Numerical conformal mapping and its inverse of unbounded multiply connected regions onto logarithmic spiral slit regions and straight slit regions

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This paper presents a boundary integral equation method with the adjoint generalized Neumann kernel for computing conformal mapping of unbounded multiply connected regions and its inverse onto several classes of canonical regions. For each canonical region, two integral equations are solved before one can approximate the boundary values of the mapping function. Cauchy’s-type integrals are used for computing the mapping function and its inverse for interior points. This method also works for regions with piecewise smooth boundaries. Three examples are given to illustrate the effectiveness of the proposed method.

1. Introduction

Conformal mappings have been a popular technique to solve several problems in the fields of science
and engineering. However, exact conformal maps are known for some regions only. One way to overcome this limitation is by means of numerical methods.

The Szegö kernel and the Bergman kernel of a simply connected region are well-known reproducing kernels and are connected to the Riemann mapping function which maps a simply connected region onto a unit disc. These kernels can be computed via Fredholm integral equations as shown in [1–3]. Boundary integral equations related to boundary relationships satisfied by functions which are analytic in simply connected regions (bounded and unbounded) or bounded doubly connected regions with smooth Jordan boundaries have been given by [4–6]. Special realizations of these integral equations are the integral equations related to the Szegö kernel, Bergmann kernel, Riemann map, exterior mapping and Ahlfors map. The kernels of these integral equations are the Neumann kernel and the Kerzman–Stein kernel. Extensions of this approach to conformal mappings of bounded and unbounded multiply connected regions onto some canonical regions are given in [7–12]. For some other approaches of conformal mappings of multiply connected regions, e.g. [13–31]. Various applications of conformal mappings in science and engineering are considered in, e.g., [22,32–35].

There exist many canonical regions with regards to conformal mapping of multiply connected regions, e.g. [18,22,36–39]. Koebe [36] gives an example of 39 types of canonical regions. However, most of the works mentioned above are for the first type of Koebe’s canonical slit region as illustrated in [36, figs 1–5]. In this paper, we present a new unified method for univalent conformal mapping of unbounded finitely connected regions and its inverse onto several classes of canonical regions via a boundary integral equation method with the adjoint generalized Neumann kernel. We consider the first 13 canonical regions shown in [36, figs 1–13]. Nasser [37] has applied the boundary integral equation method for numerical conformal mapping onto these canonical regions by reformulating the conformal mapping problem as a Riemann–Hilbert (RH) problem. Then, integral equations with the generalized Neumann kernel are constructed to solve the RH problem. Nasser [37] is an extension of the author’s two previous papers [23,24]. This method only works for solving the direct mapping problem. Recently, a fast boundary integral equation for numerical conformal mapping onto a strip with rectilinear slit has been given in [40]. The method is based on a combination of a uniquely solvable boundary integral equation with generalized Neumann kernel and the Fast Multipole Method. In this paper, by computing the mapping function via the adjoint generalized Neumann kernel, we are able to compute its inverse map and the time taken for Matlab to run the program also reduces. Amano & Okano [15] and DeLillo et al. [41] have also developed their own techniques for numerical conformal mapping onto circular and radial slit regions where Amano & Okano [15] use charge simulation methods while DeLillo et al. [41] use an explicit formula from an unbounded circular region and later solve it using the least-squares method.

In this paper, we construct two integral equations which can be used to compute the mapping function and its inverse from any unbounded multiply connected regions onto Koebe’s first 13 canonical regions. One integral equation is used to find the parameters of the canonical regions. The other integral equation is used to compute the derivative of the boundary correspondence function.

The plan of this paper is as follows: §2 presents some auxiliary materials. Section 3 presents a boundary integral equation with the adjoint generalized Neumann kernel. In §§4–7, we present the derivation for numerical conformal mapping of unbounded multiply connected regions onto canonical regions and its inverse. In §8, we give some examples to illustrate the effectiveness of our method. Finally, §9 presents a short conclusion.

2. Notations and auxiliary material

Let \( \Omega^- \) be an unbounded multiply connected region of connectivity \( m \). The boundary \( \Gamma \) consists of \( m \) Jordan curves \( \Gamma_j, j = 1, 2, \ldots, m \), i.e. \( \Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_m \). The boundaries \( \Gamma_j \) are assumed in clockwise orientation (figure 1).
Figure 1. An unbounded multiply connected region $\Omega^-$ with connectivity $m$.

Figure 2. The classes of the canonical region. (Online version in colour.)

This paper has introduced slightly different notations from our previous works [12,23,24,26,31,37,42–44] so as to give a detailed explanation of the presented method.

The curves $\Gamma_j$ are parametrized by $2\pi$-periodic twice continuously differentiable complex-valued function $\eta_j(t), t \in [0, 2\pi], j = 1, 2, \ldots, m$, $\Gamma_1 : \eta_1(t), t \in [0, 2\pi]$ 
\[ \vdots \]
$\Gamma_m : \eta_m(t), t \in [0, 2\pi] \]  
with non-vanishing first derivatives, i.e.
$$\eta_j'(t) = \frac{d\eta_j(t)}{dt} \neq 0, \quad t \in [0, 2\pi], \quad j = 1, \ldots, m.$$

In this paper, we shall consider four canonical regions which are annulus with spiral slits ($U^1$), unit disc with spiral slits ($U^2$), spiral slits region ($U^3$) and rectilinear slits region ($U^4$) (figure 2). These canonical regions are the same as the first 13 canonical regions shown in [36, figs 1–13].

Let $\Phi(z)$ be the conformal mapping function that maps $\Omega^-$ onto any of the canonical regions mentioned above, $z_1$ is a prescribed point inside $\Gamma_1$, $z_2$ is a prescribed point inside $\Gamma_2$ and $\alpha$ is a prescribed point located in $\Omega^-$. In this paper, we determine the mapping function $\Phi(z)$ by computing two real functions which are an unknown function $S_j(t)$ and a piecewise constant real function $R_j$.  

3. Adjoint generalized Neumann kernel

Let $A_{1,j}(t)$ and $A_{2,j}(t)$ be complex-valued continuously differentiable $2\pi$-periodic functions for all $t \in [0, 2\pi]$ and $j = 1, \ldots, m$. The adjoint function $\tilde{A}_{p,j}(t)$ of $A_{p,j}(t)$ is defined by
$$\tilde{A}_{p,j}(t) = \frac{\eta_j'(t)}{A_{p,j}(t)}.$$
Throughout this paper, we shall assume the functions \( A_{1,j}(t) \) and \( A_{2,j}(t) \) are constant functions, so we shall eliminate the variable \( t \), i.e. we shall define
\[
A_{1,j} = e^{i(\pi/2 - \theta_j)} \quad \text{and} \quad A_{2,j} = e^{-i(\pi/2 - \theta_j)},
\]
where \( \theta_j \) are given real constants for \( j = 1, \ldots, m \). Hence, the adjoint functions \( \tilde{A}_{1,j}(t) \) and \( \tilde{A}_{2,j}(t) \) are given by
\[
\tilde{A}_{1,j}(t) = A_{2,j} \eta_j'(t) \quad \text{and} \quad \tilde{A}_{2,j}(t) = A_{1,j} \eta_j'(t).
\]

For \( p = 1, 2 \), we consider two real kernels formed with \( A_{p,j} \) as [44]
\[
N_{p,j}(s, t) = \frac{1}{\pi} \text{Im} \left( \frac{A_{p,l}}{A_{p,j}} \frac{\eta_j'(t)}{\eta_j(t) - \eta_l(s)} \right)
\]
and
\[
M_{p,j}(s, t) = \frac{1}{\pi} \text{Re} \left( \frac{A_{p,l}}{A_{p,j}} \frac{\eta_j'(t)}{\eta_j(t) - \eta_l(s)} \right),
\]
where \( s, t \in [0, 2\pi] \) and \( j, l = 1, 2, \ldots, m \). The kernel \( N_{p,j}(s, t) \) is known as the generalized Neumann kernel formed with \( A_{p,j} \) and \( \eta_j(t) \). For \( j \neq l \), all kernels \( N_{p,j}(s, t) \) and \( M_{p,j}(s, t) \) are continuous on \([0, 2\pi] \times [0, 2\pi]\) because \( \eta_j(t) \neq \eta_l(s) \) for all \( s, t \in [0, 2\pi] \). When \( j = l \), the kernel \( N_{p,j}(s, t) \) is continuous and takes on the diagonal the values
\[
N_{p,j}(t, t) = \frac{1}{2\pi} \text{Im} \left( \frac{\eta_j''(t)}{\eta_j'(t)} \right).
\]
The kernel \( M_{p,j}(s, t) \) has a cotangent singularity where it can be written as
\[
M_{p,j}(s, t) = -\frac{1}{2\pi} \cot \frac{s - t}{2} + M_{1,p,j}(s, t),
\]
with a continuous kernel \( M_{1,p,j}(s, t) \) which takes on the diagonal the values
\[
M_{1,p,j}(t, t) = \frac{1}{2\pi} \text{Re} \left( \frac{\eta_j''(t)}{\eta_j'(t)} \right).
\]
The generalized Neumann kernel \( \tilde{N}_{p,j}(s, t) \) and the real kernel \( \tilde{M}_{p,j}(s, t) \) formed with the adjoint function \( \tilde{A}_{p,j}(t) \) and \( \tilde{\eta}_j(t) \) are defined by
\[
\tilde{N}_{p,j}(s, t) = \frac{1}{\pi} \text{Im} \left( \frac{-\tilde{A}_{p,l}(s)}{\tilde{A}_{p,j}(t)} \frac{\tilde{\eta}_j'(t)}{\tilde{\eta}_j(t) - \eta_l(s)} \right) \quad \text{and} \quad \tilde{M}_{p,j}(s, t) = \frac{1}{\pi} \text{Re} \left( \frac{-\tilde{A}_{p,l}(s)}{\tilde{A}_{p,j}(t)} \frac{\tilde{\eta}_j'(t)}{\tilde{\eta}_j(t) - \eta_l(s)} \right).
\]
Then,
\[
\tilde{N}_{p,j}(s, t) = -N_{p,j}^*(s, t) \quad \text{and} \quad \tilde{M}_{p,j}(s, t) = -M_{p,j}^*(s, t),
\]
where \( N_{p,j}^*(s, t) = N_{p,j}(t, s) \) is the adjoint kernel of the generalized Neumann kernel \( N_{p,j}(s, t) \) and \( M_{p,j}^*(s, t) = M_{p,j}(t, s) \) is the adjoint kernel of the kernel \( M_{p,j}(s, t) \).

It is known that \( \lambda = 1 \) is not an eigenvalue of the kernel \( N_{p,j} \) and \( \lambda = -1 \) is an eigenvalue of the kernel \( N_{p,j} \) with multiplicity \( m \). The eigenfunctions of \( N_{p,j} \) corresponding to the eigenvalue \( \lambda = -1 \) are \( \{\chi_1^{[1]}, \chi_2^{[1]}, \ldots, \chi_m^{[1]}\} \), where
\[
\chi^{[1]}(\xi) = \begin{cases} 1, & \xi \in \Gamma_j, \\ 0, & \text{elsewhere}, \end{cases}
\]
\( j = 1, 2, \ldots, m \) (see [44] for details).
For \( l, j = 1, 2, \ldots, m \), we define a real kernel \( J_{l,j}(s, t) \), where \( s, t \in [0, 2\pi] \) by

\[
J_{l,j}(s, t) = \begin{cases} 
\frac{1}{2\pi}, & \text{if } l = j, \\
0, & \text{if } l \neq j,
\end{cases}
\]

(3.5)

Let \( H \) be the space of all real Hölder continuous \( 2\pi \)-periodic function and \( L \) is the subspace of \( H \) which contains the piecewise real constant functions. Hence, we have the following theorems from [31,43].

**Theorem 3.1.** For any given function \( \gamma \in H \), the \( m \times m \) system of integral equations

\[
\mu_j(t) + \sum_{l=1}^{m} \int_{0}^{2\pi} [N_{p,j,l}(t, s) + J_{l,j}(t, s)] \mu_l(s) \, ds = \gamma_j(t), \quad j = 1, 2, \ldots, m,
\]

(3.6)

is uniquely solvable.

**Theorem 3.2.** For fixed integers \( p = 1, 2 \) and \( k = 1, 2, \ldots, m \), let \( \{\varphi_{1}^{[k]}, \varphi_{2}^{[k]}, \ldots, \varphi_{m}^{[k]}\} \) be the unique solution of the \( m \times m \) system of integral equations

\[
\varphi_{j}^{[k]}(t) + \sum_{l=1}^{m} \int_{0}^{2\pi} [N_{p,j,l}(t, s) + J_{l,j}(t, s)] \varphi_{l}^{[k]}(s) \, ds = - \chi_{j}(t), \quad j = 1, 2, \ldots, m.
\]

(3.7)

Let \( \gamma, \mu \in H \) and \( h, v \in L \) such that

\[
A_{p,j} F(\eta_j(t)) = \gamma_j(t) + h_j + i[\mu_j(t) + v_j], \quad j = 1, 2, \ldots, m
\]

(3.8)

are boundary values of a function \( F(z) \) analytic in \( \Omega^- \) with \( F(\infty) = 0 \). Then the elements of the piecewise constant functions \( h = (h_1, h_2, \ldots, h_m) \) and \( v = (v_1, v_2, \ldots, v_m) \) are given by

\[
h_k = \frac{1}{2\pi} \sum_{j=1}^{m} \int_{0}^{2\pi} \gamma_j(t) \varphi_j^{[k]}(t) \, dt
\]

(3.9)

and

\[
v_k = \frac{1}{2\pi} \sum_{j=1}^{m} \int_{0}^{2\pi} \mu_j(t) \varphi_j^{[k]}(t) \, dt.
\]

(3.10)

4. **An annulus with spiral slits region**

Consider the class of canonical region \( U^1 \), that is an annulus centred at the origin with \( m - 2 \) logarithmic spiral slits. We assume that \( \Phi(z) \) maps the curve \( \Gamma_1 \) onto a unit circle \( |\Phi| = 1 \), the curve \( \Gamma_2 \) onto the circle \( |\Phi(z)| = R_2 \) and the curves \( \Gamma_j, j = 3, 4, \ldots, m \) onto slits on the logarithmic spiral [39]

\[
\text{Im}(e^{-i\theta} \log \Phi(\eta_j(t))) = R_j, \quad t \in [0, 2\pi],
\]

(4.1)

where \( R_2, \ldots, R_m \) are undetermined real constants. The parameters \( \theta_j, j = 3, 4, \ldots, m \) are not predetermined by \( \Omega^- \), hence they can be given any real constants such that each \( \theta_j \) represents the angle of intersection between the logarithmic spiral and any ray emanating from the origin. Note that the slits are always traversed twice. For \( \theta = \pi/2 \), the logarithmic spiral slit is a circular
In view of (4.7) as where \( F = 0 \) and \( \theta = 0 \), the spiral is a radial slit pointing at the origin. We choose \( \theta_1 = \theta_2 = \pi / 2 \). Then, the boundary values of the mapping function \( \Phi(z) \) satisfy

\[
A_{1,j} \log \Phi(\eta_j(t)) = \hat{R}_j + iS_j(t), \quad j = 1, 2, \ldots, m, \tag{4.2}
\]

where \( S_j(t) \) is unknown real-valued function and

\[
\hat{R}_j = \begin{cases} 
0, & j = 1, \\
\ln R_2, & j = 2, \\
-R_j, & j = 3, \ldots, m.
\end{cases} \tag{4.3}
\]

The mapping function \( \Phi(z) \) can be uniquely determined by assuming that \( \Phi(\infty) > 0 \). Thus, the mapping function can be expressed as

\[
\Phi(z) = c \left( \frac{z - z_2}{z - z_1} \right)^{F(z)}, \tag{4.4}
\]

where \( F(z) \) is an analytic function with \( F(\infty) = 0 \) and \( c = \Phi(\infty) \) is an undetermined real constant.

By taking the logarithm of both sides of (4.4), we obtain

\[
\log \Phi(\eta_j(t)) = \ln c + \log \left( \frac{\eta_j(t) - z_2}{\eta_j(t) - z_1} \right) + F(\eta_j(t)), \quad j = 1, 2, \ldots, m. \tag{4.5}
\]

Then by multiplying (4.5) with \( A_{1,j} \) and applying (4.2), we get

\[
A_{1,j} F(\eta_j(t)) = \hat{R}_j + iS_j(t) - A_{1,j} \ln c - A_{1,j} \log \left( \frac{\eta_j(t) - z_2}{\eta_j(t) - z_1} \right),
\]

\[
= \hat{R}_j + iS_j(t) - \ln c (\sin \theta_j + i \cos \theta_j) + \gamma_j(t) + i \hat{\mu}_j(t), \quad j = 1, 2, \ldots, m, \tag{4.6}
\]

where

\[
\gamma_j(t) + i \hat{\mu}_j(t) = -A_{1,j} \log \left( \frac{\eta_j(t) - z_2}{\eta_j(t) - z_1} \right).
\]

By differentiating (4.6) with respect to \( t \), we get

\[
A_{1,j} F'(\eta_j(t)) \eta_j'(t) = \gamma_j'(t) + i(S_j'(t) + \hat{\mu}_j'(t)), \quad j = 1, 2, \ldots, m, \tag{4.7}
\]

where

\[
\gamma_j'(t) + i \hat{\mu}_j'(t) = -A_{1,j} \frac{z_2 - z_1}{(\eta_j(t) - z_2)(\eta_j(t) - z_1)}. \]

In view of \( \tilde{A}_{2,j} = A_{1,j} \eta_j'(t) \) and \( f(z) = F(z) + (z_2 - z_1)/(z - z_2)(z - z_1) \) is analytic in \( \Omega^- \), we rewrite (4.7) as

\[
\tilde{A}_{2,j} f(\eta_j(t)) = iS_j'(t), \quad j = 1, 2, \ldots, m.
\]

Then by [45, Theorem 1], we obtain

\[
S_j'(t) + \sum_{l=1}^{m} \int_{0}^{2\pi} N_{2,j}^s(t, s) S_l'(s) \, ds = 0, \quad j = 1, 2, \ldots, m \tag{4.8}
\]

for \( t \in [0, 2\pi] \). However, this integral equation is not uniquely solvable [44, Theorem 12]. Note that, the image of the curve \( I_1 \) is anticlockwise oriented, the image of the curve \( I_2 \) is clockwise.
oriented, and the images of the curves $\Gamma_j, j = 3, 4, \ldots, m$ are slits that are traversed twice. So, we have $S_1(2\pi) - S_1(0) = 2\pi, S_2(2\pi) - S_2(0) = -2\pi, S_j(2\pi) - S_j(0) = 0$ for $j = 3, \ldots, m$. Hence,

$$\sum_{l=1}^{m} \int_{0}^{2\pi} j_{ij}(t, s)S_j'(s) \, ds = \tilde{h}_j, \quad j = 1, 2, \ldots, m,$$

(4.9)

where

$$\tilde{h}_j = \begin{cases} 1, & j = 1, \\ -1, & j = 2, \\ 0, & j = 3, 4, \ldots, m. \end{cases} \quad (4.10)$$

By adding (4.9) to (4.8), we conclude that the unknown functions $S'_j(t), \ldots, S'_m(t)$ satisfy the following system integral equations:

$$S'_j(t) + \sum_{l=1}^{m} \int_{0}^{2\pi} [N^*_{2j,l}(t, s) + j_{ij}(t, s)]S'_j(s) \, ds = \tilde{h}_j, \quad t \in [0, 2\pi], \quad j = 1, \ldots, m,$$

(4.11)

which in view of theorem 3.1 is uniquely solvable.

The function $S_j(t)$ can be computed as an antiderivative of its derivative $S'_j(t)$. The function $S'_j(t)$ is $2\pi$-periodic. Thus, it can be represented by a Fourier series

$$S'_j(t) = a^{[j]}_0 + \sum_{k=1}^{\infty} a^{[j]}_k \cos kt + \sum_{k=1}^{\infty} b^{[j]}_k \sin kt.$$  

(4.12)

The values of $a^{[j]}_0, a^{[j]}_k$ and $b^{[j]}_k$ are computed by using Matlab’s function ‘$\texttt{fft}$’. Hence, the function $S_j(t)$ can be written as

$$S_j(t) = \int S'_j(t) \, dt + \hat{v}_j = \rho_j(t) + \hat{v}_j, \quad j = 1, 2, \ldots, m,$$

(4.13)

where the function $\rho_j(t)$ can be calculated by Fourier series representation as

$$\rho_j(t) = \int S'_j(t) \, dt = a^{[j]}_0 t + \sum_{k=1}^{\infty} a^{[j]}_k \sin kt - \sum_{k=1}^{\infty} \frac{b^{[j]}_k}{k} \cos kt$$  

(4.14)

and $\hat{v}_j$ is undetermined real integration constant and should be calculated.

The boundary values of the function $F$ in (4.6) can be written as

$$A_{1j}F(\eta_j(t)) = h_j + \gamma_j(t) + i(\mu_j(t) + v_j), \quad j = 1, 2, \ldots, m,$$

(4.15)

where

$$\mu_j(t) = \rho_j(t) + \hat{\mu}_j(t), \quad h_j = \hat{R}_j - \sin \theta_j \ln c, \quad v_j = \hat{v}_j - \cos \theta_j \ln c.$$  

(4.16)

Note that the functions $\gamma_j(t)$ and $\mu_j(t)$ in (4.15) are known, and the function $F$ satisfies the assumptions of theorem 3.2. Hence, the constants $h_j$ and $v_j$ can be computed from theorem 3.2.

Hence, the constants $c, \hat{R}_2, \hat{R}_3, \ldots, \hat{R}_m$ and $\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_m$ can be computed from (4.16). Then the parameters $R_2, R_3, \ldots, R_m$ can be computed from (4.3) and the values of the function $S_j(t)$ can be computed from (4.13).

By obtaining all the information above, the mapping function at the boundary points can be calculated from (4.2) as

$$\Phi(\eta_j(t)) = e^{A_{1j}R_j(t) + iS_j(t))}, \quad j = 1, 2, \ldots, m.$$
Then for all \( z \in \Omega^− \), by Cauchy’s integral formula [46] we have

\[
w = \Phi(z) = c + \frac{1}{2\pi i} \sum_{j=1}^{m} \int_{0}^{2\pi} \frac{\Phi(\eta_j(t))}{\eta_j(t) - z} \eta_j'(t) \, dt.
\] (4.17)

For computing the inverse map, note that the mapping function \( \Phi^{-1}(w) = z \) is analytic in the region \( U^1 \) with a simple pole at \( w = c \). Let \( G(w) \) be an analytic function in \( U^1 \) defined as

\[G(w) = (w - c)\Phi^{-1}(w).\]

Hence,

\[G(w) = \frac{1}{2\pi i} \int_{\partial U^1} \frac{(\zeta - c)\Phi^{-1}(\zeta)}{\zeta - w} \, d\zeta.
\]

By introducing \( \zeta_j(t) = \Phi(\eta_j(t)) \), then by Cauchy’s integral formula, we have

\[(w - c)\Phi^{-1}(w) = \frac{1}{2\pi i} \sum_{j=1}^{m} \int_{0}^{2\pi} \frac{\Phi(\eta_j(t)) - c}{\Phi(\eta_j(t)) - w} \frac{\Phi^{-1}(\Phi(\eta_j(t)))}{\Phi(\eta_j(t)) - w} A_{2j} iS_j'(t) \Phi(\eta_j(t)) \, dt,
\]

which implies

\[\Phi^{-1}(w) = \frac{1}{(w - c)2\pi} \sum_{j=1}^{m} \int_{0}^{2\pi} \frac{\Phi(\eta_j(t)) - c}{\Phi(\eta_j(t)) - w} \frac{\Phi^{-1}(\Phi(\eta_j(t)))}{\Phi(\eta_j(t)) - w} A_{2j} S_j'(t) \Phi(\eta_j(t)) \, dt.
\]

5. The unit disc with spiral slits region

Consider the class of canonical region \( U^2 \), i.e. the unit disc with \( m - 1 \) logarithmic spiral slits. We assume that \( \Phi(z) \) maps the curve \( \Gamma^1 \) onto the unit circle \( |\Phi| = 1 \) and the curves \( \Gamma^j, j = 2, 3, \ldots, m \) onto slits on the logarithmic spirals [39]

\[\text{Im}(e^{-i\theta_j} \log \Phi(\eta_j(t))) = R_j, \]

(5.1)

where \( R_2, \ldots, R_m \) are undetermined real constants. The real constants \( \theta_j, j = 2, 3, \ldots, m \), have the same geometrical meaning as in §4. We choose \( \theta_1 = \pi/2 \). Then, as in §4, the boundary values of the mapping function \( \Phi(z) \) satisfy

\[A_{1j} \log \Phi(\eta_j(t)) = \hat{R}_j + iS_j(t), \quad j = 1, 2, \ldots, m,\]

(5.2)

where \( S_j(t) \) is unknown real-valued function and

\[\hat{R}_j = \begin{cases} 0, & j = 1, \\ -R_j, & j = 2, 3, \ldots, m. \end{cases}\]

(5.3)

The mapping function \( \Phi(z) \) can be uniquely determined by assuming \( \Phi(\infty) = 0 \) and \( \lim_{z \to \infty} z \Phi(z) > 0 \). Thus, the mapping function can be expressed as [37]

\[\Phi(z) = \frac{c}{z - z_1} e^{F(z)},\]

(5.4)

where \( F(z) \) is an analytic function in \( \Omega^− \) with \( F(\infty) = 0 \) and \( c = \lim_{z \to \infty} z \Phi(z) \) is an undetermined positive real constant.
By taking the logarithm of both sides of (5.4), we obtain
\[
\log \Phi(\eta_j(t)) = \ln c - \log(\eta_j(t) - z_1) + F(\eta_j(t)), \quad j = 1, 2, \ldots, m.
\] (5.5)

Multiplying (5.5) with $A_{1,j}$ and applying (5.2), we get
\[
A_{1,j}F(\eta_j(t)) = \tilde{R}_j + iS_j(t) - A_{1,j} \ln c + A_{1,j} \log(\eta_j(t) - z_1),
\]
\[
= \tilde{R}_j + iS_j(t) - \ln c(\sin \theta_j + i \cos \theta_j) + \gamma_j(t) + i \hat{\mu}_j(t), \quad j = 1, 2, \ldots, m,
\] (5.6)

where
\[
\gamma_j(t) + i \hat{\mu}_j(t) = A_{1,j} \log(\eta_j(t) - z_1).
\]

By differentiating (5.6) with respect to $t$, we obtain
\[
A_{1,j}F'(\eta_j(t))\eta_j'(t) = \gamma_j'(t) + i(S_j'(t) + \hat{\mu}_j'(t)), \quad j = 1, 2, \ldots, m,
\] (5.7)

where
\[
\gamma_j'(t) + i \hat{\mu}_j'(t) = A_{1,j} \frac{\eta_j'(t)}{\eta_j(t) - z_1}.
\]

As $\tilde{A}_{2,j}(t) = A_{1,j} \eta_j'(t)$ and $f(z) = F'(z) - 1/(z - z_1)$ is analytic in $\Omega^-$, we rewrite (5.7) as
\[
\tilde{A}_{2,j}(t)f(\eta_j(t)) = iS_j'(t), \quad j = 1, 2, \ldots, m.
\]

Then by [45, Theorem 1], we get
\[
S_j'(t) + \sum_{l=1}^{m} \int_{0}^{2\pi} N_{2,l,j}(t,s)S_l'(s) \, ds = 0, \quad t \in [0, 2\pi], \quad j = 1, 2, \ldots, m.
\] (5.8)

This integral equation is not uniquely solvable [44]. Note that, the image of the curve $I_1$ is anticlockwise oriented and the images of the curves $I_j, j = 2, 3, \ldots, m$ are slits that traversed twice. So we have $S_1(2\pi) - S_1(0) = 2\pi$ and $S_j(2\pi) - S_j(0) = 0$ for $j = 2, 3, \ldots, m$, which implies that
\[
\sum_{l=1}^{m} \int_{0}^{2\pi} J_{j,l}(t,s)S_l'(s) \, ds = \tilde{h}_j, \quad j = 1, 2, \ldots, m,
\] (5.9)

where
\[
\tilde{h}_j = \begin{cases} 1, & j = 1, \\ 0, & j = 2, 3, \ldots, m. \end{cases}
\] (5.10)

Hence, the unknown function $S_j'(t)$ is the unique solution of the system of following integral equations:
\[
S_j'(t) + \sum_{l=1}^{m} \int_{0}^{2\pi} [N_{2,j,l}(t,s) + J_{j,l}(t,s)]S_l'(s) \, ds = \tilde{h}_j, \quad t \in [0, 2\pi], \quad j = 1, 2, \ldots, m.
\] (5.11)

For $j = 1, 2, \ldots, m$, the function $S_j(t)$ can be written as
\[
S_j(t) = \int S_j'(t) \, dt + \hat{\nu}_j = \rho_j(t) + \gamma_j(t).
\]

Hence, the boundary values of the function $F$ in (5.6) can be written as
\[
A_{1,j}F(\eta_j(t)) = h_j + \gamma_j(t) + i(\mu_j(t) + v_j),
\] (5.12)

where
\[
\mu_j(t) = \rho_j(t) + \hat{\mu}_j(t), \quad h_j = \tilde{R}_j - \sin \theta_j \ln c \quad \text{and} \quad v_j = \hat{\nu}_j - \cos \theta_j \ln c.
\]
Then, the values of \( S_j(t) \) and \( \hat{R}_j \) in (5.2) can be obtained by using the same procedure as in the previous section. By obtaining all the information above, the mapping function at the boundary points can be calculated by

\[
\Phi(\eta_j(t)) = e^{A_2(\hat{R}_j + iS_j(t))}, \quad j = 1, 2, \ldots, m.
\]

Then for all \( z \in \Omega^− \), by Cauchy’s integral formula [46], we have

\[
w = \Phi(z) = \frac{1}{2\pi i} \sum_{j=1}^{m} \int_{0}^{2\pi} \frac{\Phi(\eta_j(t))}{\eta_j(t) - z} \eta_j'(t) \, dt. \tag{5.13}
\]

For computing the inverse map, note that the mapping function \( \Phi^{-1}(w) = z \) is analytic in the region \( U^2 \) with a simple pole at \( w = 0 \). Let \( G(w) \) be an analytic function in \( U^2 \) defined as \( G(w) = w\Phi^{-1}(w) \). For computing the inverse map, note that the mapping function \( \Phi^{-1}(w) = z \) is analytic in the region \( U^2 \) with a simple pole at \( w = 0 \). Let \( G(w) \) be an analytic function in \( U^2 \) defined as \( G(w) = w\Phi^{-1}(w) \). Then, by using the same reasoning as in §4, we get

\[
\Phi^{-1}(w) = \frac{1}{2\pi i} \sum_{j=1}^{m} \int_{0}^{2\pi} \frac{\Phi(\eta_j(t))\eta_j(t)}{\Phi(\eta_j(t)) - w} A_{2,j} S_j'(t) \Phi(\eta_j(t)) \, dt.
\]

6. Spiral slits region

Consider the canonical region \( U^3 \), i.e. the \( m \) logarithmic spirals on the entire \( w \)-plane, where the logarithmic spiral have the following representation [39]

\[
\text{Im}(e^{-i\theta} \log \Phi(\eta_j(t))) = R_j, \quad j = 1, 2, \ldots, m, \tag{6.1}
\]

where \( R_1, \ldots, R_m \) are real constants. The parameters \( \theta_j, j = 1, 2, \ldots, m \) are given real constants and have the same geometrical meaning as in §4. The boundary values of the mapping function \( \Phi(z) \) can be written as

\[
A_{1,j} \log \Phi(\eta_j(t)) = \hat{R}_j + iS_j(t), \tag{6.2}
\]

where \( S_j(t) \) is unknown real-valued function and

\[
\hat{R}_j = -R_j, \quad j = 1, 2, \ldots, m. \tag{6.3}
\]

The mapping function \( \Phi(z) \) can be uniquely determined by assuming \( \Phi(\alpha) = 0, \Phi(\infty) = \infty \) and \( \Phi(z) \) has the Laurent series expansion in a neighbourhood of \( z = \infty \) as [39, p. 112]

\[
\Phi(z) = z + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots.
\]

The mapping function can be expressed as [37]

\[
\Phi(z) = (z - \alpha) e^{F(z)}, \tag{6.4}
\]

where \( F(z) \) is an analytic in \( \Omega^- \) with \( F(\infty) = 0 \). Using the same procedure as in §4, we obtain

\[
A_{1,j} F(\eta_j(t)) = \hat{R}_j + iS_j(t) - A_{1,j} \log(\eta_j(t) - \alpha)
\]

\[
= \hat{R}_j(t) + iS_j(t) + \gamma_j(t) + i\hat{\mu}_j(t), \quad j = 1, 2, \ldots, m, \tag{6.5}
\]

where

\[
\gamma_j(t) + i\hat{\mu}_j(t) = -A_{1,j} \log(\eta_j(t) - \alpha).
\]

Differentiating (6.5) with respect to \( t \), we obtain

\[
A_{1,j} F'(\eta_j(t)) \eta_j'(t) = \gamma_j'(t) + i(S_j'(t) + \hat{\mu}_j'(t)), \quad j = 1, 2, \ldots, m, \tag{6.6}
\]

where

\[
\gamma_j'(t) + i\hat{\mu}_j'(t) = A_{1,j} \frac{\eta_j'(t)}{\alpha - \eta_j(t)}.
\]
As \( \tilde{A}_{2,j}(t) = A_{1,j} \eta_j(t) \), \( f(z) = F'(z) \) is analytic in \( \Omega^- \) with \( f(\infty) = 0 \) and \( g(z) = 1/(\alpha - z) \) is analytic in \( \Omega^+ \) (the complement of \( \Omega^- \) with respect to the extended complex plane \( \mathbb{C} \cup \{ \infty \} \)), we rewrite (6.6) as

\[
\tilde{A}_{2,j}(t)f(\eta_j(t)) = \gamma'_j(t) + i(S'_j(t) + \tilde{\mu}'_j(t)), \quad j = 1, 2, \ldots, m.
\]

Then by [44, Theorem 2(c)] and [45], we get

\[
S'_j(t) + \tilde{\mu}'_j(t) + \sum_{l=1}^{m} \int_{0}^{2\pi} N^s_{2,j,l}(t,s)(S'_l(s) + \tilde{\mu}'_l(s)) \, ds = \sum_{l=1}^{m} \int_{0}^{2\pi} M^s_{2,j,l}(t,s)\gamma'_l(s) \, ds. \tag{6.7}
\]

We have also

\[
\tilde{A}_{2,j}(t)g(t) = \gamma'_j(t) + i\tilde{\mu}'_j(t), \quad j = 1, 2, \ldots, m.
\]

Then by [44, Theorem 2(d)] and [45], we get

\[
\tilde{\mu}'_j(t) - \sum_{l=1}^{m} \int_{0}^{2\pi} N^s_{2,j,l}(t,s)\tilde{\mu}'_l(s) \, ds = -\sum_{l=1}^{m} \int_{0}^{2\pi} M^s_{2,j,l}(t,s)\gamma'_l(s) \, ds. \tag{6.8}
\]

Then, by adding (6.7) and (6.8), we have

\[
S'_j(t) + \sum_{l=1}^{m} \int_{0}^{2\pi} N^s_{2,j,l}(t,s)S'_l(s) \, ds = -2\text{Im} \left[ \frac{\tilde{A}_{2,j}(t)}{\alpha - \eta_j(t)} \right], \quad t \in [0, 2\pi], \quad j = 1, 2, \ldots, m. \tag{6.9}
\]

Integral equation (6.9) is not uniquely solvable [44, Theorem 12]. Note that the images of the curves \( \Gamma_1, \Gamma_2, \ldots, \Gamma_m \) are slits that are traversed twice. So we have \( S_j(2\pi) - S_j(0) = 0 \), which implies

\[
\sum_{l=1}^{m} \int_{0}^{2\pi} I_{j,l}(t,s)S'_l(s) \, ds = 0, \quad t \in [0, 2\pi], \quad j = 1, 2, \ldots, m. \tag{6.10}
\]

Hence, the unknown function \( S'_j(t) \) is the unique solution of the following system of integral equations

\[
S'_j(t) + \sum_{l=1}^{m} \int_{0}^{2\pi} [N^s_{2,j,l}(t,s) + I_{j,l}(t,s)]S'_l(s) \, ds = -2\text{Im} \left[ \frac{\tilde{A}_{2,j}(t)}{\alpha - \eta_j(t)} \right] \tag{6.11}
\]

for \( t \in [0, 2\pi], \ j = 1, 2, \ldots, m. \)

For \( j = 1, 2, \ldots, m \), the function \( S_j(t) \) can be written as

\[
S_j(t) = \int S'_j(t) \, dt + \tilde{v}_j = \rho_j(t) + \tilde{v}_j(t).
\]

Hence, the boundary values of the function \( F \) in (6.5) can be written as

\[
A_{1,j}F(\eta_j(t)) = h_j + \gamma_j(t) + i(\mu_j(t) + v_j), \tag{6.12}
\]

where

\[
\mu_j(t) = \rho_j(t) + \tilde{\mu}_j(t), \quad h_j = \hat{R}_j \quad \text{and} \quad v_j = \tilde{v}_j.
\]

Then, the values of \( S_j(t) \) and \( \hat{R}_j \) in (6.2) can be obtained by using the same procedure as in §4.
The mapping function at the boundary points can be calculated by
\[ \Phi(\eta_j(t)) = e^{A_2(jR_j + iS_j(t))}, \quad j = 1, 2, \ldots, m. \]

For computing the mapping of exterior point, let \( K(z) \) be an analytic function for \( z \in \Omega^- \) defined as
\[ K(z) = \frac{\Phi(z)}{z - \alpha}, \quad \text{where} \quad \lim_{z \to \infty} K(z) = 1. \]

Then, by Cauchy’s integral formula [46], we have
\[ w = z - \alpha + \frac{z - \alpha}{2\pi i} \sum_{j=1}^{m} \oint_{C_j} \frac{\Phi(\eta_j(t))}{(\eta_j(t) - \alpha)(\eta_j(t) - z)} \eta_j(t) \, dt. \tag{6.13} \]

For computing the inverse map, the inverse of the Laurent series expansion for \( \Phi(z) \) near \( \infty \) has the following representation [39, p. 114]
\[ \Phi^{-1}(w) = w + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots. \]

Let \( G(w) \) be analytic in \( U^3 \) defined by
\[ G(w) = \frac{\Phi^{-1}(w) - \alpha}{w}, \quad \text{where} \quad \lim_{w \to \infty} G(w) = 1. \]

Then, by Cauchy’s integral formula, we have
\[ G(w) = G(\infty) + \frac{1}{2\pi i} \oint_{\partial U^3} \frac{G(\zeta)}{\zeta - w} \, d\zeta. \]

By introducing \( \zeta_j(t) = \Phi(\eta_j(t)) \), we have
\[ z = w + \alpha + \frac{w}{2\pi} \sum_{j=1}^{m} \oint_{C_j} \frac{\eta_j(t) - \alpha}{\Phi(\eta_j(t))(\Phi(\eta_j(t)) - w)} A_2(jS_j(t)\Phi(\eta_j(t))) \, dt. \]

### 7. Rectilinear slits region

Consider the canonical region \( U^4 \), that is the entire complex plane with \( m \) rectilinear slits on the straight lines [39]
\[ \text{Im}(e^{-i\theta_j} \Phi) = R_j, \quad j = 1, 2, \ldots, m, \tag{7.1} \]
where \( R_1, \ldots, R_m \) are undetermined real constants and \( \theta_j, j = 1, 2, \ldots, m \) are the given angles of intersections between the straight lines (7.1) and the real axis. The boundary values of the mapping function \( \Phi(z) \) can be written as
\[ A_{1,j} \Phi(\eta_j(t)) = \hat{R}_j + iS_j(t), \quad j = 1, 2, \ldots, m. \tag{7.2} \]

The mapping function \( \Phi(z) \) can be uniquely determined by assuming \( \Phi(\infty) = \infty \) and \( \lim_{z \to \infty} (z - \Phi(z)) = 0 \). Thus, the mapping function can be expressed as [37]
\[ \Phi(z) = z + F(z), \tag{7.3} \]
where \( F(z) \) is analytic in \( \Omega^- \) with \( F(\infty) = 0 \). By multiplying (7.3) with \( A_{1,j} \) and applying (7.2), we get
\[ A_{1,j} F(\eta_j(t)) = \hat{R}_j + iS_j(t) - A_{1,j} \eta_j(t) \]
\[ = \hat{R}_j + iS_j(t) + \gamma_j(t) + i\hat{\mu}_j(t), \quad j = 1, 2, \ldots, m, \tag{7.4} \]
where
\[ \gamma_j(t) + i\hat{\mu}_j(t) = -A_{1,j} \eta_j(t). \]
Then, by differentiating (7.4) with respect to \( t \), we have

\[
A_{1,j}F'(\eta(t))\eta_j'(t) = \gamma_j'(t) + i(S_j'(t) + \mu_j'(t)), \quad j = 1, 2, \ldots, m, \tag{7.5}
\]

where

\[
\gamma_j'(t) + i\mu_j'(t) = -A_{1,j}\eta_j'(t).
\]

As \( \tilde{A}_{2,j}(t) = A_{1,j}\eta_j'(t), \) \( f(z) = F(z) \) is analytic in \( \Omega^- \) with \( f(\infty) = 0 \) and \( g(z) = -1 \) is analytic in \( \Omega^+ \), we rewrite (7.5) as

\[
\tilde{A}_{2,j}(t)f(\eta_j(t)) = \gamma_j'(t) + i(S_j'(t) + \mu_j'(t)), \quad j = 1, 2, \ldots, m.
\]

Then by [44, Theorem 2(c)] and [45], for \( t \in [0, 2\pi], j = 1, 2, \ldots, m \) we get

\[
S_j'(t) + \mu_j'(t) + \sum_{l=1}^{m} \int_{0}^{2\pi} N_{2,j,l}(t,s)(S_j(s) + \mu_j(s)) ds = \sum_{l=1}^{m} \int_{0}^{2\pi} M_{2,j,l}(t,s)\gamma_j'(s) ds.
\tag{7.6}
\]

As

\[
\tilde{A}_{2,j}(t)g(\eta_j(t)) = \gamma_j'(t) + i\mu_j'(t), \quad j = 1, 2, \ldots, m,
\]

by [44, Theorem 2(d)] and [45], we have

\[
\mu_j'(t) - \sum_{l=1}^{m} \int_{0}^{2\pi} N_{2,j,l}(t,s)\mu_j(s) ds = -\sum_{l=1}^{m} \int_{0}^{2\pi} M_{2,j,l}(t,s)\gamma_j'(s) ds.
\tag{7.7}
\]

Adding (7.6) with (7.7), yields

\[
S_j'(t) + \sum_{l=1}^{m} \int_{0}^{2\pi} N_{2,j,l}(t,s)S_j(s) ds = 2\text{Im}[A_{1,j}\eta_j'(t)], \quad t \in [0, 2\pi], \quad j = 1, 2, \ldots, m.
\tag{7.8}
\]

This integral equation is not uniquely solvable. Note that, the images of the curve \( \Gamma_1, \Gamma_2, \ldots, \Gamma_m \) are slits that traversed twice. So we have \( S_j(2\pi) - S_j(0) = 0 \), which implies

\[
\int_{0}^{2\pi} f_{j,l}(t,s)S_j(s) ds = 0, \quad j = 1, 2, \ldots, m. \tag{7.9}
\]

Hence, the unknown function \( S_j'(t) \) is the unique solution of the following integral equation:

\[
S_j'(t) + \sum_{l=1}^{m} \int_{0}^{2\pi} [N_{2,j,l}(t,s) + f_{j,l}(t,s)]S_j(s) ds = 2\text{Im}[A_{1,j}\eta_j'(t)], \quad t \in [0, 2\pi],
\tag{7.10}
\]

\( j = 1, 2, \ldots, m. \)

For \( j = 1, 2, \ldots, m \), the function \( S_j(t) \) can be written as

\[
S_j(t) = \int S_j'(t) dt + \hat{v}_j = \rho_j(t) + \hat{v}_j(t).
\]

Hence, the boundary values of the function \( F \) in (7.4) can be written as

\[
A_{1,j}F(\eta_j(t)) = h_j + \gamma_j(t) + i(\mu_j(t) + \nu_j), \tag{7.11}
\]

where

\[
\mu_j(t) = \rho_j(t) + \hat{\mu_j}(t), \quad h_j = \hat{\rho_j}, \quad \nu_j = \hat{v}_j \quad \text{and} \quad j = 1, 2, \ldots, m.
\]
Then the values of $S_j(t)$ and $\hat{R}_j$ in (7.2) can be obtained by using the same procedure as in §4. By obtaining all the information above, the mapping function at the boundary points can be calculated by

$$
\Phi(\eta_j(t)) = A_{2,j}(\hat{R}_j + iS_j(t)), \quad j = 1, 2, \ldots, m.
$$

For calculating the mapping of exterior points, note that $\Phi(z)$ has the Laurent expansion series near $\infty$ as [39, p. 104]:

$$
\Phi(z) = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3} + \cdots.
$$

Let $K(z)$ be an analytic function for $z \in \Omega^-$ defined as

$$
K(z) = \Phi(z) - z, \quad \text{where} \quad \lim_{z \to \infty} K(z) = 0.
$$

Then by Cauchy’s integral formula [46], we have

$$
w = z + \frac{1}{2\pi i} \sum_{j=1}^{m} \int_{0}^{2\pi} \frac{\Phi(\eta_j(t)) - \eta_j(t)}{\eta_j(t) - z} \eta_j'(t) \, dt. \quad (7.12)
$$

For computing the inverse map, the inverse of the Laurent series expansion for $\Phi(z)$ near $\infty$ has the following representation [39, p. 114]

$$
\Phi^{-1}(w) = w + \frac{b_1}{w} + \frac{b_2}{w^2} + \frac{b_3}{w^3} + \cdots.
$$

Let $G(w)$ be an analytic function in $U^4$ defined as

$$
G(w) = \Phi^{-1}(w) - w, \quad \text{where} \quad \lim_{w \to \infty} G(w) = 0.
$$

Then,

$$
G(w) = \frac{1}{2\pi i} \int_{\partial U^4} \frac{\Phi^{-1}(\zeta) - \zeta}{\zeta - w} \, d\zeta.
$$

By introducing $\zeta_j(t) = \Phi(\eta_j(t))$, then by the Cauchy’s integral formula, we have

$$
z = w + \frac{1}{2\pi} \sum_{j=1}^{m} \int_{0}^{2\pi} \frac{\eta_j(t) - \Phi(\eta_j(t))}{\Phi(\eta_j(t)) - w} A_{2,j} S_j'(t) \, dt.
$$

### 8. Numerical examples

As the boundaries $I_j$ are parametrized by $\eta_j(t)$ which are $2\pi$-periodic function, the reliable method to solve the integral equations are by means of Nyström method with trapezoidal rule [47]. Each boundary will be discretized by $n$ number of equidistant points. The resulting linear system is then solved by using Gaussian elimination method. The integral equations presented in §§4–8 can be also used to compute the conformal mapping for the region that contains corner points. However, the integral equation needs to be modified slightly when $\eta_j(t)$ is a corner point (see [42] for more details). For this case, the integral equations will be solved using the method presented in [48] (see also [42]). Suppose that each boundary component $I_j$ contains $p_j \geq 1$ corner points located at $2k\pi/n$, $k = 0, 1, \ldots, p_j - 1$. Suppose $\sigma_j(t)$ be a function which is bijective, strictly monotonically increasing and infinitely differentiable and is defined by [48]

$$
\sigma(t) = 2\pi \frac{[v(t)]^3}{[v(t)]^3 + [v(2\pi - t)]^3} \quad (8.1)
$$

and

$$
v(t) = \left( \frac{1}{3} - \frac{1}{2} \right) \left( \frac{\pi - t}{\pi} \right)^3 + \frac{1}{3} \frac{\pi - t}{\pi} + \frac{1}{2}, \quad t \in [0, 2\pi], \quad (8.2)
$$

where the functions $\sigma_j(t)$ and $v_j(t)$ satisfy

$$
\sigma'(0) = \sigma'(2\pi) = 0, \quad v(0) = 0, \quad v(2\pi) = 1.
$$
Then a function \( \delta_j(t) \) which satisfies

\[
\delta_j(t) : [0, 2\pi] \to [0, 2\pi] \quad \text{and} \quad \delta_j' \left( \frac{2k\pi}{p} \right) = 0
\]

can be defined by

\[
\delta_j(t) = \begin{cases} 
\frac{1}{p_j} \sigma(p_j t), & t \in \left[ 0, \frac{2\pi}{p_j} \right), \\
\frac{1}{p_j} \sigma(p_j t - 2\pi) + \frac{2\pi}{p_j}, & t \in \left[ \frac{2\pi}{p_j}, \frac{4\pi}{p_j} \right), \\
\vdots \\
\frac{1}{p_j} \sigma(p_j t - 2(p_j - 2)\pi) + \frac{2(p_j - 2)p_j}{p_j}, & t \in \left[ \frac{2(p_j - 2)p_j}{p_j}, \frac{2(p_j - 1)p_j}{p_j} \right), \\
\frac{1}{p_j} \sigma(p_j t - 2(p_j - 1)\pi) + \frac{2(p_j - 1)p_j}{p_j}, & t \in \left[ \frac{2(p_j - 1)p_j}{p_j}, 2\pi \right]. 
\end{cases}
\]

For \( j = 1, 2, \ldots, m \), we define a new parametrization \( \zeta_j(t) \), \( t \in [0, 2\pi] \), of the boundary component \( \Gamma_j \) by

\[
\zeta_j(t) = \eta_j(\delta_j(t)), \quad (8.3)
\]

i.e. the boundary \( \Gamma \) will be re-parametrized by

\[
\Gamma_1 : \zeta_1(t), \quad t \in [0, 2\pi], \\
\vdots \\
\Gamma_m : \zeta_m(t), \quad t \in [0, 2\pi]. 
\]

(8.4)

Let \( f^*_j \) be the parametrization interval \([0, 2\pi]\) minus the points at which \( \delta_j'(t) = 0 \) for \( j = 1, \ldots, m \). Then

\[
\delta_j'(t) \neq 0 \quad \text{for all } t \in f^*_j. 
\]

We have

\[
\zeta_j'(t) = \eta_j'(\delta_j(t))\delta_j'(t), \quad t \in [0, 2\pi], \quad j = 1, 2, \ldots, m. 
\]

(8.5)

Thus, \( \zeta_j'(t) = 0 \) at each corner point and \( \zeta_j'(t) \neq 0 \) for all \( t \in f^*_j \). By introducing the new parametrization \( \zeta_j(t) \) of the boundary \( \Gamma \), a modification of the integral equations (4.11), (5.11), (6.11), (7.10) and (3.7) in theorem 3.2 are required. The integral equations (4.11), (5.11), (6.11) and (7.10) can be generally written as

\[
S_j'(t) + \sum_{l=1}^{m} \int_{f^*_j} \frac{1}{\pi} \text{Im} \left( \frac{A_{p,l}(s)}{A_{p,j}(t)} \eta_j'(t) \eta_l(s) - \eta_j(s) \right) S_l'(s) ds = 2\phi_j(t) 
\]

(8.6)

and

\[
\sum_{l=1}^{m} \int_{f^*_j} f_{j,l}(t,s) S_l'(s) ds = \tilde{h}_j, 
\]

(8.7)

for \( t \in f^*_j \), where only the functions \( \phi_j(t) \) and the constants \( \tilde{h}_j \) in the right-hand sides of (8.6) and (8.7) are different from one integral equation to another, \( j = 1, 2, \ldots, m \).
By introducing $t = \delta_j(t)$ and $s = \delta_l(\sigma)$, we have

$$S'_{j}(\delta_j(t)) + \sum_{l=1}^{m} \int_{I_j^l} \frac{1}{\pi} \text{Im} \left( \frac{A_{p,l}(\delta_l(\sigma))}{A_{p,l}(\delta_j(t))} \frac{\eta_l^j(\delta_l(\sigma))}{\eta_l(\delta_j(t))} \right) S'_{l}(\delta_l(\sigma)) d\sigma = 2\phi_j(\delta(t))$$

(8.8)

and

$$\sum_{l=1}^{m} \int_{I_j^l} I_{j,l}(\delta_j(t), \delta_l(\sigma)) S'_{j}(\delta_l(\sigma)) \delta_l^j(\sigma) d\sigma = h_j,$$

(8.9)

for $\tau \in I_j^s$. Multiplying both sides of equation (8.8) by $\delta_j^j(\tau)$ for $\tau \in I_j^s$, we get

$$S_j' \delta_j^j(\tau) + \sum_{l=1}^{m} \int_{I_j^l} \frac{1}{\pi} \text{Im} \left( \frac{A_{p,l}(\delta_l(\sigma))}{A_{p,l}(\delta_j(t))} \frac{\eta_l^j(\delta_l(\sigma))}{\eta_l(\delta_j(t))} \right) S_l'(\delta_l(\sigma)) \delta_l^j(\sigma) d\sigma = 2\phi_j(\delta(t))$$

(8.10)

and

$$\sum_{l=1}^{m} \int_{I_j^l} I_{j,l}(\delta_j(t), \delta_l(\sigma)) S'_{l}(\delta_l(\sigma)) \delta_l^j(\sigma) d\sigma = h_j,$$

(8.11)

Equations (8.10) and (8.11) are valid even when $\delta_j^j(\tau) = 0$, i.e. for all $\tau \in I_j, j = 1, 2, \ldots, m$. By using (8.3) and (8.5), $\hat{S}_j' = S_j'(\delta_j(t)) \delta_j^j(\tau)$, $\hat{\phi}_j(t) = \phi_j(\delta_j(t)) \delta_j^j(\tau)$, $A_{p,j}(\delta_j(t))$ are constant and as $\delta : I \to J$ is bijective and the function $h_j(t)$ is constant in each interval $I_j, j = 1, \ldots, m$, we have

$$\hat{S}_j'(\tau) + \sum_{l=1}^{m} \int_{0}^{2\pi} \frac{1}{\pi} \text{Im} \left( \frac{A_{p,l}(\delta_l(\sigma))}{A_{p,l}(\tau)} \frac{\hat{\zeta}_l^j(\tau)}{\hat{\zeta}_l^j(\tau)} \right) \hat{S}_l'(\sigma) d\sigma = 2\hat{\phi}_j(\tau)$$

(8.12)

and

$$\sum_{l=1}^{m} \int_{0}^{2\pi} I_{j,l}(\tau, \sigma) \hat{S}_l'(\sigma) d\sigma = \hat{h}_j.$$  

(8.13)

Hence, we have

$$\hat{S}_j'(\tau) + \sum_{l=1}^{m} \int_{0}^{2\pi} \left[ N_{Z,j,l}(\tau, \sigma) + I_{j,l}(\tau, \sigma) \right] \hat{S}_l'(\sigma) d\sigma = 2\hat{\phi}_j(t) + \hat{h}_j(\tau), \quad \tau \in I_j.$$  

(8.14)

Integral equation (8.14) can be solved by means of Nyström method with trapezoidal rule.

Integral equation (3.7) can be modified by using the same procedure as described above. Hence, the piecewise constant functions $h_k$ and $u_k$ can be computed from (3.9) and (3.10).

In this paper, we choose test regions with connectivities three and four. The computations were carried out on Windows 7 64-bit operating system, Intel processor Quad-core 2.33GHz, 4GB DDR3 RAM using algorithms coded in Matlab R2011a.

**Example 8.1.** Consider an unbounded region $\Omega^-$ bounded by three circles

$$\Gamma_1 : \eta_1(t) = 2 + e^{-it},$$

$$\Gamma_2 : \eta_2(t) = -1 + i\sqrt{3} + 0.5 e^{-it},$$

and

$$\Gamma_3 : \eta_3(t) = -1 - i\sqrt{3} + 1.5 e^{-it},$$

where $0 \leq t \leq 2\pi$.

For this example, the special points are $z_1 = 2, z_2 = -1 + i\sqrt{3}$ and $a = 0$. We choose the value of $\theta_1 = \pi/2, \theta_2 = \pi/2$ and $\theta_3 = \pi/4$. Figure 3 shows the images of the conformal mappings of the original region onto the canonical regions by using our method with $n = 256$ points. This example has also been considered in [15] for $\theta_1 = \pi/2, \theta_2 = 0$ and $\theta_3 = 0$ for the last image. The values of the parameters for the canonical regions for our method and [15] are given in table 1. Table 2
Figure 3. The original region $\Omega^-$ and its images for example 8.1. (a) $\Omega^-$, (b) $U^1$, (c) $U^2$, (d) $U^3$ and (e) $U^4$. (Online version in colour.)

Table 1. The values for approximated parameters in example 1 for spiral slit region with $\theta_1 = \pi/2$, $\theta_2 = 0$ and $\theta_3 = 0$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>parameters</th>
<th>our method</th>
<th>Amano &amp; Okano [15]</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>$R_1$</td>
<td>1.466503772756291</td>
<td>1.4683</td>
</tr>
<tr>
<td></td>
<td>$R_2$</td>
<td>2.39521440312238</td>
<td>2.393</td>
</tr>
<tr>
<td></td>
<td>$R_3$</td>
<td>$-2.226956363608526$</td>
<td>$-2.2259$</td>
</tr>
<tr>
<td>32</td>
<td>$R_1$</td>
<td>1.467544586417223</td>
<td>1.46757</td>
</tr>
<tr>
<td></td>
<td>$R_2$</td>
<td>2.394496917138897</td>
<td>2.39447</td>
</tr>
<tr>
<td></td>
<td>$R_3$</td>
<td>$-2.22634968304721$</td>
<td>$-2.22633$</td>
</tr>
<tr>
<td>64</td>
<td>$R_1$</td>
<td>1.467540607962205</td>
<td>1.467540618</td>
</tr>
<tr>
<td></td>
<td>$R_2$</td>
<td>2.394498620150213</td>
<td>2.39449861</td>
</tr>
<tr>
<td></td>
<td>$R_3$</td>
<td>$-2.226352161572016$</td>
<td>$-2.226352156$</td>
</tr>
<tr>
<td>128</td>
<td>$R_1$</td>
<td>1.467540608209702</td>
<td>1.46754060820969</td>
</tr>
<tr>
<td></td>
<td>$R_2$</td>
<td>2.394498619975067</td>
<td>2.394498619975067</td>
</tr>
<tr>
<td></td>
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<td>$-2.226352161424804$</td>
<td>$-2.22635216142479$</td>
</tr>
</tbody>
</table>

shows the time taken for the computer to run the program coded in Matlab. The times taken are in seconds. The inverse transformations for each canonical region are given in figure 4a–d. The comparisons between the condition number of the matrices of our method and [15] are given in figure 5.
Figure 4. The inverse of conformal mapping. (a) The inverse of $U^1$, (b) the inverse of $U^2$, (c) the inverse of $U^3$ and (d) the inverse of $U^4$. (Online version in colour.)

Figure 5. Condition numbers of the matrices for adjoint generalized Neumann kernel formed with $A_1, A_2$ and the charge simulation method [15], for example 8.1.

Table 2. Time taken in seconds for computing the conformal mapping onto its canonical regions for example 1.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$U^1$</th>
<th>$U^2$ (s)</th>
<th>$U^3$ (s)</th>
<th>$U^4$ (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>128</td>
<td>0.603617</td>
<td>0.605556</td>
<td>0.612396</td>
<td>0.589240</td>
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<tr>
<td>256</td>
<td>1.172543</td>
<td>1.173658</td>
<td>1.149271</td>
<td>1.122841</td>
</tr>
<tr>
<td>512</td>
<td>3.025301</td>
<td>3.020078</td>
<td>3.027725</td>
<td>3.003177</td>
</tr>
<tr>
<td>1024</td>
<td>10.571163</td>
<td>10.421955</td>
<td>10.610214</td>
<td>10.432853</td>
</tr>
</tbody>
</table>

Example 8.2. Consider an unbounded region $\Omega$ bounded by four rectangles

$\Gamma_1: \eta_1(t) = \{x + iy : |x| = 2, |y - 3| \leq 1\} \cup \{x + iy : |x| \leq 2, |y - 3| = 1\}$,

$\Gamma_2: \eta_2(t) = \{x + iy : |x| = 2, |y + 3| \leq 1\} \cup \{x + iy : |x| \leq 2, |y + 3| = 1\}$,
This example has been considered in [37]. For example 8.2, the special points are $z_1 = 2, z_2 = -2$ and $\alpha = 0$ and we choose the value of $\theta_j$ by $\theta_1 = \pi/2, \theta_2 = \pi/2, \theta_3 = 0$ and $\theta_4 = 0$. Figure 6 shows the images of the conformal mappings of the original region onto its canonical regions by using our method with $n = 256$ points. The inverse transformations for each canonical region are given in figure 7. Table 3 shows the time taken for computing the conformal mapping onto its canonical region between our proposed method and by the study of Nasser [37].
Figure 8. Image transformation by the conformal mapping of unbounded multiply connected region. (a) The original region, (b) $U^1$ with $\theta = (\pi/2, \pi/2, 0, \pi/3)$, (c) $U^3$ with $\theta = (\pi/2, \pi/2, \pi/4, 0)$, (d) $U^2$ with $\theta = (\pi/2, 0, \pi/2, \pi/3)$ and (e) $U^4$ with $\theta = (\pi/2, \pi/2, \pi/4, 0)$. (Online version in colour.)

Figure 9. Inverse transformation for $U^1$. (a) Grid lines and (b) images. (Online version in colour.)

Table 3. Time taken in seconds for computing the conformal mapping onto its canonical regions for example 2.

<table>
<thead>
<tr>
<th>$n$</th>
<th>methods</th>
<th>$U^1$ (s)</th>
<th>$U^2$ (s)</th>
<th>$U^3$ (s)</th>
<th>$U^4$ (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>128</td>
<td>ours</td>
<td>0.833566</td>
<td>0.806499</td>
<td>0.829227</td>
<td>1.644461</td>
</tr>
<tr>
<td></td>
<td>Nasser [37]</td>
<td>3.545890</td>
<td>3.699722</td>
<td>3.614476</td>
<td>3.701283</td>
</tr>
<tr>
<td>256</td>
<td>ours</td>
<td>1.708097</td>
<td>1.712417</td>
<td>1.741601</td>
<td>3.409029</td>
</tr>
<tr>
<td>512</td>
<td>ours</td>
<td>4.935630</td>
<td>4.957821</td>
<td>4.917135</td>
<td>8.558020</td>
</tr>
<tr>
<td></td>
<td>Nasser [37]</td>
<td>73.11641</td>
<td>76.08164</td>
<td>73.28085</td>
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</tr>
<tr>
<td>1024</td>
<td>ours</td>
<td>18.43652</td>
<td>18.76065</td>
<td>18.43312</td>
<td>29.35082</td>
</tr>
<tr>
<td></td>
<td>Nasser [37]</td>
<td>424.1434</td>
<td>432.2561</td>
<td>427.4205</td>
<td>420.0699</td>
</tr>
</tbody>
</table>
**Figure 10.** Inverse transformation for \( U_2 \). (a) Grid lines and (b) images. (Online version in colour.)

**Figure 11.** Inverse transformation for \( U_3 \). (a) Grid lines and (b) images. (Online version in colour.)

**Figure 12.** Inverse transformation for \( U_4 \). (a) Grid lines and (b) images. (Online version in colour.)
Example 8.3. Consider an unbounded region $\Omega^-$ bounded by

$$G_1: \eta_1(t) = \{ x + iy : |x| = 0.725, |y + 0.725| \leq 0.125 \} \cup \{ x + iy : |x| \leq 0.725, |y + 0.725| = 0.125 \},$$

$$G_2: \eta_2(t) = \{ x + iy : |x + 0.75| = 0.75, |y - 0.625| \leq 0.125 \} \cup \{ x + iy : |x + 0.75| \leq 0.75, |y - 0.625| = 0.125 \},$$

$$G_3: \eta_3(t) = 1.25 + 0.7i + 0.15e^{-it}$$

and

$$G_4: \eta_4(t) = 1.4 - 0.05i + 0.2e^{-it},$$

where $0 \leq t \leq 2\pi$.

For this example, we show some simple image transformation based on the presented method. The image transformation is done by Matlab’s function `imtransform` where the conformal map and the inverse conformal map algorithms are used to obtain the desired transformation; see figure 8 for image transformation based on the conformal maps and figures 9–12 for image transformation based on the inverse conformal maps. For this example, the special points are $z_1 = -0.7i$, $z_2 = -0.8 + 0.6i$ and $\alpha = 0.4 + 0.1i$. As the image transformation is based on conformal map, some details (angles and magnitude between curves of the grid lines) are preserved although the shape of the images are different.

9. Conclusion

This paper presented two integral equations with adjoint generalized Neumann kernels for solving RH problems for certain auxiliary functions related to the conformal maps to the canonical regions. The integral equations are discretized using Nyström’s method with the trapezoidal rule. For regions with corners, we use a quadratic formula based on a graded mesh. The resulting linear systems are solved by Gaussian elimination in Matlab. Several examples are given to show the effectiveness of the present method. The advantage of this method is that it allows to compute the values of the conformal mapping as well as the values of its inverse.

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References


