the proof is complete.

(iii) Let $e^2 = e$ and $r \in R$. Suppose that $f(x) = ex^{-1} - er$ and $g(x) = (1 - e) + \alpha(r)(1 - e)x + \alpha(r)\alpha^2(r)(1 - e)x^2 + \cdots \in R[[x, x^{-1}; \alpha]]$. We have $f(x)g(x) = 0$. Since R is an α-LA ring, $er(1 - e) = 0$ and so $er = ere$. Next, let $h(x) = (1 - e)x^{-1} - (1 - e)r$ and $k(x) = e + \alpha(r)ex + \alpha(r)\alpha^2(r)ex^2 + \cdots \in R[[x, x^{-1}; \alpha]]$. We have $h(x)k(x) = 0$. Since R is an α-LA ring, it implies that $(1 - e)re = 0$. Therefore, $re = ere$ and so $re = er$, which implies that R is abelian.

(iv) It is clear by (i), (ii) and (iii).

Let α be an endomorphism of a ring R. According to Krempa [19], an endomorphism α of a ring R is said to be *rigid* if $a\alpha(a) = 0$ implies $a = 0$, for $a \in R$.

A ring R is called *α-rigid* if there exists a rigid endomorphism α of R. In [12], the authors introduced α-compatible rings and studied its properties. A ring R is *α-compatible* if for each $a, b \in R$, $ab = 0 \Leftrightarrow a\alpha(b) = 0$. Clearly, this may only happen when the endomorphism α is injective. Also by [12, Lemma 2.2], a ring R is α-rigid if and only if R is α-compatible and reduced.

Lemma 2.4. *Let α be a compatible automorphism of a ring R. Then we have the following statements:*

(i) *If $ab = 0$, then $a\alpha^k(b) = \alpha^k(a)b = 0$, for all integers k;*
(ii) *If $\alpha^k(a)b = 0$ for some integer k, then $ab = 0$.*

Proof. Since α is an automorphism, it is clear, by [12, Lemma 3.2].

In [14], Hong, Kim, and Kwak asked a question that: "Let α be a monomorphism (or automorphism) of a (commutative) reduced ring R and R be α-skew Armendariz. Is R α-rigid?" A positive answer to this question have been given in [23, Theorem A] by Matczuk, and in [6, Theorem 1] by Chen and Tong. Now, we generalize this result to α-LA rings.

Theorem 2.5. *Let α be an automorphism of a ring R. Then R is α-rigid if and only if R is reduced and α-LA ring.*

Proof. Suppose R is an α-rigid ring. Let $f(x) = \sum_{i=-m}^{\infty} a_i x^i$ and $g(x) = \sum_{j=-n}^{\infty} b_j x^j$ by elements of $R[[x, x^{-1}; \alpha]]$ with $f(x)g(x) = 0$. We prove that $a_i\alpha^i(b_j) = 0$, for each $i \geq -m$ and $j \geq -n$. The proof is by induction on $i + j$. It can be easily checked that $a_{-m}\alpha^{-m}(b_{-n}) = 0$. Now, we suppose that our claim is true for $i + j < k$. We have

$$a_{-m}\alpha^{-m}(b_{k+m}) + a_{-m+1}\alpha^{-m+1}(b_{k+m-1}) + \cdots + a_{k+n}\alpha^{k+n}(b_{-n}) = 0, \qquad (*)$$

since $\sum_{i+j=k} a_i\alpha^i(b_j)$ is the coefficient of x^k in $f(x)g(x)$. On the other hand, $a_{-m}\alpha^{-m}(b_j) = 0$, for $j < k + m$. Thus $\alpha^t(b_j)a_{-m} = 0$, for each t and $j < k + m$, since R is α-rigid. Therefore, by multiplying a_{-m} from the right-hand side of Eq. $(*)$, we have $a_{-m}\alpha^{-m}(b_{k+m})a_{-m} = 0$ and consequently $a_{-m}\alpha^{-m}(b_{k+m}) = 0$, since R is reduced. Thus, by multiplying a_{-m+1} from the right-hand side of Eq. $(*)$, we have $a_{-m+1}\alpha^{-m+1}(b_{k+m-1})a_{-m+1} = 0$ and consequently $a_{-m+1}\alpha^{-m+1}(b_{k+m-1}) = 0$. By repeating this method, we obtain $a_i\alpha^i(b_j) = 0$, for $i + j = k$ and so R is an α-LA ring. Conversely, let $a\alpha(a) = 0$. Let $f(x) = \alpha(a)x^{-1} - \alpha(a)$ and $g(x) = \alpha(a) + \alpha^2(a)x + \alpha^3(a)x^2 + \cdots \in R[[x, x^{-1}; \alpha]]$. Therefore, $f(x)g(x) = 0$ and so $\alpha(a)\alpha(a) = 0$, since R is an α-LA ring. Thus $\alpha(a^2) = 0$ and consequently $a = 0$, since α is an automorphism and R is reduced. Hence R is α-rigid, and the proof is complete.

In [14, Proposition 3] the authors showed that R is an α-rigid ring if and only if $R[x; \alpha]$ is a reduced ring. Also, in [19, Corollary 2.5], Krempa proved that R is α-rigid if and only if $R[[x; \alpha]]$ is reduced. Now, we have the following result.

Proposition 2.6. *Let α be an automorphism of a ring R. Then R is an α-rigid ring if and only if $R[[x, x^{-1}; \alpha]]$ is a reduced ring.*

Proof. Suppose that R is α-rigid. Let $f(x)=\sum_{i=-m}^{\infty} a_i x^i \in R[[x,x^{-1};\alpha]]$ and $f^2=0$. Thus $a_{-m}\alpha^{-m}(a_{-m})=0$ and so $a_{-m}=0$, since R is α-rigid. Hence $f=0$ and $R[[x,x^{-1};\alpha]]$ is reduced. Conversely, suppose that $R[[x,x^{-1};\alpha]]$ is a reduced ring and $a\alpha(a)=0$. Thus $(\alpha(a)x^{-1})^2=0$ and so $a=\alpha(a)=0$, since $R[[x,x^{-1};\alpha]]$ is reduced. Hence R is α-rigid, and the proof is complete.

Recall that for an ideal I of R, if $\alpha(I)\subseteq I$, then $\bar{\alpha}: R/I\to R/I$ defined by $\bar{\alpha}(a+I)=\alpha(a)+I$ is an endomorphism of a factor ring R/I. If α is an automorphism and $\alpha(a)\notin I$, for each $a\in R\backslash I$, then $\bar{\alpha}$ is automorphism. The following example, shows that there exists an α-LA ring R with a nonzero proper ideal I such that R/I is not an $\bar{\alpha}$-LA ring.

Example 2.7. Let $\mathbb{Z}$ be the ring of integers and $\mathbb{Z}_4$ the ring of integers modulo 4. Consider the ring $R=\{(a,b)\mid a\in\mathbb{Z} \text{ and } b\in\mathbb{Z}_4\}$ with addition point-wise and multiplication given by $(a,b)(c,d)=(ac,ad+bc)$. Let $\alpha: R\to R$ be an automorphism defined by $\alpha((a,b))=(a,-b)$. We show that R is an α-LA ring. Suppose that $f(x)=\sum_{i=-m}^{\infty}(a_i,b_i)x^i$ and $g(x)=\sum_{j=-n}^{\infty}(c_j,d_j)x^j$ in $R[[x,x^{-1};\alpha]]$ such that $f(x)g(x)=0$. We prove that $(a_i,b_i)(c_j,(-1)^i d_j)=0$, for each i and j. Let $f_1(x)=\sum_{i=-m}^{\infty} a_i x^i$ and $g_1(x)=\sum_{j=-n}^{\infty} c_j x^j\in\mathbb{Z}[x]$, $f_2(x)=\sum_{i=-m}^{\infty} b_i x^i$ and $g_2(x)=\sum_{j=-n}^{\infty} d_j x^j\in\mathbb{Z}_4[x]$. Since $f(x)g(x)=0$, we have $(f_1,f_2)(g_1,g_2)=0$, and so $f_1g_1=f_1*g_2+f_2g_1=0$, where $a_ix^i*d_jx^j=a_i(-1)^i d_j x^{i+j}$. $f_1g_1=0$ implies that $f_1=0$ or $g_1=0$. If $f_1=0$, we get $f_2g_1=0$ and so $b_i\in\{0,2\}$ and $c_j=2t_j$. So one can see that $(a_i,b_i)(c_j,(-1)^i d_j)=(0,b_i)(c_j,(-1)^i d_j)=0$, for each i and j. If $f_2=0$, we get $f_1*g_2=0$, and so $a_i=2t_i$ and $c_j\in\{0,2\}$. So one can see that $(a_i,b_i)(c_j,(-1)^i d_j)=(a_i,b_i)(0,(-1)^i d_j)=0$, for each i and j. Hence R is an α-LA ring. However, for an ideal $I=\{(a,0)\mid a\in 4\mathbb{Z}\}$ of R, the factor ring $R/I=\{(a,b)\mid a,b\in\mathbb{Z}_4\}$ is not $\bar{\alpha}$-LA, since $((2,0)+(2,1)x)^2=0$, but $(2,1)\bar{\alpha}((2,0))\neq 0$.

Let I be an ideal of a ring R with $\alpha(I)\subseteq I$ and $\alpha(a)\notin I$, for each $a\in R\backslash I$. Clearly, R/I an $\bar{\alpha}$-rigid ring if and only if $a\alpha(a)\in I$ implies that $a\in I$, for each $a\in R$. So if $a\alpha(a)\in I$ implies that $a\in I$, for each $a\in R$, then R/I is an $\bar{\alpha}$-LA ring, by Theorem 2.5. Also if $a\alpha(a)\in I$ implies that $a\in I$, for each $a\in R$, then I is completely semiprime ideal (i.e. $a^2\in I$ implies that $a\in I$). In fact, if $a^2\in I$, then $a\alpha(a)\alpha(a\alpha(a))=a\alpha(a^2)\alpha^2(a)\in I$, since $\alpha(I)\subseteq I$. Therefore, $a\alpha(a)\in I$ and so $a\in I$.

In Example 2.7, the ideal $I=(4\mathbb{Z},0)$ of R does not satisfy the condition ($a\alpha(a)\in I$ implies that $a\in I$, for each $a\in R$). In fact, $(2,0)\alpha(2,0)\in I$, but $(2,0)\notin I$. Thus the condition ($a\alpha(a)\in I$ implies that $a\in I$, for each $a\in R$) is not superfluous.

Clearly, if R is a domain with an automorphism α, then R is an α-LA ring. In particular, for a completely prime ideal P (i.e., if $ab\in P$, then $a\in P$ or $b\in P$), a factor ring R/P is $\bar{\alpha}$-LA. The following example shows that there exists an automorphism of an abelian ring R whose prime radical $N_*(R)$ is a completely semiprime ideal and an α-LA ring. Also $R/N_*(R)$ is an $\bar{\alpha}$-LA ring, but R is not α-LA.

Example 2.8. Let $R=\{\left(\begin{smallmatrix} a & c\\ 0 & b\end{smallmatrix}\right)\mid a-b\equiv c\equiv 0 \pmod 2 \text{ and } a,b,c\in\mathbb{Z}\}$. Let $\alpha: R\to R$ be an automorphism defined by $\alpha(\left(\begin{smallmatrix} a & c\\ 0 & b\end{smallmatrix}\right))=\left(\begin{smallmatrix} a & -c\\ 0 & b\end{smallmatrix}\right)$. One can see that $N_*(R)=\{\left(\begin{smallmatrix} 0 & c\\ 0 & 0\end{smallmatrix}\right)\mid c\equiv 0 \pmod 2\}$. Clearly, $N_*(R)$ is α-LA. Also, $A\alpha(A)\in N_*(R)$ implies that $A\in N_*(R)$, since $N_*(R)$ is completely semiprime and $\alpha(N_*(R))=N_*(R)$. So $R/N_*(R)$ is an $\bar{\alpha}$-LA ring. However, R is not an α-LA ring. Since for $f(x) = \left(\begin{smallmatrix} 2 & 2 \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right) x$ and $g(x) = \left(\begin{smallmatrix} 0 & 2 \\ 0 & -2 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right) x$, we have $f(x)g(x) = 0$, but $\left(\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right) \alpha(\left(\begin{smallmatrix} 0 & 2 \\ 0 & -2 \end{smallmatrix}\right)) \neq 0$.

Note that we use $M_n(R)$ and $T_n(R)$, for the ring of n-by-n matrices over R and the ring of n-by-n upper triangular matrices over R, respectively. We denote the identity matrix and unit matrices in ring $M_n(R)$, by I_n and E_{ij}, respectively.

The last example of this section shows that there exists a non-identity automorphism α of a ring R such that I is an α-LA ring and R/I is an $\bar{\alpha}$-LA ring, for any nonzero proper ideal I of R, but R is not an α-LA ring.

Example 2.9. Let F be a field, $R = T_2(F)$ and α an automorphism of R defined by $\alpha(\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)) = \left(\begin{smallmatrix} a & -b \\ 0 & c \end{smallmatrix}\right)$. We have $f(x)g(x) = 0$, where $f(x) = E_{11} + (E_{11} + E_{12})x$, $g(x) = -E_{22} + (E_{12} + E_{22})x$. But $(E_{11} + E_{12})E_{22} \neq 0$. Thus R is not an α-LA ring. Note that the only nonzero proper ideals of R are

$$I_1 = \{aE_{11} + bE_{12} \mid a, b \in F\}, \qquad I_2 = \{aE_{12} + bE_{22} \mid a, b \in F\}, \qquad I_3 = \{aE_{12} \mid a \in F\}.$$

Clearly, R/I_1 is an $\bar{\alpha}$-LA ring, since $R/I_1 \cong F$. Now, we show that I_1 is an α-LA ring. For $f(x) = \sum_{i=-m}^{\infty} A_i x^i$ and $g(x) = \sum_{i=-n}^{\infty} B_j x^j$ in $I[[x, x^-; \alpha]]$, where $A_i = a_i E_{11} + b_i E_{12}$ and $B_j = c_j E_{11} + d_j E_{12}$, assume that $f(x)g(x) = 0$. Therefore, $a_{-m}c_{-n} = a_{-m}d_{-n} = 0$, since $A_{-m}\alpha^{-m}(B_{-n}) = 0$. If $0 \neq a_{-m}$, then $B_{-n} = 0$, a contradiction; whence $a_{-m} = 0$. This implies that $A_{-m}\alpha^{-m}(B_j) = 0$, for all $j \geq -n$. On the other hand, $A_{-m+1}\alpha^{-m+1}(B_{-n}) + A_{-m}\alpha^{-m}(B_{-n+1}) = 0$ and so $A_{-m+1}\alpha^{-m+1}(B_{-n}) = 0$. This implies that $a_{-m+1} = 0$, since $0 \neq B_{-n}$ and consequently $A_{-m+1}\alpha^{-m+1}(B_j) = 0$, for each j. By continuing in this way, we have $A_i\alpha^i(B_j) = 0$, and so I_1 is an α-LA ring. Similarly, one can see that R/I_2 is an $\bar{\alpha}$-LA ring and I_2 is an α-LA ring. Finally, it can be easily checked that I_3 is an α-LA ring. Moreover, R/I_3 is an $\bar{\alpha}$-LA ring, since R/I_3 is reduced and $\bar{\alpha}$ is an identity map on R/I_3.

3. ON ANNIHILATOR IDEALS OF SKEW ARMENDARIZ OF LAURENT SERIES TYPE RINGS

In this section, we study relations between the set of annihilators in R and the set of annihilators in $R[[x, x^{-1}; \alpha]]$. We then consider the relationship between the properties of being Baer, quasi-Baer, p.p., and p.q.-Baer of a ring R, and of the skew Laurent series ring $R[[x, x^{-1}; \alpha]]$, respectively. For a nonempty subset X of R, $\ell_R(X)$ denotes the left annihilators of X in R.

In [4, Theorem 1.2], Birkenmeier, Kim, and Park proved the following theorem.

Theorem 3.1. *Let R be a ring and α an automorphism of R. If R is a quasi-Baer ring, then so is $R[[x, x^{-1}; \alpha]]$.*

The following example shows that the converse of Theorem 3.1 is not true in general.

Example 3.2. Let S be a prime ring which is not simple and assume that I is a nontrivial ideal of S. Consider the ring $R = \{(a, b) \in S \oplus S \mid b - a \in I\}$

and the automorphism α of R given by $\alpha((a,b)) = (b,a)$, for each $(a,b) \in R$. In [11, Example 2.9], it is shown that R is not quasi-Baer. Now, we show that $A = R[[x, x^{-1}; \alpha]]$ is quasi-Baer. Let L be a nonzero ideal of A. Put $0 \neq f(x) = \sum_{i=-m}^{\infty} (a_i, b_i)x^i \in L$. Since L is an ideal of A, $f'(x) = ((1,1)x)f(x) \in L$. Suppose that $0 \neq g(x) = \sum_{j=-n}^{\infty} (c_j, d_j)x^j \in \ell_A(L)$. Then $g(x)f(x) = g(x)f'(x) = 0$. So we have $(c_{-n}, d_{-n})\alpha^{-n}(a_{-m}, b_{-m}) = (c_{-n}, d_{-n})\alpha^{-n}(b_{-m}, a_{-m}) = 0$. Therefore, $(c_{-n}a_{-m}, d_{-n}b_{-m}) = (c_{-n}b_{-m}, d_{-n}a_{-m}) = (0,0)$. On the other hand, one of the a_{-m} or b_{-m} is nonzero, since $(a_{-m}, b_{-m}) \neq (0,0)$. Without loss of generality, suppose that $a_{-m} \neq 0$. Thus $c_{-n}a_{-m} = d_{-n}a_{-m} = 0$ implies that $c_{-n}Sa_{-m} = d_{-n}Sa_{-m} = 0$. Therefore, $c_{-n} = d_{-n} = 0$, since S is a prime ring. So $\ell_S(I) = 0$, and hence $R[[x, x^{-1}; \alpha]]$ is a quasi-Baer ring.

In [24], Nasr-Isfahani and Moussavi called a ring R with an endomorphism α, an α-*weakly rigid* ring if for each $a, b \in R$, $aRb = 0$ if and only if $a\alpha(Rb) = 0$. Note that, when R is α-weakly rigid, then α is injective and every prime ring with an automorphism α is α-weakly rigid. Also, for any n, a ring R is weakly rigid if and only if $T_n(R)$ is weakly rigid [24, Theorem 2.6] if and only if $M_n(R)$ is weakly rigid [24, Theorem 2.3]. Also, If R is a semiprime weakly rigid ring, then the ring of polynomials $R[X]$, for X an arbitrary nonempty set of indeterminates, is a semiprime weakly rigid ring [24, Corollary 2.12].

The next lemma appears in [24, Lemma 3.1] and will be helpful in the sequel.

Lemma 3.3. *Let R be an α-weakly rigid ring, where α is an automorphism of R, then for each $a, b \in R$ and positive integers i and j, $aRb = 0$ if and only if $\alpha^i(a)R\alpha^j(b) = 0$.*

Proposition 3.4. *Let R be an α-weakly rigid ring, where α is an automorphism of R and $A = R[[x, x^{-1}; \alpha]]$. Let $U = \{\ell_R(I) \mid I$ is an ideal of $R\}$, $V = \{\ell_A(J) \mid J$ is an ideal of $A\}$, $\phi : U \to V$ and $\psi : V \to U$, given by $\phi(L) = L[[x, x^{-1}; \alpha]]$ and $\psi(L') = L' \cap R$, respectively; then $\psi o \phi = id_U$.*

Proof. Let I be an ideal of R. We prove that $\ell_R(I)[[x, x^{-1}; \alpha]] = \ell_A(AIA)$. Let $f(x) = \sum_{i=-m}^{\infty} a_i x^i \in \ell_R(I)[[x, x^{-1}; \alpha]]$. Thus $a_i \in \ell_R(I)$, for each $i \geq -m$ and so $a_iRa = 0$, for each $a \in I$. Therefore, by Lemma 3.3, $a_iR\alpha^k(a) = 0$, for each $k \in \mathbb{Z}$. So $f(x) \in \ell_A(AIA)$ and hence $\ell_R(I)[[x, x^{-1}; \alpha]] \subseteq \ell_A(AIA)$. Now, assume that $g(x) = \sum_{j=-n}^{\infty} b_j x^j \in \ell_A(AIA)$. So $(\sum_{j=-n}^{\infty} b_j x^j)ra = 0$, for each $r \in R$ and $a \in I$. Therefore, $b_j\alpha^j(ra) = 0$, for each $j \geq -n$ and hence $b_j\alpha^j(Ra) = 0$. So $b_jRa = 0$. Hence $b_j \in \ell_R(I)$, for each $j \geq -n$ and so $g(x) \in \ell_R(I)[[x, x^{-1}; \alpha]]$. Thus, $\ell_R(I)[[x, x^{-1}; \alpha]] = \ell_A(AIA)$. Therefore, ϕ is well defined. Next assume that J is an ideal of A. Clearly, C_J is an ideal of R and $\ell_A(J) \cap R = \ell_R(C_J)$, where C_J is the set of all coefficients of elements of J. So ψ is well defined. Therefore, $\psi o \phi(U) = U[[x, x^{-1}; \alpha]] \cap R = U$, and the result follows.

The following theorem shows that if R is an α-weakly rigid ring, then the converse of Theorem 3.1 is indefeasible.

Theorem 3.5. *Let R be an α-weakly rigid ring, where α is an automorphism of R. If $R[[x, x^{-1}; \alpha]]$ is quasi-Baer, then so is R.*

Proof. Let I be an ideal of R. Since $A = R[[x, x^{-1}; \alpha]]$ is a quasi-Baer ring, $\ell_A(AIA) = Ae(x)$, for some idempotent $e(x) = \sum_{i=-m}^{\infty} e_i x^i \in A$. Thus we have $\ell_R(I)[[x, x^{-1}; \alpha]] = Ae(x)$, by Proposition 3.4. So $e_i \in \ell_R(I)$, for each $i \geq -m$. Thus $Re_0 \subseteq \ell_R(I)$. Also by Proposition 3.4, $\ell_R(I) = \ell_R(I)[[x, x^{-1}; \alpha]] \cap R = \ell_A(AIA) \cap R = Ae(x) \cap R$. So for each $a \in \ell_R(I)$, $a = ae(x)$. Thus $a = ae_0$ and so $\ell_R(I) = Re_0$. Also, since $e_0 \in \ell_R(I)$, we have $e_0 = e_0^2$. Thus $\ell_R(I) = Re_0$, and the proof is complete.

An ideal I of R is called an α-*ideal* if $\alpha(I) \subseteq I$. Let α be an automorphism of R, then $\bar{\alpha} : R[[x, x^{-1}; \alpha]] \to R[[x, x^{-1}; \alpha]]$, given by $\bar{\alpha}(\sum_{i=-m}^{\infty} a_i x^i) = \sum_{i=-m}^{\infty} \alpha(a_i) x^i$ is an automorphism.

According to Hirano [13], a ring R is called α-*quasi Baer* if the left annihilator of every α-ideal of R is generated, as a left ideal, by an idempotent. Example 3.2 enables us to construct numerous examples of α-quasi Baer rings which are not quasi-Baer; for more details see [11].

Theorem 3.6. *Let α be an automorphism of a ring R. Then we have the following statements*:

(i) *If R is an α-quasi Baer ring, then $R[[x, x^{-1}; \alpha]]$ is an $\bar{\alpha}$-quasi Baer ring*;
(ii) *If R is an α-LA ring and $R[[x, x^{-1}; \alpha]]$ is $\bar{\alpha}$-quasi Baer, then R is an α-quasi Baer ring*;
(iii) *If R is an α-weakly rigid ring and $R[[x, x^{-1}; \alpha]]$ is $\bar{\alpha}$-quasi Baer, then R is an α-quasi Baer ring.*

Proof. (i). Let I be an $\bar{\alpha}$-ideal of $R[[x, x^{-1}; \alpha]]$. It is easy to see that $I_0 = \{a \in R \mid ax^{-m} + a_{-m+1}x^{-m+1} + \cdots \in I,$ for some $a_i \in R$ and $m \in \mathbb{Z}\}$ an α-ideal of R. The rest of the proof is similar to that of Theorem 3.1.

(ii) Assume that I is an α-ideal of R and $A = R[[x, x^{-1}; \alpha]]$. Clearly, AIA is an $\bar{\alpha}$-ideal of A. Since A is an $\bar{\alpha}$-quasi Baer ring, we have $\ell_A(AIA) = Ae$ for an idempotent $e \in A$. Since R is an α-LA ring, $e \in R$, by Proposition 2.3. Thus $Re \subseteq \ell_R(I)$. Now, let $r \in \ell_R(I)$. Since $\alpha(I) \subseteq I$, $r \in \ell_A(AIA) = Ae$. Hence $\ell_R(I) = Re$ and so R is α-quasi Baer.

(iii) Let I be an α-ideal of R. Since AIA is an $\bar{\alpha}$-ideal of A, the proof is similar to that of Theorem 3.5.

In [14, Theorems 21 and 22] Hong, Kim, and Kwak proved that, when R is an α-skew Armendariz ring with $\alpha(e) = e$, for any $e^2 = e \in R$, then R is a Baer (p.p.-) ring if and only if $R[x; a]$ is a Baer (p.p.-) ring. Now, we have the following results.

Theorem 3.7. *Let R be an α-LA ring. Then R is a Baer ring if and only if $R[[x, x^{-1}; \alpha]]$ is a Baer ring.*

Proof. Assume that R is a Baer α-LA ring. Let V be a nonempty subset of $A = R[[x, x^{-1}; \alpha]]$ and C_V the set of all coefficients of elements of V. We have $\ell_R(C_V) = Re$, for some idempotent $e \in R$. So $e \in \ell_A(V)$. Thus $Ae \subseteq \ell_A(V)$. Now, let $g(x) = \sum_{i=-n}^{\infty} b_j x^j$ be a nonzero element of $\ell_A(V)$. Thus, $g(x)f(x) = 0$, for each $f(x) = \sum_{i=-m}^{\infty} a_i x^i \in V$. On the other hand, by Proposition 2.3, R is an abelian Baer ring, and so R is a reduced Baer ring. Since R is an α-LA ring and reduced ring, R is an α-rigid ring, by Theorem 2.5. Therefore, $g(x)f(x) = 0$ implies that $b_j a_i = 0$, for each i and j. So $b_j \in \ell_R(C_V) = Re$, for each $j \geq -n$. Thus $b_j = b_j e$. Hence $g(x) = g(x)e$, by Proposition 2.3 and so $\ell_A(V) = Ae$. Conversely, assume that A is a Baer ring. Let U be a nonempty subset of R. Then $\ell_A(U) = Ae$, for some $e^2 = e \in R$. Thus, $\ell_R(U) = \ell_A(U) \cap R = Ae \cap R = Re$. Hence R is Baer, and the proof is complete.

In [21, Definition 2], Liu defined the notion of generalized join for a countable subset of idempotents. Let $\{e_0, e_1, \dots\} \subseteq I(R)$. The set $\{e_0, e_1, \dots\}$ is said to have a *generalized join* e if there exists an idempotent $e \in R$ such that:

(i) $(1-e)Re_i = 0$, for all i;
(ii) if f is an idempotent and $(1-f)Re_i = 0$ for all i, then $(1-f)Re = 0$.

Theorem 3.8. *Let R be an α-LA ring. Then $R[[x, x^{-1}; \alpha]]$ is a left p.p.-ring if and only if R is a left p.p.-ring and every countable family of idempotents of R has a generalized join in $I(R)$.*

Proof. Suppose that $A = R[[x, x^{-1}; \alpha]]$ is a left p.p.-ring and $r \in R$. So $\ell_A(r) = Ae$, for some idempotent $e \in R$, by Proposition 2.3. Thus $\ell_R(r) = \ell_A(r) \cap R = Ae \cap R = Re$. Hence R is a left p.p.-ring. Now, let $\{e_0, e_1, \dots\} \subseteq I(R)$. Put $g(x) = \sum_{k=0}^{\infty} e_k x^k \in R[[x, x^{-1}; \alpha]]$. Since $R[[x, x^{-1}; \alpha]]$ is left p.p., $\ell_A(g(x)) = Af$, for some idempotent $f \in R$. Let $e = 1 - f$. Since $f \in \ell_A(g(x))$, we have $fe_k = 0$, for each k. Thus, $(1 - e)e_k = 0$ and so $R(1-e)e_k = 0$. This implies that $(1-e)Re_k = 0$, since R is abelian, by Proposition 2.3. If h is an idempotent of R such that $(1-h)Re_k = 0$, for each $k \geq 0$, then $(1-h)R \in \ell_A(g(x)) = Af$ and hence $(1-h)r = (1-h)rf$, for each $r \in R$. Thus we have $(1-h)Re = 0$. This shows that e is a join of $\{e_0, e_1, \dots\}$. Conversely, suppose that $f(x) = \sum_{i=-m}^{\infty} a_i x^i \in R[[x, x^{-1}; \alpha]]$. Since R is a left p.p.-ring, $\ell_R(a_i) = Re_i$, for each $i \geq -m$, where $e_i^2 = e_i \in R$. Let $1 - e$ be a join of $\{1 - e_i \mid i \geq -m\}$. We prove that $\ell_A(f(x)) = Ae$. Since $1 - e$ be a join of $\{1 - e_i \mid i \geq -m\}$, we have $e(1 - e_i) = 0$ and hence $e = ee_i$. Thus $ea_i = ee_i a_i = 0$, for each i. Therefore, $e \in \ell_A(f(x))$ and so $Ae \subseteq \ell_A(f(x))$. Next, if $g(x) = \sum_{j=-n}^{\infty} b_j x^j \in \ell_A(f(x))$, then $b_j \alpha^j(a_i) = 0$, for all $i \geq -m$ and $j \geq -n$. Thus $\alpha^{-j}(b_j)a_i = 0$. Therefore, $\alpha^{-j}(b_j) \in \ell_R(a_i) = Re_i$ and so $\alpha^{-j}(b_j)(1 - e_i) = 0$. This implies that $b_j(1 - e_i) = 0$, by Proposition 2.3. Since R is abelian, $(1 - e_i)b_j = 0$. Also, $\ell_R(b_j) = Rf_j$, where $f_j^2 = f_j \in R$, since R is a left p.p.-ring. So $(1 - e_i) = (1 - e_i)f_j$ and hence $(1 - e_i)(1 - f_j) = 0$, for each i, j. Since R is abelian, $(1 - f_j)(1 - e_i) = 0$. So $(1 - f_j)R(1 - e_i) = 0$, since R is abelian. Since $1 - e$ is a join of $\{1 - e_i \mid i \geq -m\}$, we have $(1 - f_j)(1 - e) = 0$, for each $j \geq -n$. Therefore, $b_j = b_j - f_j b_j = b_j(1 - f_j) = b_j(1 - f_j)e \in Re$, for each $j \geq -m$. Thus $g(x) = g(x)e$, since $\alpha(e) = e$, by Proposition 2.3. Hence $R[[x, x^{-1}; \alpha]]$ is a left p.p.-ring, and the proof is complete.

The following example shows that p.q.-Baerness property of a ring R is not inherited by $R[[x, x^{-1}; \alpha]]$, in general.

Example 3.9. Let $R = \{(a_i)_{i=1}^{\infty} \in \prod_{n=1}^{\infty} F_n \mid a_n \text{ is eventually constant}\}$, where F is a field and $F_n = F$, for each n. Then R is a von Neumann regular ring (hence p.q.-Baer). But the ring $R[[x, x^{-1}]]$ is not p.q.-Baer [3, Example 3.6].

In [22, Theorem 3.5], the author showed that if R is an α-compatible ring, then $R[[x, x^{-1}; \alpha]]$ is a left p.q.-Baer ring if and only if R is a left p.q.-Baer ring and every countable subset of $S_\ell(R)$ has a generalized countable join in R.

Although α-compatible left p.q.-Baer rings is a fairly narrow class of rings, in [24] it is shown that there are many rich classes of weakly rigid left p.q.-Baer rings. The class of α-compatible left p.q.-Baer rings does not contain the class of prime rings and is not closed under extensions to matrix rings or triangular matrix rings. However, the notion of a weakly rigid left p.q.-Baer ring overcomes these shortfalls. Now, we show that if R is an α-weakly rigid ring, then $R[[x, x^{-1}; \alpha]]$ is a left p.q.-Baer ring if and only if R is a left p.q.-Baer ring and every countable subset of $S_\ell(R)$ has a generalized countable join in R.

Following Tominaga [26], an ideal I of R is said to be *right s-unital* if, for each $a \in I$ there exists an element $b \in I$ such that $ab = a$.

Proposition 3.10. *Let R be an α-weakly rigid ring, where α is an automorphism of R. Then we have the following implications*: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

(i) $\ell_A(Af(x))$ *is right s-unital in* $A = R[[x, x^{-1}; \alpha]]$, *for each* $f(x) \in A$.
(ii) $\ell_R(Ra)$ *is right s-unital in* R, *for each* $a \in R$.
(iii) *If* $f(x)Ag(x) = 0$, *for* $f(x) = \sum_{i=-m}^{\infty} a_i x^i$ *and* $g(x) = \sum_{j=-n}^{\infty} b_j x^j \in A$, *then* $a_iRb_j = 0$, *for all* i *and* j.
(iv) $\varphi: \{\ell_R(I) \mid I$ *is an ideal of* $R\} \to \{\ell_A(J) \mid J$ *is an ideal of* $A\}$; $\varphi(U) = U[[x, x^{-1}; \alpha]]$ *is bijective.*

Proof. (i) $\Rightarrow$ (ii). Let $a \in R$. Since R is α-weakly rigid, we have $\ell_R(Ra) \subseteq \ell_A(Aa)$. So, for each $b \in \ell_R(Ra)$, there exists $f(x) = \sum_{i=-m}^{\infty} a_i x^i \in \ell_A(Aa)$ such that $bf(x) = b$, since $\ell_A(Aa)$ is right s-unital in A, by (i). Therefore, $ba_0 = b$. This implies that $\ell_R(Ra)$ is right s-unital.

(ii) $\Rightarrow$ (iii). Assume that $(\sum_{i=-m}^{\infty} a_i x^i)A(\sum_{j=-n}^{\infty} b_j x^j) = 0$. Using induction on $i + j$, we show that $a_iRb_j = 0$, for each i and j. It is clear for the case $i + j = -m - n$. Now, suppose that $a_iRb_j = 0$, for $i + j < t$. Let r be an arbitrary element of R. Then we have

$$\sum_{k=-m-n}^{\infty} \left(\sum_{i+j=k} a_i x^i r b_j x^j \right) = \sum_{k=-m-n}^{\infty} \left(\sum_{i+j=k} c_k x^k \right) = 0, \qquad (*)$$

where $c_k = a_{-m}\alpha^{-m}(rb_{k+m}) + a_{-m+1}\alpha^{-m+1}(rb_{k+m-1}) + \cdots + a_{k+n}\alpha^{k+n}(rb_{-n}) = 0$. Since $\ell_R(Rb_j)$ is right s-unital, there exists $e_{ij} \in \ell_R(Rb_j)$ such that $a_ie_{ij} = a_i$, for $i = -m, \ldots, n + t - 1$ and $j = -n, \ldots, t - i - 1$. If we put $f_i = e_{i,-n} \cdots e_{i,t-i-1}$, for $i = -m, \ldots, n + t - 1$, then $a_if_i = a_i$ and $f_i \in \ell_R(Rb_{-n}) \cap \cdots \cap \ell_R(Rb_{t-i-1})$. For $k = t$, replacing r by $\alpha^m(f_{-m})r$ in Eq. $(*)$ and using α-weakly rigidness of R, we obtain $a_{-m}Rb_{t+m} = a_{-m}f_{-m}Rb_{t+m} = 0$. Continuing this process (replacing r by $\alpha^{-i}(f_i)r$ in Eq. $(*)$, for $i = -m, \ldots, n + t - 1$, respectively, and using α-weakly rigidness of R), we obtain $a_iRb_j = 0$ for $i + j = t$. So R satisfies condition (iii).

(iii) $\Rightarrow$ (iv). Let $U = \ell_R(I)$, for some ideal I of R. We claim that $\ell_A(AIA) = U[[x, x^{-1}; \alpha]]$. Since R is α-weakly rigid and $UI = 0$, we have $U[[x, x^{-1}; \alpha]] \subseteq \ell_A(AIA)$. Let $f(x) = \sum_{i=-m}^{\infty} a_i x^i \in \ell_A(AIA)$. Then $a_i \in U = \ell_R(I)$, for each $i \geq -m$.

Hence $f(x) \in U[[x, x^{-1}; \alpha]]$. Therefore, φ is a well-defined injective map. Assume that $V = \ell_A(J)$, for some ideal J of A. Let V' and J' denote the set of coefficients of elements of V and J, respectively. Clearly, V' and J' are ideals of R. Now, we prove $\ell_R(J') = V'$. Clearly, $\ell_R(J') \subseteq V'$. Let $f(x) = \sum_{i=-m}^{\infty} a_i x^i \in J$ and $g(x) = \sum_{j=-n}^{\infty} b_j x^j \in V$. Then $g(x)Af(x) = 0$. Since R satisfies condition (iii), $b_j R a_i = 0$, for all $i \geq -m$ and $j \geq -n$. Thus $V'J' = 0$ and so $V' \subseteq \ell_R(J')$. Thus $\ell_R(J') = V'$ and so $V = \ell_R(J')[[x, x^{-1}; \alpha]]$.

The following definition is given by Huang in [15].

Definition 3.11. Let $E = \{e_0, e_1, \dots\}$ be a countable subset of $S_\ell(R)$. Then E is said to have a *generalized countable join* e if for a given $a \in R$, there exists $e \in S_\ell(R)$ such that:

(1) $ee_i = e_i$, for all positive integers i;
(2) If $ae_i = e_i$ for all positive integers i, then $ae = e$.

As it is mentioned in [15], if there exists an element $e \in R$ that satisfies conditions (1) and (2) above, then $e \in S_\ell(R)$. Indeed, the condition (1): $ee_i = e_i$, for all i implies $ee = e$ by (2), and so e is an idempotent. Further, let $a \in R$ be arbitrary. Then the element $d = e - ea + eae$ is an idempotent in R and $de_i = e_i$, for all i. Thus $de = e$ by (2). Note that $de = (e - ea + eae)e = d$. Consequently, $e = d = e - ea + eae$ or $ea = eae$. Thus $e \in S_\ell(R)$. In particular when R is a Boolean ring or a reduced p.p. ring, then the generalized countable join is indeed a join in R.

Observe that $(1-e)re_i = (1-e)e_i re_i = (e_i - ee_i)re_i$, when $e_i \in S_\ell(R)$. Thus, $e_i = ee_i$ if and only if $(1-e)re_i = 0$ for all $r \in R$ when $e_i \in S_\ell(R)$ for all i. Now, let $E = \{e_0, e_1, e_2, \dots\} \subseteq S_\ell(R)$ and e a generalized countable join of E. To show e is a generalized join (in the sense of Liu), it remains to show condition (ii) holds. Let f be an idempotent in R such that $(1-f)Re_i = 0$. Then, in particular, $(1-f)e_i = 0$, for all i. Thus $(1-f)e = 0$ by hypothesis. It follows that $(1-f)re = (1-f)ere = 0$ and thus $(1-f)Re = 0$. Therefore, e is a generalized join of E. Thus, in the content of left semicentral idempotents, a generalized countable join is a generalized join in the sense of Liu. Conversely, let $e \in S_\ell(R)$ be a generalized join (in the sense of Liu) of the set $E = \{e_0, e_1, e_2, \dots\} \subseteq S_\ell(R)$. Observe that condition (ii) is equivalent to

(ii′) if d is an idempotent and $de_i = e_i$ then $de = e$.

Let $a \in R$ be arbitrary such that $ae_i = e_i$ for all i. Then condition (ii′) and a similar argument used in the case of reduced p.p.-rings implies that $ae = e$. Thus e is a generalized countable join. Therefore, in the content of left semicentral idempotents, Liu's generalized join is equivalent to generalized countable join.

Theorem 3.12. *Let R be an α-weakly rigid ring, where α is an automorphism of R. Then $R[[x, x^{-1}; \alpha]]$ is a left p.q.-Baer ring if and only if R is left p.q.-Baer and every countable subset of $S_\ell(R)$ has a generalized countable join in R.*

Proof. Let $A = R[[x, x^{-1}; \alpha]]$ be a left p.q.-Baer ring and $a \in R$. Then $\ell_A(Aa) = Ae(x)$, for some idempotent $e(x) = \sum_{i=-m}^{\infty} e_i x^i \in A$. By Proposition 3.10, there exists an ideal I of R such that $\ell_A(Aa) = I[[x, x^{-1}; \alpha]]$. Hence $e_i \in I$, for each $i \geq -m$. Also $r = re(x)$ implies that $re_0 = r$, for all $r \in I$. Thus $I = Re_0$. This implies that $e_iRe_0 = 0$ and so $e_iR\alpha^k(e_0) = 0$, by Lemma 3.3. Therefore, $\ell_A(Aa) = Ae_0$. So $\ell_R(Ra) = Re_0$ and R is left p.q.-Baer. Now, suppose that $E = \{f_0, f_1, \dots\} \subseteq S_\ell(R)$ and $\varphi(x) = \sum_{i=0}^{\infty} f_i x^i \in A$. Since A is left p.q.-Baer, there exists a right semicentral $e(x) = \sum_{i=-m}^{\infty} e_i x^i \in A$, such that $\ell_A(A\varphi(x)) = Ae(x)$. By a similar argument, we have $\ell_A(A\varphi(x)) = Ae_0$, where e_0 is the constant term of $e(x)$. Hence $e_0 f_i = 0$, for $i \geq 0$. Let $g = 1 - e_0$. Then $gf_i = f_i$, for each i. Suppose that h is an element of R such that $hf_i = f_i$, for each i. So $(1-h) \in \ell_A(A\varphi(x))$. Thus $1 - h = (1-h)e_0$ and $(1-h)g = (1-h)(1-e_0) = 0$. Hence g is a generalized countable join of the set $\{f_0, f_1, \dots\}$. Conversely, assume that R is left p.q.-Baer and every countable family of left semicentral idempotents in R has a generalized countable join in $S_\ell(R)$. Let $f(x) = \sum_{i=-m}^{\infty} a_i x^i \in A$. Then there exist idempotents $e_i \in R$, $i \geq -m$, such that $\ell_R(Ra_i) = Re_i$. Suppose that $e \in S_\ell(R)$ is a generalized countable join of the set $\{1 - e_i\}_{i=-m}^{\infty}$. Hence $1 - e = (1-e)e_i$, for all $i \geq -m$. Therefore, $(1-e)ra_i = (1-e)e_i ra_i = 0$, each $r \in R$ and $i \geq -m$. Thus we have $(1-e)rf(x) = \sum_{i=-m}^{\infty} (1-e)ra_i x^i = 0$. So $1 - e \in \ell_A(Af(x))$, by α-weakly rigidness. Thus $A(1-e) \subseteq \ell_A(Af(x))$. Suppose that $g(x) = \sum_{j=-n}^{\infty} b_j x^j \in \ell_A(Af(x))$. Then $b_j Ra_i = 0$ for $i \geq -m$ and $j \geq -n$, by Proposition 3.10. Since α is an automorphism, for each j there exists $c_j \in R$ such that $b_j = \alpha^j(c_j)$. So $c_j Ra_i = 0$ for all i and j, by Lemma 3.3. Hence $c_j = c_j e_i$, for $i \geq -m$ and $j \geq -n$. Consequently, $c_j(1 - e_i) = 0$ or $(1-c_j)(1-e_i) = 1 - e_i$, for $i \geq -m$ and $j \geq -n$. Thus, $(1-c_j)e = e$ or $c_j(1-e) = c_j$, for $j \geq -n$. It follows that $g(x) = \sum_{j=-n}^{\infty} b_j x^j = \sum_{j=-n}^{\infty} \alpha^j(c_j(1-e))x^j = (\sum_{j=-n}^{\infty} \alpha^j(c_j)x^j)(1-e) = g(x)(1-e) \in A(1-e)$. Therefore, $\ell_A(Af(x)) = A(1-e)$, and the result follows.

4. EXTENSIONS OF SKEW ARMENDARIZ OF LAURENT SERIES TYPE RINGS

In this section, we study various types of examples of skew Armendariz of Laurent series type rings, extending the class of skew Armendariz of Laurent series rings to non-semiprime rings.

Note that the n-by-n (triangular) matrix ring over a ring R needs not to be id-LA, by [25, Remark 3.1]. Also, in the following example we show that $M_n(R)$ and $T_n(R)$ are not α-LA rings, for a nontrivial automorphism α of R.

Example 4.1. Let S be a ring, $R = M_2(S)$ and α an automorphism of R, defined by $\alpha(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) = \left(\begin{smallmatrix} a & -b \\ -c & d \end{smallmatrix}\right)$. For $f(x) = E_{11}x^{-1} - E_{12}$ and $g(x) = E_{22} - E_{12}x + E_{22}x^2 - E_{12}x^3 + \cdots$ in $R[[x, x^{-1}; \alpha]]$ we have $f(x)g(x) = 0$, but $E_{12}E_{22} \neq 0$. Thus R is not an α-AL ring. Moreover, this also shows that the upper triangular matrix ring $T_2(S)$ is not an α-AL ring.

Let R be a ring and σ denotes an endomorphism of R with $\sigma(1) = 1$. In [7] the authors introduced *skew triangular matrix* ring as a set of all triangular matrices with addition point-wise and a new multiplication subject to the condition $E_{ij}r = \sigma^{j-i}(r)E_{ij}$. So $(a_{ij})(b_{ij}) = (c_{ij})$, where $c_{ij} = a_{ii}b_{ij} + a_{i,i+1}\sigma(b_{i+1,j}) + \cdots + a_{ij}\sigma^{j-i}(b_{jj})$, for each $i \leq j$ and denoted it by $T_n(R, \sigma)$.

The subring of the skew triangular matrices with constant main diagonal is denoted by $S(R, n, \sigma)$, and the subring of the skew triangular matrices with constant diagonals is denoted by $T(R, n, \sigma)$. We can denote $A = (a_{ij}) \in T(R, n, \sigma)$ by

$(a_{11}, \ldots, a_{1n})$. Then $T(R, n, \sigma)$ is a ring with addition point-wise and multiplication given by:

$$(a_0, \ldots, a_{n-1})(b_0, \ldots, b_{n-1}) \\ = (a_0 b_0, a_0 * b_1 + a_1 * b_0, \ldots, a_0 * b_{n-1} + \cdots + a_{n-1} * b_0),$$

with $a_i * b_j = a_i \sigma^i(b_j)$, for each i and j. On the other hand, there is a ring isomorphism $\varphi : R[x; \sigma]/\langle x^n \rangle \to T(R, n, \sigma)$, given by $\varphi(\sum_{i=0}^{n-1} a_i x^i) = (a_0, a_1, \ldots, a_{n-1})$, with $a_i \in R$, $0 \leq i \leq n-1$. So $T(R, n, \sigma) \cong R[x; \sigma]/\langle x^n \rangle$, where $R[x; \sigma]$ is the skew polynomial ring with multiplication subject to the condition $xr = \sigma(r)x$ for each $r \in R$, and $\langle x^n \rangle$ is the ideal generated by x^n.

Now, we consider two following subrings of $S(R, n, \sigma)$:

$$A(R, n, \sigma) = \left\{ \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{i=1}^{n-j+1} a_j E_{i,i+j-1} + \sum_{j=\lfloor \frac{n}{2} \rfloor + 1}^{n} \sum_{i=1}^{n-j+1} a_{i,i+j-1} E_{i,i+j-1} \,\middle|\, a_j, a_{i,i+j-1} \in R \right\}$$

$$B(R, n, \sigma) = \{A + rE_{1k} \mid A \in A(R, n, \sigma) \text{ and } r \in R\} \qquad n = 2k \geq 4.$$

For example:

$$A(R, 3, \sigma) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\},$$

$$A(R, 4, \sigma) = \left\{ \begin{pmatrix} a_1 & a_2 & a & b \\ 0 & a_1 & a_2 & c \\ 0 & 0 & a_1 & a_2 \\ 0 & 0 & 0 & a_1 \end{pmatrix} \mid a_1, a_2, a, b, c \in R \right\}.$$

In the special case when $\sigma = id_R$, we use $S(R, n)$, $A(R, n)$, $B(R, n)$, and $T(R, n)$ instead of $S(R, n, \sigma)$, $A(R, n, \sigma)$, $B(R, n, \sigma)$, and $T(R, n, \sigma)$, respectively. Notice that, for a reduced ring R, Lee and Zhou [19] introduced $A(R, n)$, $B(R, n)$ and denoted them by $A_n(R)$ and $A_n(R) + RE_{1k}$, respectively, and used them to provide special Armendariz subrings of $T_n(R)$.

Let α and σ be endomorphisms of R such that $\alpha\sigma = \sigma\alpha$. Then $\bar{\alpha} : S(R, n, \sigma) \to S(R, n, \sigma)$, given by $\bar{\alpha}((a_{ij})) = (\alpha(a_{ij}))$ is an endomorphism of $S(R, n, \sigma)$. Also $\bar{\sigma} : R[[x, x^{-1}; \alpha]] \to R[[x, x^{-1}; \alpha]]$, given by $\bar{\sigma}(\sum_{i=-m}^{\infty} a_i x^i) = \sum_{i=-m}^{\infty} \sigma(a_i) x^i$ is an endomorphism of $R[[x, x^{-1}; \alpha]]$.

Proposition 4.2. *Let σ and α be endomorphism and automorphism of a ring R, respectively, such that $\alpha\sigma = \sigma\alpha$ and $k \geq 2$ a natural number. Then R is an α-LA ring, when the ring $S(R, k, \sigma)$ is an $\bar{\alpha}$-LA ring.*

Proof. Let $f(x) = \sum_{i=-m}^{\infty} a_i x^i$ and $g(x) = \sum_{j=-n}^{\infty} b_j x^j \in R[[x, x^{-1}; \alpha]]$ such that $f(x)g(x) = 0$. Let $F(x) = \sum_{i=-m}^{\infty} A_i x^i$ and $G(x) = \sum_{j=-n}^{\infty} B_j x^j \in S[[x, x^{-1}; \bar{\alpha}]]$, where $S = S(R, k, \sigma)$, $A_i = a_i I_n$ and $B_j = b_j I_n$, for each $i \geq -m$ and $j \geq -n$. It is clear that $F(x)G(x) = 0$. Thus $A_i \bar{\alpha}^i(B_j) = 0$, since S is an $\bar{\alpha}$-LA ring. Hence $a_i \alpha^i(b_j) = 0$ and R is an α-LA ring.

Proposition 4.3. *Let α be a rigid automorphism and σ an endomorphism of a ring R such that $\alpha\sigma = \sigma\alpha$. If R is a σ-rigid ring, then $R[[x, x^{-1}; \alpha]]$ is a $\bar{\sigma}$-rigid ring.*

Proof. Let $f(x) = \sum_{i=-m}^{\infty} a_i x^i \in R[[x, x^{-1}; \alpha]]$ and $f(x)\bar{\sigma}(f(x)) = 0$. So we have $a_{-m}\alpha^{-m}(\sigma(a_{-m})) = 0$ and consequently $a_{-m}\sigma(\alpha^{-m}(a_{-m})) = 0$, since $\alpha\sigma = \sigma\alpha$. Thus $a_{-m}\alpha^{-m}(a_{-m}) = 0$, since R is σ-rigid and so $a_m = 0$, since R is α-rigid. Hence $f(x) = 0$, and the proof is complete.

Theorem 4.4. *Let σ be a rigid endomorphism of a ring R and $A = (a_{ij})$, $B = (b_{ij}) \in A(R, n, \sigma)$ such that $AB = 0$, then $a_{ik}b_{kj} = 0$, for each $1 \leq i, j, k \leq n$.*

Proof. The proof is by induction on n. For $n = 1$, the proof is clear. Assume that the result is true for $A(R, n-1, \sigma)$. Let $A = (a_{ij})$ and $B = (b_{ij})$ in $A(R, n, \sigma)$ such that $AB = 0$. We have $A = \left(\begin{smallmatrix} A' & X \\ 0 & a \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} B' & Y \\ 0 & b \end{smallmatrix}\right)$, where $A', B' \in A(R, n-1, \sigma)$. Since $AB = 0$, we have $A'B' = 0$. Thus $a_{ik}b_{kj} = 0$, for $1 \leq i, j, k \leq n-1$, by induction hypothesis. Now, it is sufficient to show $a_{ik}b_{kn} = 0$, for each i, k. For $i = n$, we have $ab = 0$. If $i = n-1$, we have $ab_{n-1,n} + a_{n-1,n}\sigma(b) = 0$. By multiplying a from the right-hand side, we obtain $ab_{n-1,n}a = 0$, since $\sigma(b)a = 0$ and so $ab_{n-1,n} = 0$, since R is reduced. Thus $ab_{n-1,n} = a_{n-1,n}b = 0$. By continuing in this way, we conclude that $a_{ik}b_{kn} = 0$, for each i, k and the proof is complete.

Theorem 4.5. *Let σ be a rigid endomorphism of a ring R and $A = (a_{ij})$, $B = (b_{ij}) \in B(R, n, \sigma)$ such that $AB = 0$, then $a_{ik}b_{kj} = 0$, for each $1 \leq i, j, k \leq n$.*

Proof. Let $A = (a_{ij})$, $B = (b_{ij})$ in $B(R, n, \sigma)$ such that $AB = 0$. We have $A = \left(\begin{smallmatrix} A' & X \\ 0 & a \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} B' & Y \\ 0 & b \end{smallmatrix}\right)$, where $A', B' \in A(R, n-1, \sigma)$. Since $A'B' = 0$, we have $a_{ik}b_{kj} = 0$, for $1 \leq i, j, k \leq n-1$, by Theorem 4.4. The rest of the proof is similar to that of Theorem 4.4.

Theorem 4.6. *Let σ be a rigid endomorphism of a ring R. Then the following statements are equivalent*:

(i) *R is an α-rigid ring*;
(ii) *For each integer $k \geq 2$, $D(R, k, \sigma)$ is an $\bar{\alpha}$-$\mathtt{LA}$ ring*;
(iii) *For some integer $k \geq 2$, $D(R, k, \sigma)$ is an $\bar{\alpha}$-$\mathtt{LA}$ ring,*

where $D(R, k, \sigma)$ is one of the rings $A(R, k, \sigma)$, $B(R, k, \sigma)$ or $T(R, k, \sigma)$.

Proof. (i) $\Rightarrow$ (ii) We only prove this theorem for $D(R, k, \sigma) = A(R, k, \sigma)$. The proof of the other cases are similar. First consider the map $\phi : A(R, k, \sigma)[[x, x^{-1}; \bar{\alpha}]] \to A(R[[x, x^{-1}; \alpha]], k, \bar{\sigma})$, given by $\phi(\sum_{i=-m}^{\infty} A_i x^i) = (f_{rs}(x))$, where $A_i = (a_{rs}^{(i)})$ and $f_{rs}(x) = \sum_{i=-m}^{\infty} a_{rs}^{(i)} x^i$, for each $i \geq -m$ and $r, s \in \{1, \ldots, k\}$. It is not hard to see that ϕ is an isomorphism. Let $f(x) = \sum_{i=-m}^{\infty} A_i x^i$ and $g(x) = \sum_{j=-n}^{\infty} B_j x^j \in A(R, k, \sigma)[[x, x^{-1}; \bar{\alpha}]]$ and $f(x)g(x) = 0$, where $A_i = (a_{rs}^{(i)})$ and $B_j = (b_{rs}^{(j)})$. So we have $(f_{rs}(x))(g_{rs}(x)) = 0$, where $f_{rs}(x) = \sum_{i=-m}^{\infty} a_{rs}^{(i)} x^i$ and $g_{rs}(x) = \sum_{j=-n}^{\infty} b_{rs}^{(j)} x^j$. Since R is α-rigid, $R[[x, x^{-1}; \alpha]]$ is $\bar{\sigma}$-rigid, by Proposition 4.3. Thus $f_{rt}(x)g_{ts}(x) = 0$, for each $r, s, t \in \{1, \ldots, k\}$, by Theorem 4.4. Also, R is an α-$\mathtt{LA}$ ring, by Theorem 2.5, since R is α-rigid. So $a_{rt}^{(0)} b_{ts}^{(j)} = 0$, for each $j \geq -n$. Thus $A_0 B_j = 0$, and hence $A(R, k, \sigma)$ is an $\bar{\alpha}$-$\mathtt{LA}$ ring.

(ii) $\Rightarrow$ (iii) It is clear.

(iii) $\Rightarrow$ (i) Let $a\alpha(a) = 0$. Take $f(x) = (aI_k)x^{-1} - I_k$ and $g(x) = ((\alpha^2(a))E_{1k} + (\alpha^3(a))E_{1k}x + (\alpha^4(a))E_{1k}x^2 + \cdots \in D(R, n, \sigma)[[x, x^{-1}; \bar{\alpha}]]$. Then $h(x)k(x) = 0$. So we get $(I_k)(\alpha^2(a)E_{1k}) = 0$, since $D(R, n, \sigma)$ is an $\bar{\alpha}$-LA ring. Hence $a = \alpha^2(a) = 0$, and so R is α-rigid.

Recall that $A(R, 2) = T(R, R)$ is the trivial extension of R. So by Theorem 4.6, if R is an α-rigid ring, then $T(R, R)$ is an $\bar{\alpha}$-LA ring.

Let σ be an endomorphism of a ring R and M an (R, R)-bimodule. Consider $T(R, M, \sigma)$ as follows:

$$T(R, M, \sigma) = \left\{ \begin{pmatrix} r & m \\ 0 & r \end{pmatrix} \mid r \in R \quad \text{and} \quad m \in M \right\},$$

with the addition component-wise and the multiplication defined by $(a_{ij})(b_{ij}) = (c_{ij})$, where $c_{11} = a_{11}b_{11}$ and $c_{12} = a_{11}b_{12} + a_{12}\sigma(b_{11})$.

Theorem 4.7. *Let R be an integral domain with monomorphism σ, M a torsion free R-module, and c an invertible element of R. Suppose $\alpha : T(R, M, \sigma) \to T(R, M, \sigma)$ given by, $\alpha\left(\begin{smallmatrix} r & m \\ 0 & r \end{smallmatrix}\right) = \left(\begin{smallmatrix} r & cm \\ 0 & r \end{smallmatrix}\right)$, which is an automorphism of $T(R, M, \sigma)$. Then $T(R, M, \sigma)$ is an α-LA ring.*

Proof. Let $f(x) = \sum_{i=-p}^{\infty} A_i x^i$ and $g(x) = \sum_{j=-q}^{\infty} B_j x^j$ be two nonzero elements of $T(R, M, \sigma)[[x, x^{-1}; \alpha]]$, with $A_i = \left(\begin{smallmatrix} r_i & m_i \\ 0 & r_i \end{smallmatrix}\right)$, $B_j = \left(\begin{smallmatrix} s_j & n_j \\ 0 & s_j \end{smallmatrix}\right)$ and $f(x)g(x) = 0$. We prove that $A_i\alpha^i(B_j) = 0$, for each i and j. Clearly, $A_{-p}\alpha^{-p}(B_q) = 0$. So $r_{-p}s_{-q} = 0$, and $r_{-p}(c^{-p}n_{-q}) + m_{-p}\sigma(s_{-q}) = 0$. If $r_{-p} \neq 0$, then $s_{-q} = 0$ and so $n_{-q} = 0$. This implies that $B_{-q} = 0$, a contradiction. Thus $r_{-p} = 0$, and consequently $s_{-q} = 0$, since $A_{-p} \neq 0$. Also $r_{-p+1}s_{-q+1} = 0$, since the coefficient x^{-p-q+2} in fg is zero. On the other hand, $m_{-p}\sigma(s_{-q+1}) + r_{-p+1}(c^{-p+1}n_{-q}) = 0$, since the coefficient x^{-p-q+1} in fg is zero. Therefore, $r_{-p+1} = s_{-q+1} = 0$. By continuing in this way, we get $r_i = s_j = 0$, for each $i \geq -p$ and $j \geq -q$. Hence, $A_i\alpha^i(B_j) = 0$, for each i and j and so $T(R, M, \sigma)$ is an α-LA ring.

Corollary 4.8. *Let R be an integral domain with its field of fractions k. Suppose σ a monomorphism of R and $\alpha : T(R, k, \sigma) \to T(R, k, \sigma)$ given by, $\alpha\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & b/c \\ 0 & a \end{smallmatrix}\right)$, which is an automorphism of $T(R, k, \sigma)$. Then $T(R, k, \sigma)$ is an α-LA ring.*

Let for each $i \in I$, R_i be a ring, α_i an automorphism of R_i, and $R = \prod_{i \in I} R_i$. Then the mapping α on R given by $\alpha((r_i)) = (\alpha_i(r_i))$ is an automorphism of R. Now, if R_i is an α_i-LA ring for each i, then R is also an α-LA ring.

Theorem 4.9. *Let R be an α-LA ring and 2 an invertible element of R. Then the ring $R[x]/\langle x^2 - 1\rangle$ is an $\bar{\alpha}$-LA ring, where $\bar{\alpha}(a + bx + \langle x^2 - 1\rangle) = \alpha(a) + \alpha(b)x + \langle x^2 - 1\rangle$.*

Proof. Since 2 is invertible, it is easy to see that $f : R[x]/\langle x^2 - 1\rangle \to R \oplus R$, given by $f(a + bx + \langle x^2 - 1\rangle) = (a - b, a + b)$, is an automorphism. On the other hand, since R is an α-LA ring, $R \oplus R$ is an $\bar{\alpha}$-LA ring. So $R[x]/\langle x^2 - 1\rangle$ is an $\bar{\alpha}$-LA ring, and the proof is complete.

Let R be a ring with automorphism α and $S = \{\left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right) \mid a, b \in R\}$ a subring of $M_2(R)$. So $\bar{\alpha}$ on S given by $\bar{\alpha}(a_{ij}) = (\alpha(a_{ij}))$ is an automorphism. Now, we have the following corollary.

Corollary 4.10. *Let R be an α-LA ring and 2 an invertible element of R. Then the ring $S = \{\left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right) \mid a, b \in R\}$ is an $\bar{\alpha}$-LA ring.*

Proof. The result follows by Theorem 4.9, as $R[x]/\langle x^2 - 1\rangle \cong S$.

Theorem 4.11. *Let α be a rigid automorphism of a ring R, where α is an automorphism of R. Consider $S = \{\left(\begin{smallmatrix} aI_{n-2} & B \\ 0 & A \end{smallmatrix}\right) | a, b \in R\}$ as a subring of $T_n(R)$. Then S is an $\bar{\alpha}$-LA ring.*

Proof. The mapping $f : S \to \oplus_{i=1}^{n-2} A(R, 3)$, given by $\phi(\left(\begin{smallmatrix} aI_{n-2} & B \\ 0 & A \end{smallmatrix}\right)) = (\left(\begin{smallmatrix} a & B_1 \\ 0 & A \end{smallmatrix}\right), \left(\begin{smallmatrix} a & B_2 \\ 0 & A \end{smallmatrix}\right), \ldots, \left(\begin{smallmatrix} a & B_{n-2} \\ 0 & A \end{smallmatrix}\right))$, where B_i is the ith row of B, is a monomorphism and so S is isomorphic with a subring of $\oplus_{i=1}^{n-2} A(R, 3)$. By Theorem 4.4, $A(R, 3)$ is an $\bar{\alpha}$-LA ring, so $\oplus_{i=1}^{n-2} A(R, 3)$ is also an $\bar{\alpha}$-LA ring. Therefore, S is an $\bar{\alpha}$-LA ring.

A ring R is called *right Ore* if, for each $a, c \in R$ with c regular there exist $a_1, c_1 \in R$ with c_1 regular such that $ac_1 = ca_1$. It is well-known that R is a right Ore ring if and only if there exists the classical right quotient ring of R. Let R be a ring with a classical right quotient ring Q. Then each automorphism α of R, extends to Q, respectively, by setting $\bar{\alpha}(ac^{-1}) = \alpha(a)\alpha(c)^{-1}$, for each $a, c \in R$, assuming that $\alpha(c)$ is regular for each regular element $c \in R$.

Theorem 4.12. *Let R be an Ore ring with the classical quotient ring Q and α an automorphism of R. If R is an α-LA ring, then for each $f(x) = \sum_{i=-m}^{r} p_i x^i$ and $g(x) = \sum_{-n}^{s} q_j x^j$ in $Q[[x, x^{-1}; \bar{\alpha}]]$, $f(x)g(x) = 0$ implies that $p_i\bar{\alpha}^i(q_j) = 0$, for each $-m \le i \le r$ and $-n \le j \le s$.*

Proof. First, for every $f(x) = \sum_{i=-m}^{r} p_i x^i \in Q[[x, x^{-1}; \bar{\alpha}]]$ (all p_i are nonzero), we define $|f(x)| = m + r + 1$. We claim that for each element $f(x) \in Q[[x, x^{-1}; \bar{\alpha}]]$, there exists a regular element $c \in R$ such that $f(x) = h(x)c^{-1}$, for some $h(x) \in R[[x, x^{-1}; \alpha]]$, or equivalently, $f(x)c \in Q[[x, x^{-1}; \bar{\alpha}]]$. The proof is by induction on $|f(x)|$. If $|f(x)| = 1$, then $f(x) = ac^{-1}x^{-m}$. So $f(x) = (ax^{-m})\alpha^m(c)^{-1}$. Now, suppose that for all elements which the number of its summands is less than t, the assertion holds, and let $f(x) = \sum_{i=-m}^{r} p_i x^i \in Q[[x, x^{-1}; \bar{\alpha}]]$, where all $p_i = a_i c_i^{-1}$ are nonzero and $m + r = t - 1$. Let $\alpha^r(d) = c_r$, for some regular element $d \in R$. Then $a_r c_r^{-1} x^r d = a_r x^r$. So we have $f(x)d = (a_{-m}c_{-m}^{-1} + \cdots + a_{r-1}c_{r-1}^{-1}x^{r-1})d + a_r x^r$. By induction hypothesis, there exists some regular element e such that $(a_{-m}c_{-m}^{-1} + \cdots + a_{r-1}c_{r-1}^{-1}x^{r-1})e \in R[[x, x^{-1}; \alpha]]$. Thus we have $f(x)de = (a_{-m}c_{-m}^{-1} + \cdots + a_{r-1}c_{r-1}^{-1}x^{r-1})e + a_r x^r e \in R[[x, x^{-1}; \alpha]]$. Also de is a regular element in R and the result follows. Now, suppose that $f(x) = (a_{-m}c_{-m}^{-1})x^{-m} + \cdots + (a_r c_r^{-1})x^r$ and $g(x) = (b_{-n}d_{-n}^{-1})x^{-n} + \cdots + (b_s d_s^{-1})x^s$ in $Q[[x, x^{-1}; \bar{\alpha}]]$ such that $f(x)g(x) = 0$. Let $a_i c_i^{-1} = f^{-1}a_i'$ and $b_i d_i^{-1} = h^{-1}b_j'$ with f, h regular elements in R. Then we have $(a_{-m}'x^{-m} + \cdots + a_r' x^r)h^{-1}(b_{-n}'x^{-n} + \cdots + b_s' x^s) = 0$. By the above argument, there exist a regular element $l \in R$ and $b_{-n}''x^{-n} + \cdots + b_s'' x^s \in R[[x, x^{-1}; \alpha]]$, such that $h^{-1}(b'_{-n}x^{-n} + \cdots + b'_s x^s) = (b''_{-n}x^{-n} + \cdots + b''_s x^s)l^{-1}$. Hence $(a'_{-m}x^{-m} + \cdots + a'_r x^r)(b''_{-n}x^{-n} + \cdots + b''_s x^s) = 0$. Since R is an α-LA ring, $a'_i\alpha^i(b''_j) = 0$, for each i and j. Therefore, $(f^{-1}a'_i)\alpha^i(b''_j)\alpha^{i+j}(l)^{-1} = 0$. Hence $(a_i c_i^{-1})\bar{\alpha}^i(h^{-1}b'_j) = (a_i c_i^{-1})\bar{\alpha}^i(b_j d_j^{-1}) = 0$, for each i and j, and the proof is complete.

Proposition 4.13. *Let R be an Ore ring with the classical quotient ring Q and α an automorphism of R. If R is an α-LA ring, then we have the following statements*:

(i) $\bar{\alpha}(e) = e$, *for each* $e^2 = e \in Q$;
(ii) *If* $e^2 = e \in Q[[x, x^{-1}; \bar{\alpha}]]$, *then* $e \in Q$;
(iii) Q *is an abelian ring*;
(iv) $Q[[x, x^{-1}; \bar{\alpha}]]$ *is an abelian ring.*

Proof. (i). Let $e^2 = e \in Q$. Suppose that $f(x) = (1-e)\bar{\alpha}^{-1}(e)x^{-1} + (1-e)$ and $g(x) = -(1-e)\bar{\alpha}^{-1}(e) + ex \in Q[[x, x^{-1}; \bar{\alpha}]]$. We have $f(x)g(x) = 0$. Since R is an α-LA ring, by Theorem 4.12, $(1-e)(\bar{\alpha}^{-1}(e)) = 0$, and consequently, $e\bar{\alpha}^{-1}(e) = \bar{\alpha}^{-1}(e)$. On the other hand, $h(x)k(x) = 0$, where $h(x) = e\bar{\alpha}^{-1}(1-e)x^{-1} + e$ and $k(x) = -e\bar{\alpha}^{-1}(1-e) + (1-e)x \in Q[[x, x^{-1}; \bar{\alpha}]]$. Thus, $e(1 - \bar{\alpha}^{-1}(e)) = 0$, by Theorem 4.12. Therefore, $e = e\bar{\alpha}^{-1}(e)$. Hence $e = e\bar{\alpha}^{-1}(e) = \bar{\alpha}^{-1}(e)$ and so $\bar{\alpha}(e) = e$.

(ii) It is similar to that of Proposition 2.3, part (ii).

(iii) Let $e^2 = e$ and $p \in Q$. Suppose that $f(x) = -ep(1-e)x^{-1} + e$ and $g(x) = ep(1-e) + (1-e)x \in Q[[x, x^{-1}; \bar{\alpha}]]$. We have $f(x)g(x) = 0$. So by Theorem 4.12, $ep(1-e) = 0$. Thus $ep = epe$. Next, let $h(x) = -(1-e)pex^{-1} + (1-e)$ and $k(x) = (1-e)pe + ex \in Q[[x, x^{-1}; \bar{\alpha}]]$. We have $h(x)k(x) = 0$. So by Theorem 4.12, $(1-e)pe = 0$. Therefore, $pe = epe$ and so $pe = ep$, which implies that Q is abelian.

(iv) It is clear by (i), (ii), and (iii).

Theorem 4.14. *Let R be a semiprime right Goldie ring, α an automorphism of R, and Q the classical quotient ring of R. Then the following statements are equivalent*:

(i) R *is an* α-LA *ring*;
(ii) Q *is an* $\bar{\alpha}$-LA *ring*;
(iii) R *is an* α-*rigid ring*;
(iv) Q *is an* $\bar{\alpha}$-*rigid ring.*

Proof. We need to prove only (i) $\Rightarrow$ (iv). Let R be an α-LA ring. By Theorem 4.13, Q is abelian. Therefore, Q is an abelian semisimple ring and hence is reduced. Let $p\alpha(p) = 0$, for $p \in R$, $f(x) = \bar{\alpha}(p)x^{-1} - \bar{\alpha}(p)$ and $g(x) = \bar{\alpha}(p) + \bar{\alpha}^2(p)x$ in $Q[[x, x^{-1}; \bar{\alpha}]]$. We have $f(x)g(x) = 0$. Thus, $0 = \bar{\alpha}(p)\bar{\alpha}(p) = \bar{\alpha}(p^2)$, by Theorem 4.12. So $p^2 = 0$. Since Q is reduced, we have $p = 0$. Thus Q is $\bar{\alpha}$-rigid, and the proof is complete.

Note that by [18, Proposition 18], semiprime right and left Goldie rings are Armendariz if and only if it is reduced. But the following example shows that this is no longer true for α-LA rings.

Example 4.15. Let $R = \mathbb{Z} \oplus \mathbb{Z}$ and $\alpha : R \to R$ given by $\alpha(a, b) = (b, a)$. Then α is an automorphism of R. Note that R is a reduced Goldie ring. On the other hand, $(1, 0)\alpha(1, 0) = (0, 0)$ but $(1, 0) \neq 0$. Thus R is not α-rigid, and so R is not an α-LA ring, by Theorem 2.5.

Theorem 4.16. *Let R be a Von Neuman regular ring and α an automorphism of R. Suppose that there exists the classical quotient ring Q of the ring R. Then the following statements are equivalent*:

(i) *R is an α-LA ring*;
(ii) *Q is an $\bar{\alpha}$-LA ring*;
(iii) *R is an α-rigid ring*;
(iv) *Q is an $\bar{\alpha}$-rigid ring.*

Proof. The proof is similar to that of Theorem 4.14, since abelian von Neuman regular rings are reduced.

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