Dynamics and control of the 2-d Navier–Stokes equations

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\textbf{A B S T R A C T}

This paper deals with the dynamics and control of the two-dimensional (2-d) Navier–Stokes (N–S) equations with a spatially periodic and temporally steady forcing term. First, we construct a dynamical system of nine nonlinear differential equations by Fourier expansion and truncation of the 2-d N–S equations. Then, we study the dynamics of the obtained reduced order system by analyzing the system’s attractors for different values of the Reynolds number, \( R_e \). By applying the symmetry of the equations on one of the system’s attractors, a symmetric limit trajectory that is part of the dynamics is obtained. Moreover, a Lyapunov based control strategy to control the dynamics of the system for a given \( R_e \) is designed. Finally, numerical simulations are undertaken to validate the theoretical developments.

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1. Introduction

Recently, the study of dynamical systems that arise from solving partial differential equations (PDEs) has been the focus of many studies in nonlinear dynamics. In general, solving nonlinear parabolic PDEs such as the Navier–Stokes equations numerically requires tremendous computational time. Techniques for solving these PDEs numerically include finite difference, finite element, finite volume, spectral methods, etc. However, these techniques are computationally expensive. Therefore, several techniques have been introduced to obtain reduced order models to approximate the dynamics of these nonlinear PDEs [1–9,11–13,20–25]. Using the reduced order models reduces the computational time tremendously.

The finite dimensionality of the global attractors of these PDEs suggests that the dynamics of the attractors can be captured by a set of ordinary differential equations (ODEs). Hence, making the long-term dynamics of the PDEs equivalent in some sense to the dynamics of the system of ODEs. The notion of inertial manifold for nonlinear evolutionary equations was introduced by Foias et al. [1] as a way to obtain such a system of ODEs. Other attempts to obtain systems of ODEs to approximate PDEs were made by other researchers [2,22–25]. Christofides and Daoutidis [25] introduced a methodology for the synthesis of nonlinear finite-dimensional systems of quasi-linear parabolic PDEs. In [25], they used the concept of approximate inertial manifolds (AIM) (see [24] and the references therein for a complete theory of AIM) to derive a system of ODEs that yields solutions that are very close to the ones of the PDE system. Smaoui and Armbruster [2] used
Karhunen–Loève (K–L) decomposition, also known as Proper Orthogonal Decomposition (POD) and symmetries to obtain a system of ODEs. In [2], a system of twelve ODEs was constructed to capture the dynamics of the two-dimensional Navier–Stokes equations with a monochromatic force. The system captured the dynamics for a fixed Reynolds number. Upon increasing the Reynolds number, the system of ODEs had a different bifurcation from the original simulation. In [3], a system of twelve ODEs was constructed to capture the dynamics of the two-dimensional Navier–Stokes equations with a monochromatic force. The system captured the dynamics for a given Reynolds number.

Several finite dimensional approximations of the Navier–Stokes equations based on truncation of the Fourier modes can be found in the literature, for example see [3–9, 20, 21]. In [9], a system of nine ODEs that approximates the dynamics of the 2-d Navier–Stokes equations was constructed using nine Fourier modes. The dynamics of the system exhibits four limit trajectories that transform simultaneously from a periodic solution to chaotic attractors through a sequence of bifurcations including a period doubling scenario. The control problem of these finite dimensional approximation is not completely carried out, although a lot of work on the control problem of parabolic PDEs such as Burgers equation, Kuramoto–Sivashinsky equation and Korteweg–de Vries–Burgers equations are studied (see [10–18] and the references therein).

The main contribution of this paper is the construction using the truncated Fourier expansion method of a system of nine ODEs that approximates the long time behavior (dynamics) of the 2-d Navier–Stokes equations for various wave numbers \( k \). We show that for \( k = 4 \), the dynamics of the reduced-order model transforms from periodic solutions to chaotic attractors through a period doubling bifurcation. Moreover, a control strategy is designed for the reduced nine ODE system to stabilize the dynamics for a given Reynolds number.

The paper is organized as follows: In Section 2, the 2-d Navier–Stokes equations is presented and a system of ODEs based on a nine Fourier mode truncation of the 2-d N–S equations is obtained for various values of wave numbers \( k \). The dynamics describing steady state solutions, periodic solutions, quasi-periodic solutions and chaotic solutions of the nine ODE system is also presented. Section 3 presents a Lyapunov based control technique applied to the ODE system with the task of stabilizing the dynamics to the origin. Numerical results are also presented in that section to show the effectiveness of the control strategy. Finally, some concluding remarks are given in Section 4.

2. The 2-d Navier–Stokes equation

2.1. The reduced order ODE model

The 2-d Navier–Stokes equations with periodic boundary conditions in two directions \( 0 \leq x, y \leq 2\pi \) are given by:

\[
\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} + \nabla p = v \nabla^2 \vec{u} + \vec{f},
\]

\[
\nabla \cdot \vec{u} = 0,
\]

where \( \vec{f} \) is a forcing term in the \( x, y \) coordinates; the kinematic viscosity is \( v = \frac{1}{\rho} \) and the pressure is \( p \). In 1958, Kolmogorov introduced the “basic 2-d Kolmogorov flow” \( \vec{u} = (k \sin ky, 0) \) as an example on which to study transition to turbulence [19]. This flow is the solution of the Navier–Stokes equations with force \( \vec{f} = (k^2 v \sin ky, 0) \), which is assumed to be stationary and spatially biperiodic.

If we let \( \vec{u} = (u_1, u_2) \) and \( \vec{f} = (f_1, f_2) \) in Eq. (1), the \( x \) and \( y \) components of \( \vec{u} \) become

\[
\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} + \frac{\partial p}{\partial x} = v \left( \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} \right) + f_1,
\]

\[
\frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} + \frac{\partial p}{\partial y} = v \left( \frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} \right) + f_2.
\]

Also, by letting

\[
u_1 = \frac{\partial \Phi}{\partial y},
\]

\[
u_2 = -\frac{\partial \Phi}{\partial x}
\]

and taking the partial derivative with respect to \( y \) of Eq. (3), the partial derivative with respect to \( x \) of Eq. (4), and subtracting the two equations, we get

\[
\frac{\partial^2 \Phi}{\partial t} + \frac{\partial}{\partial x} \left( \Delta \Phi \frac{\partial \Phi}{\partial y} \right) - \frac{\partial}{\partial y} \left( \Delta \Phi \frac{\partial \Phi}{\partial x} \right) = v \Delta^2 \Phi + \frac{\partial f_1}{\partial y} - \frac{\partial f_2}{\partial x}.
\]
Eq. (7) is called the stream function equation. It can also be written in terms of the nondimensional vorticity $\omega = -\Delta \phi$ formulation:

$$\frac{\partial \Delta \phi}{\partial t} - \Delta^2 \phi + R_e \left[ \frac{\partial}{\partial x} \left( \Delta \phi \frac{\partial \phi}{\partial y} \right) - \frac{\partial}{\partial y} \left( \Delta \phi \frac{\partial \phi}{\partial x} \right) \right] - k^4 \cos(ky) = 0$$

and $R_e = \frac{1}{4}$.

To analyze the stability of the basic flow equation, we perturb it $\phi$ such that:

$$\phi = \phi' - \cos ky.$$  \hfill (9)

Then Eq. (8) reduces to

$$\frac{\partial \Delta \phi}{\partial t} - \Delta^2 \phi' - R_e \left[ \frac{\partial}{\partial x} \left( \Delta \phi' \frac{\partial \phi'}{\partial y} \right) - \frac{\partial}{\partial y} \left( \Delta \phi' \frac{\partial \phi'}{\partial x} \right) \right] - kR_e \sin ky \frac{\partial}{\partial x} \left[ \Delta \phi' + k^2 \phi' \right].$$

where $R_e = \frac{1}{4}$ is the Reynolds number and the scaled time $\bar{t} = vt$.

The solution of the above equation is of the form

$$\phi'(t, x, y) = e^{\rho \bar{t}} \tilde{\phi}(x, y),$$

which when substituted into Eq. (10) results in the following linear spectral equation:

$$\rho \Delta \tilde{\phi} - \Delta^2 \tilde{\phi} + kR_e \sin ky \frac{\partial}{\partial x} \left[ \Delta \tilde{\phi} + k^2 \tilde{\phi} \right] = 0.$$  \hfill (11)

Eq. (11) defines two non-real eigenvalues $\rho_1$ and $\rho_2$ with two non-real eigenfunctions $\phi_1$ and $\phi_2$ such that [9]:

$$\phi_1 = \sum_{n=-\infty}^{\infty} \zeta_n \cos(2x + 2y + kny)$$

and

$$\phi_2 = \sum_{n=-\infty}^{\infty} \eta_n \cos(x + y + kny).$$

Now if we expand the stream function $\phi(x, y)$ in terms of the Fourier expansion,

$$\phi(x, y) = \phi_{0,1}(t) \cos ky + \sum_{m=-1}^{1} \sum_{n=-2}^{2} \phi_{m,n}(t) \cos(mx + my + kny),$$

then the following system of nine ODEs is obtained (see Appendix A for the complete derivation of the ODE system):

$$\begin{align*}
\dot{x}_1 &= p_{11} x_1 + p_{12} x_2 x_5 + p_{13} x_3 x_4 + p_{14} x_4 x_5 + p_{15} x_6 x_7 + p_{16} x_7 x_8 + p_{17} x_8 x_9 + p_{11}, \\
\dot{x}_2 &= p_{21} x_2 + p_{22} x_1 x_3 + p_{23} x_4 x_6 + p_{24} x_5 x_7, \\
\dot{x}_3 &= p_{31} x_3 + p_{32} x_1 x_2 + p_{33} x_1 x_4 + p_{34} x_4 x_7 + p_{35} x_5 x_8, \\
\dot{x}_4 &= p_{41} x_4 + p_{42} x_1 x_5 + p_{43} x_1 x_5 + p_{44} x_2 x_8 + p_{45} x_3 x_7 + p_{46} x_5 x_9, \\
\dot{x}_5 &= p_{51} x_5 + p_{52} x_1 x_4 + p_{53} x_2 x_7 + p_{54} x_3 x_8 + p_{55} x_4 x_9, \\
\dot{x}_6 &= p_{61} x_6 + p_{62} x_1 x_7 + p_{63} x_2 x_4, \\
\dot{x}_7 &= p_{71} x_7 + p_{72} x_1 x_6 + p_{73} x_1 x_8 + p_{74} x_2 x_5 + p_{75} x_1 x_4, \\
\dot{x}_8 &= p_{81} x_8 + p_{82} x_1 x_7 + p_{83} x_1 x_9 + p_{84} x_3 x_5, \\
\dot{x}_9 &= p_{91} x_9 + p_{92} x_1 x_6 + p_{93} x_4 x_5,
\end{align*}$$

where $x_i, i = 1, \ldots, 9$ are the state variables with $x = [x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6 \ x_7 \ x_8 \ x_9]^T$ and the parameters $p_{ij}$ are defined such that:
\[ p_{11} = -k^2; \quad p_{12} = 0.5(2 - 3k)Re; \quad p_{13} = 0.5(2 - k)Re; \quad p_{14} = 0.5(2 + k)Re; \]
\[ p_{15} = (4 - 3k)Re; \quad p_{16} = (4 - k)Re; \quad p_{17} = (4 + k)Re; \]
\[ p_{21} = -1 - (1 - 2k)^2; \quad p_{22} = \frac{1}{1 + (1 - 2k)^2} k(k - 1)Re; \]
\[ p_{23} = \frac{-2}{1 + (1 - 2k)^2} k(2k^2 - 4k + 3)Re; \quad p_{24} = \frac{9}{1 + (1 - 2k)^2} k(k - 1)Re; \]
\[ p_{31} = -1 - (1 - k)^2; \quad p_{32} = \frac{0.5}{1 + (1 - k)^2} k(3k^2 - 4k + 2)Re; \]
\[ p_{33} = \frac{0.5}{1 + (1 - k)^2} k(k^2 - 2)Re; \quad p_{34} = \frac{-0.5}{1 + (1 - k)^2} k(2 - 4k + 6)Re; \]
\[ p_{35} = \frac{1}{1 + (1 - k)^2} k(k^2 + 2k - 6)Re; \]
\[ p_{41} = -2; \quad p_{42} = -0.5k(k - 1)Re; \quad p_{43} = -0.5k(k + 1)Re; \]
\[ p_{44} = k(3 - 2k)Re; \quad p_{45} = -0.5k(k - 3)Re; \quad p_{46} = -0.5k(k + 3)Re; \quad p_{51} = -1 - (1 + k)^2; \]
\[ p_{52} = \frac{-0.5}{1 + (1 + k)^2} k^2(2 - 2k)Re; \quad p_{53} = \frac{-4.5}{1 + (1 + k)^2} k(k^2 - 2)Re; \quad p_{54} = \frac{-1}{1 + (1 + k)^2} k^2(2 - k - 6)Re; \]
\[ p_{55} = \frac{0.5}{1 + (1 + k)^2} k^2(4k + 6)Re; \quad p_{56} = -4 - (2 - 2k)^2; \quad p_{57} = \frac{4}{1 + (2 - 2k)^2} k(2k - 2)Re; \]
\[ p_{58} = \frac{2}{1 + (2 - 2k)^2} k^2(2k - 2)Re; \quad p_{59} = \frac{4}{1 + (2 - 2k)^2} k^2(2k - 2)Re; \]
\[ p_{60} = -8; \quad p_{61} = -0.5k(2 - 2k)Re; \quad p_{62} = -0.5k(k + 2)Re; \quad p_{63} = -0.5k^2Re; \]
\[ p_{64} = -4 - (2 + k)^2; \quad p_{65} = \frac{-1}{1 + (2 + k)^2} k^2(2 - 8)Re; \quad p_{66} = \frac{-0.5}{1 + (2 + k)^2} k^2(2 + 2k)Re. \]

Note that the system given by the equations in (13) can be written in the following form:
\[ \dot{x} = Ax + f(x), \]
where the diagonal matrix \(A\) is such that:
\[ A = \text{diag}(p_{11}, p_{21}, p_{31}, p_{41}, p_{51}, p_{61}, p_{71}, p_{81}, p_{91}) \]
and the nonlinear vector \(f(x)\) is such that:
\[ f = [f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9]^T \]
with,
\[ f_1 = p_{12}x_2x_3 + p_{13}x_3x_4 + p_{14}x_4x_5 + p_{15}x_5x_6 + p_{16}x_6x_7 + p_{17}x_7x_8 + p_{18}x_8x_9 + p_{19}x_9x_1 + p_{11}; \]
\[ f_2 = p_{22}x_2x_3 + p_{23}x_3x_4 + p_{24}x_4x_5 + p_{25}x_5x_6 + p_{26}x_6x_7; \]
\[ f_3 = p_{32}x_2x_3 + p_{33}x_3x_4 + p_{34}x_4x_5 + p_{35}x_5x_6; \]
\[ f_4 = p_{42}x_2x_3 + p_{43}x_3x_4 + p_{44}x_4x_5 + p_{45}x_5x_6 + p_{46}x_6x_7 + p_{47}x_7x_8 + p_{48}x_8x_9; \]
\[ f_5 = p_{52}x_2x_3 + p_{53}x_3x_4 + p_{54}x_4x_5 + p_{55}x_5x_6 + p_{56}x_6x_7 + p_{57}x_7x_8 + p_{58}x_8x_9; \]
\[ f_6 = p_{62}x_2x_3 + p_{63}x_3x_4; \]
\[ f_7 = p_{72}x_2x_3 + p_{73}x_3x_4 + p_{74}x_4x_5 + p_{75}x_5x_6 + p_{76}x_6x_7 + p_{77}x_7x_8 + p_{78}x_8x_9; \]
\[ f_8 = p_{82}x_2x_3 + p_{83}x_3x_4 + p_{84}x_4x_5; \]
\[ f_9 = p_{92}x_2x_3 + p_{93}x_3x_4. \]

**Remark 1.** It is noted that the parameters \(p_{ij}\) are functions of the wave number \(k\) and the Reynolds numbers \(Re\). Hence, for a given wave number \(k\) and a given Reynolds number \(Re\), the parameters \(p_{ij}\) are constants. Also, it is noted that \(p_{1i}, \ (i = 1, \ldots, 9)\) are negative.
**Remark 2.** It should be noted that the point \((x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) = (-1, 0, 0, 0, 0, 0, 0, 0)\) is the basic state solution for system (13).

**Remark 3.** It is noted that system (13) is invariant under the following symmetry property: 
\((x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9)\) is a solution of (13) if and only if \((x_1, -x_2, -x_3, -x_4, -x_5, x_6, x_7, x_8, x_9)\) is also a solution of (13).

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**Fig. 1.** (a) Phase portrait of one of the limit trajectory at \(R_\ell = 16.4\); (b) Phase portraits of the symmetric limit trajectory obtained by applying the following transformation \(T\): \((x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) \rightarrow (x_1, -x_2, -x_3, -x_4, -x_5, x_6, x_7, x_8, x_9)\) to the trajectory in (a) at \(R_\ell = 16.4\); (c) Phase portraits of the two limit trajectories at \(R_\ell = 16.4\).

**Fig. 2.** Phase portrait of a single limit trajectory at (a) \(R_\ell = 20\); (b) \(R_\ell = 20.5\); (c) \(R_\ell = 21\) and (d) \(R_\ell = 21.1\).
2.2. Simulation results of the reduced order ODE system

In this subsection, we present the results of the ODE system (13) for the case when \( k = 4 \). The numerical simulations presented are derived using the DsTool software [26] where the 4th-order Runge-Kutta method is used as the numerical integrator with the time step \( dt = 0.0001 \).

Fig. 1(a) depicts the phase portrait of one of the limit trajectory when \( Re = 16.4 \); Fig. 1(b) depicts the symmetric limit trajectory obtained after applying the following transformation:

\[
T: (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) \rightarrow (x_1, -x_2, -x_3, -x_4, -x_5, x_6, x_7, x_8, x_9).
\]

Fig. 1(c) shows the phase portrait of the two limit trajectories obtained in Fig. 1(a) and (b).

Fig. 2 presents a single limit trajectory at different Reynolds numbers. Again, if we apply the transformation \( T \), a symmetric limit trajectory will appear. The two symmetric limit cycles at \( Re = 20 \) (Fig. 2(a)) undergo a period doubling bifurcation leading to a pair of two loop limit cycles at \( Re = 20.5 \) (Fig. 2(b)) which in turn bifurcates into four doublings represented by a pair of four loop limit cycles when \( Re = 21 \) (Fig. 2(c)). As the Reynolds number increases further, it experiences a period doubling bifurcation leading to a pair of symmetric chaotic trajectories at \( Re = 21.1 \) (Fig. 2(d)).

Increasing the Reynolds number a little bit more (i.e., \( Re = 21.12 \)), the chaotic trajectory bursts into the other chaotic symmetric counterpart trajectory and this bursting phenomena is itself chaotic (see Fig. 3). Fig. 3(b) shows the time evolution of the state \( x_4 \) at \( Re = 21.12 \). The figure highlights the typical phase space dynamics that represents the stream function \( \phi \). The dynamics follow a chaotic regime similar to the one observed at \( Re = 21 \), then it undergoes a strong turbulent (chaotic) explosion, then it seems to settle down to another chaotic regime which is symmetric to the previously observed one, then other explosions follow. Intervals between explosions are not constants. This behavior can be described in terms of heteroclinic connections and homoclinic loop in phase space. The heteroclinic connections are structurally stable, where the intervals between bursts (explosions) become shorter as the Reynolds number increases.

3. A Lyapunov based controller for the ODE system

3.1. Design of the controller

In this section, we design a Lyapunov based controller to drive the states of the system to converge to the origin as \( t \) tends to infinity. Note that a controller is added to the first ode of system (13) as follows:

\[
\dot{x}_1 = p_{11}x_1 + p_{12}x_2x_3 + p_{13}x_2x_4 + p_{14}x_2x_5 + p_{15}x_2x_7 + p_{16}x_2x_8 + p_{17}x_2x_9 + p_{11} + u,
\]

\[
\dot{x}_2 = p_{21}x_2 + p_{22}x_2x_3 + p_{23}x_2x_5 + p_{24}x_2x_7,
\]

\[
\dot{x}_3 = p_{31}x_3 + p_{32}x_1x_2 + p_{33}x_2x_4 + p_{34}x_2x_5 + p_{35}x_2x_8,
\]

\[
\dot{x}_4 = p_{41}x_4 + p_{42}x_1x_3 + p_{43}x_1x_5 + p_{44}x_2x_5 + p_{45}x_3x_7 + p_{46}x_5x_9,
\]

\[
\dot{x}_5 = p_{51}x_5 + p_{52}x_1x_4 + p_{53}x_1x_7 + p_{54}x_3x_7 + p_{55}x_5x_8 + p_{56}x_4x_9,
\]

\[
\dot{x}_6 = p_{61}x_6 + p_{62}x_1x_7 + p_{63}x_3x_4,
\]

\[
\dot{x}_7 = p_{71}x_7 + p_{72}x_1x_6 + p_{73}x_1x_8 + p_{74}x_2x_5 + p_{75}x_3x_4,
\]

\[
\dot{x}_8 = p_{81}x_8 + p_{82}x_1x_7 + p_{83}x_1x_9 + p_{84}x_5x_3,
\]

\[
\dot{x}_9 = p_{91}x_9 + p_{92}x_1x_8 + p_{93}x_4x_5,
\]

where the parameters \( p_{ij} \) are the same as defined earlier.
Let $a_i$ ($i = 1, \ldots, 9$) be positive scalars such that the following algebraic equations are satisfied:

\begin{align}
& a_2 p_{23} + a_9 p_{44} + a_6 p_{63} = 0, \\
& a_2 p_{24} + a_3 p_{33} + a_7 p_{74} = 0, \\
& a_1 p_{34} + a_4 p_{45} + a_5 p_{57} = 0, \\
& a_3 p_{35} + a_5 p_{54} + a_8 p_{84} = 0, \\
& a_4 p_{46} + a_5 p_{55} + a_9 p_{93} = 0. \\
\end{align}

In addition, let the gain $g_1$ be a positive scalar.

**Theorem 1. The control law:**

\begin{align}
\mathbf{u} &= -\frac{1}{\mathbf{a}_i} [a_1 p_{12} x_2 x_3 + a_1 p_{13} x_3 x_4 + a_1 p_{14} x_4 x_5 + a_1 p_{15} x_5 x_6 + a_1 p_{16} x_6 x_7 + a_1 p_{17} x_7 x_8 + a_1 p_{18} x_8 x_9 + a_2 p_{11} + a_2 p_{22} x_2 x_3 + a_3 p_{32} x_2 x_3 \\
&+ a_4 p_{42} x_3 x_4 + a_5 p_{53} x_4 x_5 + a_6 p_{64} x_5 x_6 + a_7 p_{72} x_6 x_7 + a_8 p_{82} x_7 x_8 + a_9 p_{92} x_8 x_9 + a_3 p_{33} x_3 x_4 \\
&+ a_4 p_{43} x_4 x_5 + a_5 p_{57} x_7 x_8 + g_1 x_1],
\end{align}

when applied to the nonlinear system (14) guarantees the asymptotic convergence of the states $x_i$, ($i = 1, \ldots, 9$) to the origin, $\bar{x} = 0$, as $t$ tends to infinity.

**Proof.** Let the Lyapunov function candidate $V$ be such that:

\begin{align}
V &= \frac{1}{2} a_1 x_1^2 + \frac{1}{2} a_2 x_2^2 + \frac{1}{2} a_3 x_3^2 + \frac{1}{2} a_4 x_4^2 + \frac{1}{2} a_5 x_5^2 + \frac{1}{2} a_6 x_6^2 + \frac{1}{2} a_7 x_7^2 + \frac{1}{2} a_8 x_8^2 + \frac{1}{2} a_9 x_9^2.
\end{align}

Using the dynamic model of the system in (14), the derivative of $V$ with respect to time is such:

\begin{align}
\dot{V} &= a_1 x_1 \dot{x}_1 + a_2 x_2 \dot{x}_2 + a_3 x_3 \dot{x}_3 + a_4 x_4 \dot{x}_4 + a_5 x_5 \dot{x}_5 + a_6 x_6 \dot{x}_6 + a_7 x_7 \dot{x}_7 + a_8 x_8 \dot{x}_8 + a_9 x_9 \dot{x}_9 \\
&= a_1 x_1 (p_{11} x_1 + p_{12} x_2 + p_{13} x_3 + p_{14} x_4 + p_{15} x_5 + p_{16} x_6 + p_{17} x_7 + p_{18} x_8 + p_{19} x_9) \\
&+ a_2 x_2 (p_{21} x_1 + p_{22} x_2 + p_{23} x_3 + p_{24} x_4 + p_{25} x_5 + p_{26} x_6 + p_{27} x_7 + p_{28} x_8 + p_{29} x_9) \\
&+ a_3 x_3 (p_{31} x_1 + p_{32} x_2 + p_{33} x_3 + p_{34} x_4 + p_{35} x_5 + p_{36} x_6 + p_{37} x_7 + p_{38} x_8 + p_{39} x_9) \\
&+ a_4 x_4 (p_{41} x_1 + p_{42} x_2 + p_{43} x_3 + p_{44} x_4 + p_{45} x_5 + p_{46} x_6 + p_{47} x_7 + p_{48} x_8 + p_{49} x_9) \\
&+ a_5 x_5 (p_{51} x_1 + p_{52} x_2 + p_{53} x_3 + p_{54} x_4 + p_{55} x_5 + p_{56} x_6 + p_{57} x_7 + p_{58} x_8 + p_{59} x_9) \\
&+ a_6 x_6 (p_{61} x_1 + p_{62} x_2 + p_{63} x_3 + p_{64} x_4 + p_{65} x_5 + p_{66} x_6 + p_{67} x_7 + p_{68} x_8 + p_{69} x_9) \\
&+ a_7 x_7 (p_{71} x_1 + p_{72} x_2 + p_{73} x_3 + p_{74} x_4 + p_{75} x_5 + p_{76} x_6 + p_{77} x_7 + p_{78} x_8 + p_{79} x_9) \\
&+ a_8 x_8 (p_{81} x_1 + p_{82} x_2 + p_{83} x_3 + p_{84} x_4 + p_{85} x_5 + p_{86} x_6 + p_{87} x_7 + p_{88} x_8 + p_{89} x_9) \\
&+ a_9 x_9 (p_{91} x_1 + p_{92} x_2 + p_{93} x_3 + p_{94} x_4 + p_{95} x_5 + p_{96} x_6 + p_{97} x_7 + p_{98} x_8 + p_{99} x_9) \\
&= a_1 p_{11} x_1^2 + a_2 p_{21} x_1^2 + a_3 p_{31} x_1^2 + a_4 p_{41} x_1^2 + a_5 p_{51} x_1^2 + a_6 p_{61} x_1^2 + a_2 p_{12} x_1 x_2 + a_3 p_{13} x_1 x_3 + a_4 p_{14} x_1 x_4 \\
&+ a_5 p_{15} x_1 x_5 + a_6 p_{16} x_1 x_6 + a_7 p_{17} x_1 x_7 + a_8 p_{18} x_1 x_8 + a_9 p_{19} x_1 x_9 + a_3 p_{23} x_2 x_3 + a_4 p_{43} x_3 x_4 + a_5 p_{53} x_4 x_5 \\
&+ a_6 p_{63} x_5 x_6 + a_7 p_{73} x_6 x_7 + a_8 p_{83} x_7 x_8 + a_9 p_{93} x_8 x_9.
\end{align}

Now, we use conditions given by (15) so that the terms involving the state variables $x_i x_j x_k$ in the derivative of the Lyapunov function $V$ are canceled out. Next, Using the control law given by (16), the derivative of $V$ with respect to time reduces to

\begin{align}
\dot{V} &= a_1 p_{11} x_1^2 + a_2 p_{21} x_1^2 + a_3 p_{31} x_1^2 + a_4 p_{41} x_1^2 + a_5 p_{51} x_1^2 + a_6 p_{61} x_1^2 + a_7 p_{71} x_1^2 + a_8 p_{81} x_1^2 + a_9 p_{91} x_1^2 - g_1 x_1^2.
\end{align}

Note from the definition of the $p_{ij}$ that $p_{i1} < 0$ for ($i = 1, \ldots, 9$). Also, the scalars $a_i$ for ($i = 1, \ldots, 9$) and $g_1$ are positive scalars. Therefore, $V$ is negative definite. Clearly, $V$ is a continuously differentiable positive definite function defined over $\mathbb{R}^9$ such that $V$ is negative definite in $\mathbb{R}^9$. Also, $V$ is radially unbounded. Therefore, it can be concluded that the controller (16) when applied to the system (14) guarantees the asymptotic convergence of the states $x_i$, ($i = 1, \ldots, 9$) to the origin as $t$ tends to infinity. $\square$

4. Simulation results of the controlled system

We apply the control law given in Theorem 1 to the nonlinear system of ODE in (14), and keeping in mind that the gains $a_i$ are positive scalars such that the linear equations given by (15) are satisfied. For example, the linear equations given by (15)
are satisfied when $a_1 = 1$, $a_2 = 0.7113$, $a_3 = 1.2048$, $a_4 = 0.3719$, $a_5 = 2.1916$, $a_6 = 2$, $a_7 = 1.0947$, $a_8 = 1$, and $a_9 = 1$. Also $g_1 = 50$.

First, the system given by (14) is simulated without control (i.e. $u = 0$) for the first 3 s; then the controller given by (16) is applied to the system for 5 s and then the controller is switched off. The simulation results for the case when $Re = 16.4$ are

![Fig. 4. Phase portrait of a single limit trajectory at $Re = 16.4$ when the control was switched on and then off after $t = 5$ s.](image1)

![Fig. 5. (a) Time series solution $x_3(t)$ of one of the limit trajectory at $Re = 21.1$; (b) Time series solution $x_4(t)$ of one of the limit trajectory at $Re = 21.1$; (c) Phase portrait of the limit trajectory at $Re = 21.1$.](image2)
presented in Fig. 4. Fig. 4(a) depicts the state $x_3$ versus time; while Fig. 4(b) shows the phase plot of $x_3$ versus $x_4$. It is clear from the figures that the proposed controller was able to drive the state of the system to the origin. Then, we concentrate on the chaotic attractor when $Re = 21.1$. A time series solution of the states $x_3$ and $x_4$ are depicted in Fig. 5(a) and (b), respectively. The phase space $x_3$ vs. $x_4$ is presented in Fig. 5(c). Recall that a symmetric solution can be easily obtained via the $T$ transformation previously defined. Note that at this Reynolds number no bursting (i.e., no explosions) is observed. The controlled system is designed to observe the bursting phenomena between the two symmetric chaotic attractors. Fig. 6(a) and (b) depict the states $x_3$ and $x_4$ versus time, respectively with the controller switched on and off. Fig. 6(c) shows the corresponding phase plot of $x_3$ versus $x_4$. Applying the control law after running the simulations for five seconds without control, one can easily see that the system is driven to the origin. Then switching the control off and on periodically for five seconds, the system seems to burst to one of the two symmetrized solutions. The choice where the system gets attracted to either one of the attractor is chaotic, and unpredictable. From a dynamical system point of view, again this behavior is described in terms of a heteroclinic connection in phase space. The heteroclinic connections is now a controlled structurally stable one (see Fig. 6).

5. Concluding remarks

In this paper, a ninth order system of ODEs is constructed using the truncated Fourier series expansion to approximate the dynamical behavior of the 2-d Navier–Stokes equations. Numerical simulations of the constructed system of odes are presented to show that the dynamics of the system transform from periodic solutions to chaotic attractors through a period doubling bifurcation. Also, by applying the symmetry of the system on one of the dynamics attractor, a symmetrized solution is obtained. In addition, a Lyapunov based controller is proposed to regulate the states of the system to the origin. Simulation results are presented to illustrate the behavior of the controlled system.

A reduced order system of less than nine ODEs whose dynamics are similar to the dynamics of the 2-d Navier–Stokes equation with a monochromatic forcing for different wave numbers $k$ and for various values of Reynolds numbers $Re$ will be the subject of future studies.
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Appendix A. The derivation of the system of nine ODEs

In this appendix we will derive a system of nine ODEs from the stream function representation of the 2-d Navier Stokes equation:

$$\frac{\partial}{\partial t} \Delta \phi + \partial_x (\Delta \phi \partial_y \phi) - \partial_y (\Delta \phi \partial_x \phi) = v \Delta^2 \phi + v k^4 \cos ky. \quad (A1)$$

Dividing by \(v\) and letting \(Re = \frac{1}{\nu}\), Eq. (A1) can be written as follows:

$$\frac{\partial}{\partial t} \Delta \phi + Re \left[ \frac{\partial_x (\Delta \phi \partial_y \phi) - \partial_y (\Delta \phi \partial_x \phi)}{v} \right] = \Delta^2 \phi + k^4 \cos ky. \quad (A2)$$

Expanding the stream function \(\phi(x, y)\) in terms of Fourier expansion as shown in Section 2

$$\phi = \phi_{0,1} \cos ky + \sum_{m=1}^{2} \sum_{n=-2}^{1} \phi_{m,n} \cos(mx + my + kny). \quad (A3)$$

then,

$$\Delta \phi = -k^2 \phi_{0,1} \cos ky + \sum_{m=1}^{2} \sum_{n=-2}^{1} - (m^2 + (m + kn)^2) \phi_{m,n} \cos(mx + my + kny), \quad (A4)$$

$$\frac{\partial}{\partial t} \Delta \phi = -k^2 \phi_{0,1} \cos ky + \sum_{m=1}^{2} \sum_{n=-2}^{1} - (m^2 + (m + kn)^2) \phi_{m,n} \cos(mx + my + kny), \quad (A5)$$

$$\Delta^2 \phi = k^4 \phi_{0,1} \cos ky + \sum_{m=1}^{2} \sum_{n=-2}^{1} (m^2 + (m + kn)^2)^2 \phi_{m,n} \cos(mx + my + kny), \quad (A6)$$

$$\frac{\partial}{\partial y} \phi = -k \phi_{0,1} \sin ky + \sum_{m=1}^{2} \sum_{n=-2}^{1} (m + kn) \phi_{m,n} \sin(mx + my + kny), \quad (A7)$$

$$\frac{\partial}{\partial x} \Delta \phi = \sum_{m=1}^{2} \sum_{n=-2}^{1} m(m^2 + (m + kn)^2) \phi_{m,n} \sin(mx + my + kny), \quad (A8)$$

$$\frac{\partial}{\partial y} \phi = \sum_{m=1}^{2} \sum_{n=-2}^{1} m \phi_{m,n} \sin(mx + my + kny) \quad (A9)$$

and,

$$\frac{\partial}{\partial x} \Delta \phi = k^3 \phi_{0,1} \sin ky + \sum_{m=1}^{2} \sum_{n=-2}^{1} (m + kn)(m^2 + (m + kn)^2) \phi_{m,n} \sin(mx + my + kny) \quad (A10)$$

Therefore,

$$\frac{\partial}{\partial y} \phi \frac{\partial}{\partial x} \Delta \phi - \frac{\partial}{\partial x} \phi \frac{\partial}{\partial y} \Delta \phi = \sum_{m=1}^{2} \sum_{n=-2}^{1} \left[-km(m^2 + (m + kn)^2) + k^3 m \right] \phi_{0,1} \phi_{m,n} \sin ky \sin(mx + my + kny)$$

$$+ \sum_{m=1}^{2} \sum_{n=-2}^{1} \left[ -(1 - 2k) m(m^2 + (m + kn)^2) + m(1 - 2k)(1 + (1 - 2k)^2) \right] \phi_{1,1} \phi_{m,n}$$

$$\times \sin(x + y - 2ky) \sin(mx + my + kny)$$

$$+ \sum_{m=1}^{2} \sum_{n=-2}^{1} \left[ -(1 - k) m(m^2 + (m + kn)^2) + m(1 - k)(1 + (1 - k)^2) \right] \phi_{1,1} \phi_{m,n}$$

$$\times \sin(x + y - ky) \sin(mx + my + kny)$$

$$+ \sum_{m=1}^{2} \sum_{n=-2}^{1} \left[ -m(m^2 + (m + kn)^2) + 2m \right] \phi_{1,0} \phi_{m,n} \sin(x + y) \sin(mx + my + kny)$$
\[
\begin{align*}
&+ \sum_{m=1}^{2} \sum_{n=1}^{1} \left[ -(1 + k) m (m^2 + (m + k n)^2) + m(1 + k)(1 + (1 + k)^2) \right] \Phi_{1,1} \Phi_{m,n} \\
&\times \sin(x + y + k y) \sin(mx + my + k n y)
\end{align*}
\]
\[
\begin{align*}
&+ \sum_{m=1}^{2} \sum_{n=1}^{1} \left[ -(2 - 2k) m (m^2 + (m + k n)^2) + m(2 - 2k)(4 + (2 - 2k)^2) \right] \Phi_{2,1} \Phi_{m,n} \\
&\times \sin(2x + 2y - k y) \sin(mx + my + k n y)
\end{align*}
\]
\[
\begin{align*}
&+ \sum_{m=1}^{2} \sum_{n=1}^{1} \left[ -(2 - k) m (m^2 + (m + k n)^2) + m(2 - k)(4 + (2 - k)^2) \right] \Phi_{2,1} \Phi_{m,n} \\
&\times \sin(2x + 2y - k y) \sin(mx + my + k n y)
\end{align*}
\]
\[
\begin{align*}
&+ \sum_{m=1}^{2} \sum_{n=1}^{1} \left[ -2m (m^2 + (m + k n)^2) + m(2) (2) \right] \Phi_{2,0} \Phi_{m,n} \sin(2x + 2y) \sin(mx + my + k n y)
\end{align*}
\]
\[
\begin{align*}
&+ \sum_{m=1}^{2} \sum_{n=1}^{1} \left[ -(2 + k) m (m^2 + (m + k n)^2) + m(2 + k)(4 + (2 + k)^2) \right] \Phi_{2,1} \Phi_{m,n} \\
&\times \sin(2x + 2y + k y) \sin(mx + my + k n y).
\end{align*}
\]  

Or,
\[
\begin{align*}
&\partial_y \partial_x \Delta \phi - \partial_x \phi \partial_y \Delta \phi = 2 \sum_{m=1}^{2} \sum_{n=1}^{1} \frac{m^2}{2} \left[ (m^2 - (m + k n)^2) + k x \right] \phi_{1,0} \phi_{m,n} \\
&\times [\cos(mx + my + k(n - 1)y) - \cos(mx + my + k(n + 1)y)]
\end{align*}
\]
\[
\begin{align*}
&+ 2 \sum_{m=1}^{2} \sum_{n=1}^{1} \frac{(1 - 2k)m^2}{2} \left[ -m^2 - (m + k n)^2 + 1 + (1 - 2k)^2 \right] \phi_{1,2} \phi_{m,n} \\
&\times [\cos((m - 1)x + (m - 1)y + k(n + 2)y) - \cos((m + 1)x + (m + 1)y + k(n - 2)y)]
\end{align*}
\]
\[
\begin{align*}
&+ 2 \sum_{m=1}^{2} \sum_{n=1}^{1} \frac{(1 - k)m^2}{2} \left[ -m^2 - (m + k n)^2 + 1 + (1 - k)^2 \right] \phi_{1,1} \phi_{m,n} \\
&\times [\cos((m - 1)x + (m - 1)y + k(n + 1)y) - \cos((m + 1)x + (m + 1)y + k(n - 1)y)]
\end{align*}
\]
\[
\begin{align*}
&+ 2 \sum_{m=1}^{2} \sum_{n=1}^{1} \frac{m^2}{2} \left[ -m^2 - (m + k n)^2 + 2 \right] \phi_{1,0} \phi_{m,n} \\
&\times [\cos((m - 1)x + (m - 1)y + kny) - \cos((m + 1)x + (m + 1)y + kny)]
\end{align*}
\]
\[
\begin{align*}
&+ 2 \sum_{m=1}^{2} \sum_{n=1}^{1} \frac{(1 + k)m^2}{2} \left[ -m^2 - (m + k n)^2 + 1 + (1 + k)^2 \right] \phi_{1,1} \phi_{m,n} \\
&\times [\cos((m - 1)x + (m - 1)y + k(n - 1)y) - \cos((m + 1)x + (m + 1)y + k(n + 1)y)]
\end{align*}
\]
\[
\begin{align*}
&+ 2 \sum_{m=1}^{2} \sum_{n=1}^{1} \frac{(1 - k)m^2}{2} \left[ -m^2 - (m + k n)^2 + 4 + (2 - 2k)^2 \right] \phi_{2,1} \phi_{m,n} \\
&\times [\cos((m - 2)x + (m - 2)y + k(n + 2)y) - \cos((m + 2)x + (m + 2)y + k(n - 2)y)]
\end{align*}
\]
\[
\begin{align*}
&+ 2 \sum_{m=1}^{2} \sum_{n=1}^{1} \frac{(2 - k)m^2}{2} \left[ -m^2 - (m + k n)^2 + 4 + (2 - k)^2 \right] \phi_{2,1} \phi_{m,n} \\
&\times [\cos((m - 2)x + (m - 2)y + k(n - 1)y) - \cos((m + 2)x + (m + 2)y + k(n + 1)y)]
\end{align*}
\]
\[
\begin{align*}
&+ 2 \sum_{m=1}^{2} \sum_{n=1}^{1} \frac{m^2}{2} \left[ -m^2 - (m + k n)^2 + 8 \right] \phi_{2,0} \phi_{m,n} \\
&\times [\cos((m - 2)x + (m - 2)y + kny) - \cos((m + 2)x + (m + 2)y + kny)]
\end{align*}
\]
\[
\begin{align*}
&+ 2 \sum_{m=1}^{2} \sum_{n=1}^{1} \frac{(2 + k)m^2}{2} \left[ -m^2 - (m + k n)^2 + 4 + (2 + k)^2 \right] \phi_{2,1} \phi_{m,n} \\
&\times [\cos((m - 2)x + (m - 2)y + k(n - 1)y) - \cos((m + 2)x + (m + 2)y + k(n + 1)y)]
\end{align*}
\]

(A11)

To construct the system of nine ODEs, we solve the following integral:

\[
\int_{0}^{2\pi} \int_{0}^{2\pi} \left[ \partial_t \Delta \phi - \Delta^2 \phi + R_e (\partial_t \phi \partial_t \Delta \phi - \partial_x \phi \partial_y \Delta \phi) - k^4 \cos ky \right] \psi \, dx dy = 0.
\]  

(A12)
where, $\psi_0 = \cos ky$ and $\psi_{m,n} = \cos(mx + my + kny)$ for $m = 1, 2$ and $n = -2, -1, 0, 1, \text{ and } \phi_{0,1} = x_1, \phi_{1,-2} = x_2, \phi_{1,-1} = x_3, \phi_{1,0} = x_4, \phi_{1,1} = x_5, \phi_{2,-2} = x_6, \phi_{2,-1} = x_7, \phi_{2,0} = x_8 \text{ and } \phi_{2,1} = x_9$. Then, the following system of ODEs is obtained:

\[
\begin{align*}
\dot{x}_1 &= -k^2 x_1 - k^2 + R \frac{1}{2} (2 - 3k)x_2 x_3 + \frac{1}{2} (2 - k)x_2 x_4 + \frac{1}{2} (k + 2)x_4 x_5 - (3k - 4)x_6 x_7 - (k - 4)x_7 x_8 + (k + 4)x_8 x_9, \\
\dot{x}_2 &= -(1 + (1 - 2k^2)) x_2 + \frac{Re}{(1 + (1 - 2k^2))} \left[ k(k - 1)x_1 x_3 - 2k(2k^2 - 4k + 3)x_4 x_6 + 9k(k - 1)x_5 x_7 \right], \\
\dot{x}_3 &= -(1 + (1 - k^2)) x_3 + \frac{Re}{(1 + (1 - k^2))} \left[ k(k^2 - 2)x_1 x_4 - k^2 - 4k - 3k^2)x_1 x_2 - \frac{k}{2} (k^2 - 4k + 6)x_4 x_7 + k^2 + 2k - 6)x_5 x_8 \right], \\
\dot{x}_4 &= -2x_2 + \frac{Re}{2} \left[ -k(k - 1)x_1 x_5 - k(k - 1)x_1 x_3 + 2k(3 - 2k)x_4 x_6 - k(k - 3)x_1 x_7 - k(k - 3)x_5 x_9 \right], \\
\dot{x}_5 &= -(1 + (1 + k^2)) x_5 + \frac{Re}{(1 + (1 + k^2))} \left[ -k(k^2 - 2)x_1 x_3 - \frac{k}{2} (k^2 - 2)x_2 x_7 - k(k^2 - 2k - 6)x_3 x_8 + \frac{k}{2} (k^2 + 4k + 6)x_4 x_9 \right], \\
\dot{x}_6 &= -(4 + 2 - 2k^2)x_6 + \frac{Re}{(4 + 2 - 2k^2)} \left[ 4k(k - 1)x_1 x_7 + 4k^2(k - 1)x_2 x_4 \right], \\
\dot{x}_7 &= -(4 + 2 - k^2)x_7 + \frac{Re}{(4 + 2 - k^2)} \left[ k(k^2 - 8)x_1 x_8 + k(3k^2 - 8k + 8)x_1 x_6 + \frac{9}{2} k^2(k - 2)x_3 x_5 + \frac{1}{2} k^2(k - 2)x_3 x_4 \right], \\
\dot{x}_8 &= -8x_6 + \frac{Re}{2} \left[ -k(k - 2)x_1 x_9 - k(k - 2)x_1 x_7 - k^2 x_3 x_5 \right], \\
\dot{x}_9 &= -(4 + 2 + k^2)x_9 + \frac{Re}{(4 + 2 + k^2)} \left[ -k(k^2 - 8)x_1 x_8 - \frac{1}{2} k^2(k + 2)x_4 x_5 \right].
\end{align*}
\] (A14)

References


