Higher order supertwisting algorithm for perturbed chains of integrators of arbitrary order

Yacine Chitour, Mohamed Harmouche, Salah Laghrouche

July 29, 2015

Abstract
In this paper, we present a generalization for the supertwisting algorithm for perturbed chains of integrators of arbitrary order. This Higher Order Supertwisting (HOST) controller, which extends the approach of [MO08], is either continuous or quasi-continuous and its design is derived from a first result obtained for pure chains of integrators, the latter relying on a geometric condition introduced in [HLC, LHC15]. The complete result is established using a weak Lyapunov function and homogeneity arguments. The effectiveness of the controller is finally illustrated with simulations for a chain of integrator of order four where we compare the performances of a continuous HOST with those of a quasi-continuous one.

Contents
1 Introduction 2
2 Notations and study of the one-dimensional case 4
  2.1 Notations and definitions 4
  2.2 The one-dimensional case 5
3 Higher order super-twisting continuous feedback for a chain of integrator 7
  3.1 Stabilization of a pure chain of integrator of arbitrarily order 7
  3.2 Stabilization of a perturbed chain of integrators: case of $\gamma$ constant 10
  3.3 Stabilization of a perturbed chain of integrators: general case 12
  3.4 Examples for feedbacks $u_0$ and Lyapunov functions $V_1$ verifying the assumptions of Theorems 2 and 3 15
    3.4.1 Hong’s controller 15
    3.4.2 Quasi-continuous modified Hong’s Controller 16

*This research was partially supported by the iCODE institute, research project of the Idex Paris-Saclay
†Y. Chitour is with L2S, Universite Paris XI, CNRS 91192 Gif-sur-Yvette, France. yacine.chitour@lss.supelec.fr
‡M. Harmouche is with Actility, Paris, France. mohamed.harmouche@actility.com
§S. Laghrouche is with OPERA Laboratory, UTBM, Belfort, France. salah.laghrouche@utbm.fr
1 Introduction

The control of nonlinear industrial systems is a challenging task because these systems are difficult to characterize and suffer from parametric uncertainty. Parametric uncertainty arises from varying operating conditions and external perturbations that affect the physical characteristics of systems. This needs to be considered during control design so that the controller counteracts the effect of variations and guarantees performance under different operating conditions. Sliding Mode Control (SMC) [Utk92,Slo84,Lev93] is well-known for control of nonlinear systems and renowned for its insensitivity to bounded parametric uncertainty and external disturbance. This technique is based on applying discontinuous control on a system which ensures convergence of the output function (sliding variable) in finite time to a manifold of the state-space, called the sliding manifold. In practice, SMC suffers from chattering; the phenomenon of finite-frequency, finite-amplitude oscillations in the output which appear because the high-frequency switching excites unmodeled dynamics of the closed loop system [UGS99]. Higher Order Sliding Mode Control (HOSMC) is an effective method for chattering attenuation [EKL96], where the discontinuous control is applied on a higher time derivative of the sliding variable. In this way, the sliding variable and also its higher time derivatives converge to the origin. As the discontinuous control does not act upon the system input directly, chattering is automatically reduced.

Many HOSMC algorithms exist in contemporary literature for control of nonlinear systems with bounded uncertainty. These algorithms are robust because they preserve the insensitivity of classical sliding mode, and maintain the performance characteristics of the closed loop system. Levant for example, has presented a method of designing arbitrary order sliding mode controllers for Single Input Single Output (SISO) systems in [Lev01b]. Laghrouche et al. [LPG07] have proposed a two part integral sliding mode based control to deal with the finite time stabilization problem and uncertainty rejection problem separately. Dinuzzo et al. have proposed another method in [DF09], where the problem of HOSM has been treated as Robust Fuller’s problem. Defoort et al. [DFKP09] have developed a robust multi-input multi-output HOSM controller, using a constructive algorithm with geometric homogeneity based finite time stabilization of an integrator chain. Harmouche et al. have presented their homogeneous controller in [HLB11] based on the work of Hong [Hon02]. Sliding mode with homogeneity approach was also used in [Lev03,Lev05], to demonstrate finite time stabilization of the arbitrary order sliding mode controllers for SISO systems [Lev01b]. A Lyapunov-based approach for arbitrary HOSMC controller design was presented in [HLC, LHC15]. In this work, it was shown that a class of homogeneous controllers that satisfies certain conditions, could be used to stabilize perturbed integrator chains. However, there is no complete argument for chains of integrator of order greater than or equal to four.

The main drawback of these controllers is that they produce a discontinuous control signal. To overcome this problem and to also get finite time convergence, Kamal et
al. [KCM+14] propose a generalization of the well-known continuous Super-Twisting algorithm for high order relative degree system with respect to the output, ensuring finite time convergence of the output variable and its \(r-1\) first derivatives to zero using continuous control signal for order \(r \leq 3\) and quasi-continuous control signal in case of order \(r > 3\).

The convergence conditions and Lyapunov analysis have been only given for order three and a higher order controller is just suggested. In this paper, we present such a HOST controller for arbitrary order with a complete argument as well as other HOST which are just quasi-continuous. Our analysis is based on the use of a weak Lyapunov function and homogeneity arguments for an extended system. The resulting HOST controllers are either continuous or quasi-continuous and both ensure finite-time stabilization, first for a pure chain of integrators and then for a perturbed one.

To describe our results, recall that a perturbed chain of integrators of length \(r\) reads
\[
\dot{z}_i = z_{i+1} \quad \text{for} \quad 1 \leq i \leq r-1 \quad \text{and} \quad \dot{z}_r = \gamma u + \phi \quad \text{where} \quad \gamma \quad \text{is a positive measurable signal lower and upper bounded with known positive constants and both} \quad \dot{\gamma} \quad \text{and} \quad \dot{\phi} \quad \text{are bounded by known positive constants. Note that we do not assume that the additive perturbation} \quad \phi \quad \text{is bounded. We first provide a HOST controller for a pure chain of integrators (i.e.} \quad \gamma \quad \text{constant and} \quad \phi \quad \text{equal to zero) based on standard controllers for a pure chain of integrators verifying in addition a geometric condition. We then prove these HOST controllers can be used for perturbed chains of integrators. In the case} \quad \gamma \quad \text{constant (let say equal to one), one must recall that one can stabilize in finite time such a perturbed chain of integrators without using HOST controllers. Indeed, by setting} \quad z_{r+1} := u + \phi \quad \text{and} \quad v := \dot{u}, \quad \text{one gets} \quad \dot{z}_{r+1} = v + \dot{\phi}. \quad \text{Hence, one is lead to stabilize a perturbed chain of length} \quad r + 1 \quad \text{with the control} \quad v \quad \text{and the bounded uncertainty} \quad \dot{\phi}. \quad \text{This can be done at the price of increasing the length of the chain of integrators (i.e., the relative degree of the output), which can be a serious drawback in some applications. One should therefore see our HOST solution as an alternative to the increase of the length of the chain of integrators. In the case of non constant} \quad \gamma \quad \text{and assuming that} \quad 0 \leq \gamma_m \leq |\gamma| \leq \gamma_M, \quad \text{we provide a solution for arbitrary length under a smallness condition on} \quad 1 - \gamma_m/\gamma_M \quad \text{that we cannot quantify. The controller we propose depends on both the state and time and presents a finite number of discontinuities in the time variable only and becomes continuous if} \quad \phi \quad \text{is assumed to be bounded.}

The paper is organized as follows. The first part of Section 2 gathers the main definitions regarding differential inclusions, homogeneity and finite time convergence for homogeneous differential inclusions. In the second part of the section, we treat in detail the one-dimensional case, reinterpreting the solution proposed by [MO08] while describing our strategy for higher order. In Section 3, we present our results for the general case, first addressing the stabilization by HOST of a pure chain of integrators of arbitrary order and then explaining how to generalize to a perturbed chain by means of homogeneity technics. We prove two results, one in the case of constant \( \gamma \) and the second one for non constant \( \gamma \). We close Section 3 by providing explicit examples of standard controllers for pure chains of integrators which do verify the required geometric condition. We finally demonstrate in Section 4 the efficiency of our HOST algorithm for a perturbed chain of integrators of order four.
2 Notations and study of the one-dimensional case

2.1 Notations and definitions

In this paper, we use \( \mathbb{R} \) and \( | \cdot | \) to denote the set of real numbers and a fixed norm on \( \mathbb{R}^r \) respectively, where \( r \) is a positive integer. For \( \lambda > 0 \), let \( D_\lambda \) be the \( r \times r \) diagonal matrix defined by

\[
D_\lambda = \text{diag}(\lambda, \cdots, \lambda).
\]  

(1)

If \( M \) is a subset of \( \mathbb{R}^r \), we use \( \overline{M} \) to denote its closure. If \( x \in \mathbb{R} \), we denote by \( [x] \) the integer part of \( x \) i.e., the smallest integer not greater than \( x \). We define the function \( \text{sign} \) as the multivalued function defined on \( \mathbb{R} \) by \( \text{sign}(x) = \frac{x}{|x|} \) for \( x \neq 0 \) and \( \text{sign}(0) = [-1, 1] \).

Similarly, for every \( a \geq 0 \) and \( x \in \mathbb{R} \), we use \( |x|^a \) to denote \( |x|^a \text{sign}(x) \). Note that \( |\cdot|^a \) is a continuous function for \( a > 0 \) and of class \( C^1 \) with derivative equal to \( a |\cdot|^{a-1} \) for \( a \geq 1 \).

Definition 1 (Differential inclusion.)

Let \( r \) be a positive integer and \( F : \mathbb{R}^r \rightarrow \mathbb{R}^r \) be a multivalued function. The differential inclusion \( \dot{z} \in F(z) \), \( z \in \mathbb{R}^r \) is said to be a Filippov differential inclusion \([\text{Lev01a}] \), if the vector set \( F(z) \) is non-empty, closed, convex, locally bounded and upper-semicontinuous.

Solutions of the differential inclusion are defined as absolutely-continuous functions of time satisfying the inclusion almost everywhere. Such solutions always exist and have most of the well-known standard properties except the uniqueness \([\text{Lev01a}] \). Similarly, a differential equation \( \dot{z} = f(z) \), \( z \in \mathbb{R}^r \), with a locally bounded Lebesgue-measurable right-hand side is said to be understood in the Filippov sense \([\text{Lev01a}] \) if it is replaced by the Filippov differential inclusion \( \dot{z} \in F(z) \), where one has, for \( z \in \mathbb{R}^r \)

\[
F(z) = \bigcap_{\delta > 0} \bigcap_{N \subset \mathbb{R}^r, \mu(N) = 0} \overline{\sigma f(O_\delta(z) \setminus N)},
\]

with \( \mu \), \( O_\delta(z) \) and \( \overline{\sigma M} \) denoting the Lebesgue measure, the set of points \( y \in \mathbb{R}^r \) so that \( |y - z| < \delta \) and the convex closure of any set \( M \subset \mathbb{R}^r \) respectively. In case \( f \) is continuous almost everywhere, the previous procedure reduces to take \( F(z) \) as the convex closure of the set of all limit values of \( f(y) \) as \( y \) tends to \( z \).

Definition 2 (Homogeneity. cf. [Lev01a].)

Let \( r \) be a positive integer. A function \( f : \mathbb{R}^r \rightarrow \mathbb{R} \) (a vector field \( f : \mathbb{R}^r \rightarrow \mathbb{R}^r \) or a differential inclusion \( F : \mathbb{R}^r \Rightarrow \mathbb{R}^r \) respectively) is said to be homogeneous of degree \( q \in \mathbb{R} \) with respect to the family of dilations \( \delta_\varepsilon(z) \), \( \varepsilon > 0 \), defined by

\[
\delta_\varepsilon(z) = (z_1, \cdots, z_r) \mapsto (\varepsilon^{p_1}z_1, \cdots, \varepsilon^{p_r}z_r),
\]

where \( p_1, \cdots, p_r \) are positive real numbers (the weights), if for every positive \( \varepsilon \) and \( z \in \mathbb{R}^r \), one has

\[
f(\delta_\varepsilon(z)) = \varepsilon^q f(z) \quad (f(\delta_\varepsilon(z)) = \varepsilon^q \delta_\varepsilon(f(z)) \text{ or } F(\delta_\varepsilon(z)) = \varepsilon^q \delta_\varepsilon(F(z)) \text{ respectively}).
\]

Definition 3 (Asymptotic and Finite time stability. cf. [Lev01a].)
(i) A differential inclusion $\dot{z} \in F(z), z \in \mathbb{R}^r$ (a differential equation $\dot{z} = f(z)$) is globally uniformly finite-time stable at 0, if $x(\cdot) \equiv 0$ is a Lyapunov-stable solution and, for every $R > 0$, there exists $T > 0$ such that every trajectory starting at $z_*$ with $|z_*| \leq R$ stabilizes at zero in the time $T$.

(ii) A differential inclusion $\dot{z} \in F(z), z \in \mathbb{R}^r$ (a differential equation $\dot{z} = f(z)$) is further globally uniformly asymptotically stable at 0, if it is Lyapunov stable and for every $R, \delta > 0$, there exists $T > 0$ such that, for every trajectory starting at $z_*$ with $|z_*| \leq R$, one has that $|z(t)| \leq \delta$ for $t \geq T$.

A set $D \in \mathbb{R}^r$ is called dilation-retractable if $\delta \varepsilon(D) \subset D$ for every $\varepsilon \in [0, 1]$.

(iii) A homogeneous differential inclusion $\dot{z} \in F(z), z \in \mathbb{R}^r$ (a differential equation $\dot{z} = f(z)$) is said to be contractive if there exist two compact sets $D_1, D_2$ and a time $T > 0$, with $D_1$ is dilation-retractable, $D_2$ containing the origin and contained in the interior of $D_1$ such that all trajectories starting within $D_1$ lie in $D_2$ at time $T$.

We will use repeatedly the result of [Lev01a, BEPP13] saying that, for a differential inclusion $\dot{z} \in F(z), z \in \mathbb{R}^r$, with negative degree, then Properties (i), (ii) and (iii) are equivalent.

### 2.2 The one-dimensional case

In this section, we revisit the one-dimensional case to describe our strategy. It must be stressed that this case has been solved by [MO08] in the unperturbed case and in the perturbed case.

We first recall the construction of a super-twisting feedback as given in [MO08] as well as its argument. Consider the unperturbed first-order integrator given by

$$\dot{z}_1 = u. \quad (2)$$

The continuous positive definite function $V_1(z_1) = k_I |z_1|$ is homogeneous of degree one with respect to the dilation $\delta \varepsilon(z_1) = \varepsilon z_1$ and the continuous feedback law $u_0(z_1) = -[z_1]^{1/2}$ renders the above closed loop system finite-time converging to zero. Notice that

$$u_0 \frac{\partial V_1}{\partial z_r} \leq 0,$$

and $\frac{\partial V_1}{\partial z_r}$ is homogeneous of degree zero.

For positive constants $k_P$ and $k_I$, [MO08] proposes the continuous time-varying feedback law

$$u_{ST}(z_1, t) = k_P u_0(z_1) - k_I \int_0^t \text{sign}(z_1) dt.$$

This feedback is made of a proportional term $u_0(z_1)$ and an integral one $- \int_0^t \text{sign}(z_1) dt$ with corresponding gains $k_P$ and $k_I$. By introducing the variable

$$\xi := -k_I \int_0^t \text{sign}(z_1) dt,$$
the related closed-loop system can be written as the following planar differential inclusion

\[ \begin{align*}
\dot{z}_1 &= -k_P \lfloor z_1 \rfloor + \xi, \\
\dot{\xi} &= -k_I \text{sign}(z_1).
\end{align*} \] (3)

Then [MO08] proceeds by considering the weak Lyapunov function given by

\[ V = V_1(z_1) + \frac{1}{2} \xi^2. \]

(Recall also that [MO08] also provide a strong Lyapunov function.) The time derivative of \( V(z_1, \xi) \) along trajectories of (3) is equal to

\[ \dot{V} = k_I \text{sign}(z_1) \left( -k_P \lfloor z_1 \rfloor + \xi \right) + \xi (-k_I \text{sign}(z_1)) = -k_I k_P \lfloor z_1 \rfloor. \]

By an adapted Barbalat type of argument, one shows that \( z_1 \) must tend to zero asymptotically and then, using the first equation in (3), \( \xi \) also converges to zero at infinity since it is absolutely continuous with a bounded derivative.

Our reinterpretation of the results of [MO08] starts with the following remark: for every positive \( p \), the differential inclusion defined by System (3) is homogeneous with degree equal to \( -p \) with respect to the family of two-dimensional dilations defined on \( \mathbb{R}^2 \) by

\[ \psi_\varepsilon(z_1, \xi) = (\varepsilon^p z_1, \varepsilon^p). \]

and it satisfies the standard assumptions as characterized in [Lev05] or [BEPP13]. Therefore, by using [Lev05] Theorem 7.1, one deduces that the convergence to \((0,0)\) of trajectories of System (3) occurs in finite-time and [BEPP13] Theorem 4.1 and Corollary 4.3 provide us with a continuous, positive-definite function \( W: \mathbb{R}^2 \to \mathbb{R}_+ \) which is \( C^1 \) except at the origin and homogeneous such that the time derivative of \( W \) along non trivial trajectories of System (3) verifies

\[ \dot{W} \leq -c W^{1/2}, \]

for some positive constant \( c \). Moreover notice that

\[ \dot{W} = \frac{\partial W}{\partial z_1} \dot{z}_1 + \frac{\partial W}{\partial \xi} \dot{\xi}. \]

Every term is the previous sum is homogeneous with the same degree as \( W^{1/2} \), i.e., \( -p/4 \) (to see that for instance take appropriate choices of \((z_1, \xi)\)). Since \( \dot{\xi} = -k_I \frac{\partial V_1}{\partial z_1} \) is homogeneous of degree zero, the quantity \( \frac{\partial W}{\partial \xi} \dot{\xi} \) has the same homogeneous degree as \( W^{1/2} \).

We now turn to the perturbed case namely the finite-time stabilization with respect to zero of

\[ \dot{z}_1 = \gamma u + \varphi, \]

where \( \gamma(\cdot) \) and \( \varphi(\cdot) \) are time-varying functions verifying \( 0 < \gamma_m \leq \gamma(\cdot) \leq \gamma_M \), \( |\dot{\gamma}(\cdot)| \leq \tilde{\gamma} \) and \( |\dot{\varphi}(\cdot)| \leq \tilde{\varphi} \). The feedback \( u \) must be devised with the sole knowledge of the bounds \( \gamma_m, \gamma_M, \tilde{\gamma} \) and \( \tilde{\varphi} \).
For the rest of the section, we only treat the case $\gamma(\cdot) \equiv 1$ for simplicity of the exposition. We choose the feedback $u$ to be equal to $u_0$ defined in (3). By setting now the integral variable $\xi$ to

$$\xi = -k_I \int_0^t \text{sign}(z_1) dt + \varphi,$$

the dynamics (3) becomes

$$\dot{z}_1 = -k_P \lfloor z_1 \rfloor^{\frac{1}{2}} + \xi,$$
$$\dot{\xi} = -k_I \text{sign}(z_1) + \dot{\varphi},$$

and the time derivative of $W$ along non trivial trajectories of System (4) verifies

$$\dot{W} \leq -cW^{1/2} + \frac{\partial W}{\partial \xi} \dot{\varphi}.$$ 

The second term in the above sum can be upper bounded by $\varphi \left\| \frac{\partial W}{\partial \xi} \right\|$ which has the same homogeneous degree as $W^{1/2}$. Therefore, by using [Lev05, Corollary 1] and assuming that $\varphi$ is small enough, we conclude to the finite-time stability of the origin for the perturbed first-order integrator using a super-twisting feedback law. We finally remove the restriction on $\varphi$ by using a classical time-coordinate transformation.

3 Higher order super-twisting continuous feedback for a chain of integrator

In this section, we essentially generalize what has been done for the first-order integrator. We first build an appropriate feedback for a pure chain of integrator and we tackle the perturbed case by a homogeneity argument.

3.1 Stabilization of a pure chain of integrator of arbitrarily order

Let $r$ be a positive integer. The $r$-th order chain of integrator is the single-input control system given by

$$\begin{cases}
\dot{z}_1 = z_2, \\
\vdots \\
\dot{z}_r = u,
\end{cases}$$

where the state $z = (z_1, \cdots, z_r)^T$ belongs to $\mathbb{R}^r$ and the control $u$ is real.

Define the following parameters:

$$\kappa \in (-\frac{1}{r}, 0), \quad p_i = 1 + (i - 1)\kappa, \quad 1 \leq i \leq r,$$

In the spirit of [LHC15][HLC], we put forwards geometric conditions on the feedback $u_0(\cdot)$ and the Lyapunov function $V_1$ which will be instrumental for building a super-twisting feedback law. We thus have the following theorem.

**Theorem 1** Let $r$ be a positive integer. Consider the $r$-th order chain of integrator defined in (5). Assume that there exists a continuous homogeneous feedback law $u_0 : \mathbb{R}^r \to \mathbb{R}$, such that the closed-loop System (5) with $u_0$ is finite time stable and such that the following conditions are satisfied:
(i) The closed-loop System (5) is homogeneous of degree \( \kappa \in (0, 1/r) \) with respect to the family of dilations \( \delta_\varepsilon \) associated with \((p_1, \cdots, p_r)\) and there exists a continuous positive definite function \( V_1 : \mathbb{R}^n \to \mathbb{R}_+ \) homogeneous with respect to the family of dilations \( \delta_\varepsilon(\cdot) \) associated with \((p_1, \cdots, p_r)\), such that there exists \( c > 0 \) and \( \alpha \in (0, 1) \) for which the time derivative of \( V_1 \) along non trivial trajectories of the closed-loop system System (5) with \( u_0 \) verifies
\[
\dot{V}_1 \leq -cV_1^\alpha.
\]

(ii) The following condition holds for \( z \in \mathbb{R}^n \)
\[
\frac{\partial V_1}{\partial z_r}(z)u_0(z) \leq 0, \text{ on } \mathbb{R}^n.
\]

(iii) The function \( \frac{\partial V_1}{\partial z_r} \) is homogeneous with non negative degree with respect to the family of dilations \( \delta_\varepsilon \) associated with \((p_1, \cdots, p_r)\).

Then, for every \( k_P \geq 1 \) and \( k_I > 0 \), the higher order supertwisting (HOST) controller defined by
\[
 u_{ST}(z,t) = k_Pu_0(z) - k_I \int_0^t \frac{\partial V_1}{\partial z_r}(z(s))ds,
\]
(7)
stabilizes System (5) in finite-time.

**Proof of Theorem 1.** The closed loop System (5) with the controller \( u_{ST} \), can be written as
\[
\begin{align*}
\dot{z}_1 &= z_2, \\
& \vdots \\
\dot{z}_r &= k_Pu_0 + \xi, \\
\dot{\xi} &= -k_I \frac{\partial V_1}{\partial z_r},
\end{align*}
\]
(8)
where we consider the integral variable \( \xi \) defined by
\[
\xi := -k_I \int_0^t \frac{\partial V_1}{\partial z_r}dt.
\]
Consider the following positive definite function \( V \) defined as
\[
V = V_1 + \frac{1}{2k_I} \xi^2.
\]
(9)
Its time derivative along trajectories of System (8) verifies the following
\[
\dot{V} = \sum_{i=1}^{r-1} \frac{\partial V_1}{\partial z_i} z_{i+1} + \frac{\partial V_1}{\partial z_r} (k_Pu_0 + \xi) - \xi \frac{\partial V_1}{\partial z_r} = \dot{V}_1 + (k_P - 1) \frac{\partial V_1}{\partial z_r}u_0 \leq -cV_1^\alpha(z),
\]
where we have used Items (i) and (ii). We deduce that \( z \) and \( \xi \) remain bounded along trajectories of System (8), \( V(z(\cdot), \xi(\cdot)) \) tends to a non negative limit and the integral
\[
\int_0^\infty V_1^\alpha(z(\cdot))
\]
is finite. As a consequence, \(\dot{V}_1\) remains bounded as well. Since one has for every \(t_1 \leq t_2\) that
\[
|V_1^{\alpha+1}(z(t_2)) - V_1^{\alpha+1}(z(t_1))| = (\alpha + 1)\left|\int_{t_1}^{t_2} \dot{V}_1 V_1^\alpha(z(t)) dt\right| \leq C \int_{t_1}^{t_2} V_1^\alpha(z(t)) dt,
\]
we conclude that \(V_1^{\alpha+1}(z(\cdot))\) admits a limit as \(t\) tends to infinity, which is hence necessarily equal to zero. Therefore, as \(t\) tends to infinity, \(\xi(\cdot)\) admits a finite limit \(\tilde{\xi}\) (since it is continuous) and \(z(\cdot)\) must tend to zero. Note that, since the closed-loop [5] is finite stable with respect to the origin and \(u_0\) is continuous, it follows that \(u_0(0) = 0\). As a consequence, \(u_0(z(\cdot))\) must tend to zero as \(t\) tends to infinity. It follows that \(\dot{z}_r(\cdot)\) converges to \(\tilde{\xi}\). Finally, \(\tilde{\xi} = 0\) is equal to zero since otherwise \(z_r/t\) would converge to \(\tilde{\xi}\), contradicting the convergence to zero of \(z(\cdot)\) as \(t\) tends to infinity.

We deduce that System (8) is globally asymptotically stable with respect to the origin. As for the first-order integrator, we use [Lev05, Theorem 7.1] to deduce that the convergence to \((0,0)\) of trajectories System (8) occurs in finite-time.

\[\Box\]

**Remark 1** In the above argument, we actually only need \(u_0\) to be continuous at zero (implying that \(u_0(0) = 0\)).

Note that, in Item (iii), the homogeneous function \(\frac{\partial V_1}{\partial z_r}\) may be of zero homogeneity degree and thus not necessarily continuous. We do ask instead \(\frac{\partial V_1}{\partial z_r}\) to be bounded in an open neighborhood of the origin in order to get the following proposition.

**Proposition 1** Assume that \(\kappa \geq -\frac{1}{r+1}\). Consider the differential inclusion defined on \(\mathbb{R}^{r+1}\) by System (8). Then, there exists a continuous, positive-definite function \(W : \mathbb{R}^{r+1} \rightarrow \mathbb{R}_+\) which is \(C^1\) except at the origin and homogeneous with respect to the dilations \((\delta_\varepsilon(z), \varepsilon^{p_{r+1}} \xi)\), where \(p_{r+1} = 1 + r\kappa\) such that the time derivative of \(W\) along non-trivial trajectories of System (8) verifies
\[
\dot{W} \leq -cW^{1/2},
\]
for some positive constant \(c\).

**Proof of Proposition 1**. The proof of the theorem consists of checking that the differential inclusion defined on \(\mathbb{R}^{r+1}\) by System (8) verifies all the conditions required to apply [BEPP13, Theorem 4.1]. Indeed, one clearly verifies that the right-hand side of System (8) is a locally essentially bounded vector field giving rise to a multivalued function \(F\) which is non-empty, compact and convex. Therefore \(F\) satisfies the standard assumptions through the Filippov regularization procedure (cf. [Fil88] for precise results) and is in addition homogeneous of degree \(\kappa\) with respect to the family of dilations \(\psi_\varepsilon\) defined on \(\mathbb{R}^{r+1}\) by \(\psi_\varepsilon(z, \xi) = (\delta_\varepsilon(z), \varepsilon^{p_{r+1}} \xi)\) where \(p_{r+1} = 1 + r\kappa\).

\[\Box\]
3.2 Stabilization of a perturbed chain of integrators: case of $\gamma$ constant

In this subsection, we apply the previous results to get finite-time convergence of the perturbed chain of integrators defined next by

$$
\begin{align*}
\dot{z}_1 &= z_2, \\
\vdots \\
\dot{z}_r &= \gamma u + \varphi,
\end{align*}
$$

(11)

where the time-varying functions $\gamma(\cdot)$ and $\varphi(\cdot)$ are measurable over $\mathbb{R}_+$ and verify the following hypotheses: there exists positive constants $\gamma_m, \gamma_M$ and non negative constants $\gamma, \varphi$ such that, for every $t \geq 0$ it holds

$$
0 < \gamma_m \leq \gamma(t) \leq \gamma_M, \\
|\dot{\gamma}(t)| \leq \bar{\gamma}, |\dot{\varphi}(t)| \leq \bar{\varphi}.
$$

(12)

(13)

In particular, notice that both $\gamma$ and $\varphi$ are globally Lipschitz with Lipschitz constants upper bounded by $\bar{\gamma}$ and $\bar{\varphi}$ respectively. One could alternatively define the previous system using a differential inclusion (cf. [Lev05] for instance.)

We now want to derive conditions under which the super-twisting feedback defined in Eq. (7) stabilizes System (11) in finite time. We obtain the following theorem.

**Theorem 2** Consider the perturbed chain of integrators defined by (11), where the time-varying function $\gamma(\cdot)$ and $\varphi(\cdot)$ verify $\gamma \equiv \gamma_m$ and $\varphi$ respectively. Assume that there exists a continuous homogeneous feedback law $u_0$ and a Lyapunov function $V_1$ verifying the assumptions (i), (ii) and (iii) of Theorem 1 with $\kappa = -\frac{1}{r+1}$ and so that $\frac{\partial V_1}{\partial z_r}$ is homogeneous of degree zero with respect to the family of dilations $\delta_\varepsilon(\cdot)$ associated with $(p_1, \ldots, p_r)$.

Then, for every positive gains $k_P \geq 1$ and $k_I > 0$, there exists $\lambda > 0$ only depending on the gains and $\bar{\varphi}$ such that the feedback law $u_{ST}(D_\lambda z, \lambda t)/\gamma_m$, where the matrix $D_\lambda$ is defined in Eq. (1) and $u_{ST}$ is given in (7), stabilizes (11) in finite-time. In particular, if the feedback $u_0$ is chosen to be continuous, then $u_{ST}(D_\lambda z, \lambda t)/\gamma_m$ is continuous as well.

**Proof of Theorem 2.** Fix now some $k_P \geq 1$ and $k_I > 0$. Associate to every absolutely continuous function $z: \mathbb{R}_+ \to \mathbb{R}^r$ the following function for $t \geq 0$

$$
\xi(t) := -k_I \int_0^t \frac{\partial V_1}{\partial z_r}(z(s))ds + \varphi(t).
$$

The closed-loop system obtained by inserting $u_{ST}$ in System (11) can be written as

$$
\begin{align*}
\dot{z}_1 &= z_2, \\
\vdots \\
\dot{z}_r &= k_P u_0(z) + \xi, \\
\dot{\xi} &= -k_I \frac{\partial V_1}{\partial z_r} + \dot{\varphi}(t).
\end{align*}
$$

(14)
It is clear that System (14) corresponds to the differential inclusion (8) perturbed by the time-varying vector field over $\mathbb{R}^{r+1}$ given by $(0, \cdots, 0, \dot{\varphi}(t))^T$ or, equivalently, by the multifunction $(0, \cdots, 0, [-\varphi, \varphi])^T$ taking values in the subsets of $\mathbb{R}^{r+1}$.

Consider the Lyapunov function $W$ associated with System (8) furnished by Theorem 1 and compute its time derivative along non trivial trajectories of System (14). One gets, for every $t \geq 0$, that

$$\dot{W} \leq -cW^{1/2} + \varphi\left|\partial W / \partial \xi\right|.$$  \hspace{1cm} (15)

We next prove that the homogeneous function $\left|\partial W / \partial \xi\right|$ has the same homogeneous degree with respect to the family of dilations $(\delta_\varepsilon(z), \varepsilon^{r+1}\xi)$ as $W^{1/2}$. Indeed, if $F : \mathbb{R}^{r+1} \rightarrow \mathbb{R}^{r+1}$ denotes the differential inclusion defined by the closed loop System (5), then Eq. (10) says that

$$\max_{y \in F(z, \xi)} \nabla W(z, \xi)y \leq -cW^{1/2},$$

for every $(z, \xi) \in \mathbb{R}^{r+1}$. Moreover, by eventually reducing $1/2$ in the previous inequality (and also using the argument in [BB05, Theorem 7.1] adapted for homogeneous differential inclusions), one can assume furthermore that there exists $d > 0$ such that, for every $(z, \xi) \in \mathbb{R}^{r+1}$, one has

$$-dW^{1/2} \leq \max_{y \in F(z, \xi)} \nabla W(z, \xi)y.$$  \hspace{1cm} (16)

We now rewrite $\nabla W(z, \xi)y$ for $y \in F(z, \xi)$ as

$$\nabla W(z, \xi)y = \sum_{i=1}^{r-1} \frac{\partial W}{\partial z_i}z_{i+1} + \frac{\partial W}{\partial z_r}(k_Pu_0 + \xi) - k_I \frac{\partial W}{\partial \xi} \frac{\partial V_1}{\partial z_r}.$$  \hspace{1cm} (17)

By a standard homogeneity argument, one sees that every term in the previous sum is a homogeneous function with the same degree with respect to the family of dilations $(\delta_\varepsilon(z), \varepsilon^{r+1}\xi)$ and thus one concludes that $\left|\partial W / \partial \xi\right|$ has the same homogeneous degree as $W^{1/2}$.

One deduces from Eq. (15) that there exists $\varphi_\ast > 0$ such that

$$\dot{W} \leq -\frac{c}{2}W^{1/2},$$

along trajectories of System (14) if $\varphi \leq \varphi_\ast$. We thus have proved the theorem under the previous restriction on $\varphi$.

To remove that restriction, we consider the following standard time-coordinate change of variable along trajectories of System (11) defined, for every $\lambda > 0$ by

$$y(t) = D_\lambda z(t/\lambda).$$  \hspace{1cm} (16)

Under the hypotheses of the theorem, one gets that System (11) can be rewritten

$$\begin{cases} \dot{y}_1 = y_2, \\ \vdots \\ \dot{y}_r = \frac{u_\lambda}{\gamma_m} + \varphi_\lambda, \end{cases}$$  \hspace{1cm} (17)
where one has set, for \( t \geq 0 \),

\[
u_\lambda(t) = u(t) \frac{\lambda}{\gamma}, \quad \varphi_\lambda(t) = \varphi(t) \frac{\lambda}{\gamma}.
\]

Note that, for almost every \( t \geq 0 \),

\[|\dot{\varphi}_\lambda| \leq \frac{\varphi}{\lambda}.
\]

By taking \( \lambda \geq \frac{\varphi}{\varphi} \), one gets that \( |\dot{\varphi}_\lambda| \leq \varphi \). We can now apply the previous stabilization result and conclude.

\[■\]

### 3.3 Stabilization of a perturbed chain of integrators: general case

In this subsection, we apply the results of Subsection 3.1 to get finite-time convergence of System (11) where the time-varying functions \( \gamma(\cdot) \) and \( \varphi(\cdot) \) are measurable over \( \mathbb{R}_+ \) and verify Eqs. (12) and (13).

We now want to derive conditions under which the super-twisting feedback defined in Eq. (7) stabilizes System (11) in finite time. We obtain the following theorem.

**Theorem 3** Consider the perturbed chain of integrators defined by (11), where the time-varying function \( \gamma(\cdot) \) and \( \varphi(\cdot) \) verify Eqs. (12) and (13). Assume that there exists a continuous homogeneous feedback law \( u_0 \) and a Lyapunov function \( V_1 \) verifying the assumptions (i), (ii) and (iii) of Theorem 1 with \( \kappa = -\frac{1}{r+1} \) and so that \( \frac{\partial V_1}{\partial z_r} \) is homogeneous of degree zero with respect to the family of dilations \( \delta_x(\cdot) \) associated with \( (p_1, \cdots, p_r) \).

Set \( \gamma_d = (\gamma_M + \gamma_m)/2 \) and \( \delta_\gamma = 1 - \gamma_m/\gamma_M \in (0, 1) \). Then, for every positive gains \( k_P \geq 1 \) and \( k_I > 0 \), there exists \( \delta_0 \in (0, 1) \) and \( \lambda_0 > 0 \) only depending on the gains and the constants in Eqs. (12) and (13) such that, if \( \delta_\gamma \leq \delta_0 \) and \( \lambda \geq \lambda_0 \), the feedback law \( u_{st} \) defined by

\[
u_{st}(z, t) = \frac{1}{\gamma_d}\left( k_P u_0(D_\lambda z) - k_I \int_{\lambda t}^{\lambda t} \frac{\partial V_1}{\partial z_r}(D_\lambda z(s))ds \right),
\]

where the matrix \( D_\lambda \) is defined in Eq. (1) stabilizes (11) in finite-time.

**Proof of Theorem 3.** As in the proof of Theorem 2, fix \( k_P \geq 1 \) and \( k_I > 0 \). Define the following function for \( t \geq 0 \)

\[
\xi(t) := -\frac{\gamma(t)}{\gamma_d} k_I \int_{\lambda t}^{\lambda t} \frac{\partial V_1}{\partial z_r}(z(s))ds + \varphi(t).
\]

The closed-loop system obtained by inserting \( u_{st} \) in System (11) can be written as

\[
\begin{aligned}
\dot{z}_1 &= z_2, \\
\vdots \\
\dot{z}_r &= k_P u_0(z) + \xi + \frac{\gamma(t) - \gamma_d}{\gamma_d} k_P u_0, \\
\dot{\varphi} &= -k_I \frac{\partial V_1}{\partial z_r} + \varphi(t) - k_I \frac{\gamma(t) - \gamma_d \frac{\partial V_1}{\partial z_r} - \frac{\gamma(t)}{\gamma_d} \int_{\lambda t}^{\lambda t} \frac{\partial V_1}{\partial z_r}(z(s))ds}.
\end{aligned}
\]
To pursue the argument, we first need the following estimates and notation: for every $t \geq 0$,

$$\left| \gamma(t) - \gamma_d \right| \leq \frac{\delta_\gamma}{2 - \delta_\gamma} \leq \delta_\gamma,$$

$$\left| \int_0^t \frac{\partial V_1}{\partial z_r}(z(s)) ds \right| \leq \max_{s \in [t,t]} \left| \frac{\partial V_1}{\partial z_r}(z(s)) \right|,$$

$$v_1 = \max_{z \in \mathbb{R}^n \setminus \{0\}} \left| \frac{\partial V_1}{\partial z_r}(z) \right|.$$

Consider now the Lyapunov function $W$ associated with System (8) furnished by Theorem 1. Computing the time derivative of $W$ along non trivial trajectories of System (25), one gets that for every $t \geq 0$,

$$\dot{W} \leq -cW^{1/2} + \delta_d k_P \left| \frac{\partial W}{\partial z_r}(z) \right| + \left| \frac{\partial W}{\partial \xi} \right| \left( \varphi + k_I v_1 (\delta_d + \frac{\tau}{\gamma_d}) \right).$$  (21)

Exactly as for $\left| \frac{\partial W}{\partial \xi} \right|$, we prove that also the homogeneous function $\left| \frac{\partial W}{\partial z_r}(z) \right|$ has the same homogeneous degree with respect to the family of dilations $(\delta_e(z), e^{\delta e+1} \xi)$ as $W^{1/2}$.

One deduces that there exist $\delta_0, \varphi_0 > 0$ and $\tilde{\gamma}$ such that, if the following extra assumptions hold true,

$$\delta_d \leq \delta_0, \quad \varphi \leq \varphi_*, \quad \tilde{\gamma}/\gamma_d \leq \tilde{\gamma},$$  (22)

then one has, for every $t \geq 0$,

$$\dot{W} \leq -\frac{c}{2} W^{1/2},$$  (23)

along non trivial trajectories of System (14).

We first assume that (22) holds and we prove the theorem. One cannot conclude immediately as in the proof of Theorem 2 since the integral variable defined in (19) is not continuous. Therefore, the evaluation of $W$ along trajectories of (25) presents discontinuity jumps at times taking integer values. To address that issue, we introduce some notations: for $n$ non negative integer, set

$$Y_n = \lim_{t \to n, t < n} W(z(t), \xi(t)) \quad Z_n = \lim_{t \to n, t > n} W(z(t), \xi(t)).$$

Dividing Eq. (23) by $2W^{1/2}(t)$ and integrating, one gets, for every non negative integer $n$ and $t \in (n, n+1)$ so that the trajectory of (25) remains non trivial, that

$$Z_n > W(t) > Y_{n+1},$$

$$Y_{n+1}^{1/2} - Z_n^{1/2} \leq -\frac{c}{4}. $$

Moreover, to estimate the jump of $W$ at discontinuity times, we notice that

$$Z_{n+1} - Y_{n+1} = W(z(n+1), \varphi(n+1)) - W(z(n+1), \xi_{n+1}^-),$$

where

$$|\xi_{n+1}^- - \varphi(n+1)| = \left| \frac{\gamma(n+1)}{\gamma_d} k_I \int_n^{n+1} \frac{\partial V_1}{\partial z_r}(z(s)) ds \right| \leq D_0 \tilde{\gamma},$$

13
with the positive constant $D_0$ only depending on the gains $k_P$ and $k_I$. Because $\frac{\partial W}{\partial \xi}$ has the same homogeneous degree with respect to the family of dilations $(\delta_{\epsilon}(z), \epsilon^{p+1} \xi)$ as $W^{1/2}$, we deduce that there exists a positive constant $D_1$ only depends on the gains $k_P$ and $k_I$ such that, for every non negative integer $n$ so that the trajectory of (25) remains non trivial on $(n, n+1)$, one gets

$$|Z_{n+1} - Y_{n+1}| = \left| \int_{\phi(n+1)}^{\phi(n+1)} \frac{\partial W}{\partial \xi}(z(n+1), \eta) d\eta \right| \leq D_1 \gamma \max(Z_{n+1}^{1/2}, Y_{n+1}^{1/2}).$$

By dividing the previous inequality by $Z_{n+1}^{1/2} + Y_{n+1}^{1/2}$, we deduce that $|Z_{n+1}^{1/2} - Y_{n+1}^{1/2}| \leq \frac{D_1 \gamma}{8D_1}$. The finite-time convergence to the origin of non trivial trajectories of (25) follows at once.

To remove most of these restrictions, we proceed as in the proof of Theorem 2, i.e., by considering the time-coordinate transformation defined in Eq. (16).

For $t \geq 0$ and $\lambda > 0$, set

$$u_{\lambda}(t) = u\left(\frac{t}{\lambda}\right), \quad \gamma_{\lambda}(t) = \gamma\left(\frac{t}{\lambda}\right), \quad \varphi_{\lambda}(t) = \varphi\left(\frac{t}{\lambda}\right).$$

Note that, for almost every $t \geq 0$,

$$|\dot{\gamma}_{\lambda}| \leq \frac{\bar{y}}{\lambda}, \quad |\dot{\varphi}_{\lambda}| \leq \frac{\bar{y}}{\lambda}.$$

By taking $\lambda \geq \lambda_0 := \min\left(\frac{\bar{y}}{\varphi_\ast}, \frac{1}{\gamma d} \right)$, one gets that $|\dot{\varphi}_{\lambda}| \leq \varphi_\ast$ and $|\dot{\gamma}_{\lambda}/\gamma d| \leq \bar{y}$. We can now apply the previous stabilization result and conclude. Note though that we are not able with this trick to remove the restriction on $\delta_d$.

If we impose an extra restriction on $\varphi$, we can get a continuous HOST feedback in case the pure integrator chain feedback $u_0$ is chosen continuous. This is explained in the following property.

**Proposition 2** Consider the same hypotheses as in Theorem 3 and, in addition suppose that $|\varphi| \leq \varphi_M$ for some known non negative constant $\varphi_M$. Then the same conclusion as in Theorem 3 is reached with the feedback law $u_{st}$ defined by

$$u_{st}(z, t) = \frac{1}{\gamma d} \left(k_P u_0(D_{\lambda}z) - k_I \int_0^t \frac{\partial V_1}{\partial z_r}(D_{\lambda}z(s)) ds\right).$$

(24)

In particular, if $u_0$ is chosen to be continuous, then $u_{st}$ will be continuous.

**Proof of Proposition 2** We define now the integral variable $\xi$ as follows for $t \geq 0$

$$\xi(t) := -k_I \int_0^t \frac{\partial V_1}{\partial z_r}(z(s)) ds + \frac{\gamma a \varphi(t)}{\gamma(t)}.$$
The closed-loop system obtained by inserting $u_{st}$ in System (11) can be written as

$$\begin{align*}
\dot{z}_1 &= z_2, \\
\vdots \\
\dot{z}_r &= k_P u_0(z) + \xi + \gamma(t) - \gamma_d(k_P u_0(z) + \xi), \\
\dot{\xi} &= -k_I \frac{\partial V_1}{\partial z_r} + \frac{\gamma_d}{\gamma^2} \dot{\phi}_\gamma - \dot{\phi}_\gamma \gamma_d k_P u_0(z) + \xi \\
\end{align*}$$

Eq. (21) becomes

$$\dot{W} \leq -c W^{1/2} + \delta_d \left| \frac{\partial W}{\partial z_r} \right| k_P u_0(z) + \xi + \frac{\partial W}{\partial \xi} \frac{\gamma_d (\varphi_M + \varphi_{M'})}{\gamma^2}.$$ 

From the above inequality, one finished as in the proof of Theorem 3.

3.4 Examples for feedbacks $u_0$ and Lyapunov functions $V_1$ verifying the assumptions of Theorems 2 and 3

In this subsection, we provide several examples of continuous controllers $u_0$ (or quasi-continuous and continuous at $z = 0$) together with Lyapunov functions $V_1$ satisfying all the conditions of Theorem 2 in the case $r \geq 2$.

3.4.1 Hong’s controller

Such a controller is simply borrowed from [Hon02]. In that reference, the convergence is proved by using a Lyapunov function $V$ explicitly constructed for that purpose. The latter function does not match the the assumptions of Theorem 2 and we have to modify it to get the required Lyapunov function $V_1$.

The controller provided in [Hon02] is defined as follows.

Let $\kappa = \frac{-1}{r+1}$ and $l_1, \ldots, l_r$ positive real numbers. We define, for $i = 0, \ldots, r + 1$, $p_i = 1 + (i - 1)\kappa$ and the functions

$$\begin{align*}
v_0 &= 0, \\
v_{i+1} &= -l_{i+1} \left[ [z_{i+1}]^{\beta_i} - [v_{i}]^{\beta_i} \right]^{\alpha_{i+1}/\beta_i}, \\
v_r &= -l_r \left[ [z_r]^{\beta_{r-1}} - [v_{r-1}]^{\beta_{r-1}} \right]^{\alpha_r/\beta_{r-1}},
\end{align*}$$

where $\alpha_i = \frac{p_{i+1}}{p_i}$, for $i = 1, \ldots, r$, and

$$\beta_0 = p_2, \quad (\beta_i + 1)p_{i+1} = \beta_0 + 1 > 0, \quad i = 1, \ldots, r - 1.$$ (27)

Then the controller $u_0$ is taken to be equal to $v_r$ and it is clearly continuous. The fact that such $u_0$ stabilises System (5) in finite-time is demonstrated by using the Lyapunov function $V$ defined as

$$V = \sum_{i=1}^r W_i,$$
where the positive real-valued functions $W_i$, $1 \leq i \leq r$ are given by

$$W_i = \int_{v_{i-1}}^{z_i} w_i(z_1, \cdots, z_{i-1}, s) ds,$$

with $w_i$ is defined as

$$w_i = [z_i]^{\beta_i-1} - [v_{i-1}]^{\beta_i-1}, \quad i = 1, \cdots, r.$$  

We use $\dot{V}$ to denote the time derivative of $V$ along every non trivial trajectory of the closed-loop system obtained from (5) and $u = u_0$. Then, there exists a choice of constants $l_1, \cdots, l_r$ such that the following inequality holds

$$\dot{V} \leq -lV^{\frac{2\lambda+2\alpha}{\lambda}},$$

for some positive constant $l$. Let $\lambda := \frac{2}{2r-1} < 1$. Note that

$$1 - \lambda = \frac{\beta_{r-1}}{1 + \beta_{r-1}}.$$  

Take now $V_1 = \frac{V^\lambda}{\lambda}$. A simple computation yields

$$\frac{\partial V_1}{\partial z_r} = V^{\lambda-1} \frac{\partial V}{\partial z_r} = V^{\lambda-1} \frac{\partial W_r}{\partial z_r} = \frac{[z_r]^{\beta_r-1} - [v_{r-1}]^{\beta_{r-1}}}{V^{\frac{\beta_{r-1}}{\beta_{r-1}+1}}}.$$  

One then checks that $\frac{\partial V_1}{\partial z_r}$ is indeed homogeneous of degree zero with respect the family of dilations $\delta_\epsilon$ associated with $(p_1, \cdots, p_r)$.

**Remark 2** In the previous controller, the term $\frac{\partial V_1}{\partial z_r}$ is globally bounded, and continuous except at the origin.

### 3.4.2 Quasi-continuous modified Hong’s Controller

The following quasi-continuous controller is a hybrid form between the continuous controller presented by Hong [Hon02] and a terminal sliding mode approach also presented by Hong et al. in [HYCS04]. Note that its form is very close to the controller proposed in [HYCS05].

Let $k = \frac{-1}{r+1}$ and $l_1, \cdots, l_r$ positive real numbers. We define, for $i = 0, \ldots, r - 1$, $p_i = 1 + (i - 1)k$

$$v_0 = 0,$$

$$v_{i+1} = l_{i+1} [z_{i+1}]^{\beta_i} + [v_i]^{\beta_i} \alpha_{i+1}/\beta_{i-1}, \quad i = 1, \cdots, r - 2,$$

where $\alpha_i = \frac{p_{i+1}}{p_i}$, for $i = 1, \ldots, r$, and the $\beta_i$’s verify Eq. (27). Then, for $i = 0, \ldots, r - 1$, we define

$$v_0 = 0,$$

$$v_{i+1} = -l_{i+1} [z_{i+1}]^{\beta_i} - [v_i]^{\beta_i} \alpha_{i+1}/\beta_{i-1}, \quad i = 1, \cdots, r - 2,$$

$$v_r = -l_r \left( [z_r]^{\beta_{r-1}} + [v_{r-1}]^{\beta_{r-1}} \right)^{\alpha_r/\beta_{r-1}} \text{sign}(z_r - v_{r-1}).$$

16
The controller $u_0$ is then taken equal to $v_r$ and it stabilises System (5) in finite-time. To see that, one considers the positive definite function $\tilde{V}_1 = \sum_{i=1}^{r} W_i$ where, for $i = 1, \ldots, r$, one has

$$W_i = \int_{v_{i-1}}^{z_i} w_i(z_1, \ldots, z_{i-1}, s) ds,$$  \hspace{1cm} (32)

with

$$w_i = [z_i]^{\beta_i - 1} - [v_{i-1}]^{\beta_i - 1}, \quad i = 1, \ldots, r - 1,$$

$$w_r = \left( |z_r|^{\beta_r - 1} + |\bar{v}_{r-1}|^{\beta_r - 1} \right) \text{sign} (z_r - v_{r-1}).$$

Then we get

$$W_i = \frac{|z_i|^{\beta_i - 1 + 1} - |v_{i-1}|^{\beta_i - 1 + 1}}{\beta_i + 1} - |v_{i-1}|^{\beta_i - 1} (z_i - v_{i-1}), \quad i = 1, \ldots, r - 1,$$

$$W_r = \frac{1}{\beta_r + 1} \left( |z_r|^{\beta_r - 1 + 1} - |v_{r-1}|^{\beta_r - 1 + 1} \right) + |\bar{v}_{r-1}|^{\beta_r - 1} |z_r - v_{r-1}|$$  \hspace{1cm} (33)

One deduces from an argument entirely similar to that of [Hon02] that the time derivative of $\tilde{V}_1$ along non trivial trajectories of the closed-loop system satisfies the following differential inequality

$$\dot{\tilde{V}}_1 \leq -l V_1^{2+2K},$$

for some positive constant $l$. Finite-time convergence to the origin follows immediately. Finally remark that the feedback control law $v_r$ is only quasi-continuous but continuous at zero. One can then apply the results given in Section 3.

The actual Lyapunov function $V_1$ is again taken of the form $\tilde{V}_1^\delta / \delta$ with $\lambda = \frac{1}{1 + \beta_r - 1}$. A simple computation yields

$$\frac{\partial V_1}{\partial z_r} = \tilde{V}_1^{\lambda - 1} w_r = \left( \frac{|z_r|^{\beta_r - 1} + |\bar{v}_{r-1}|^{\beta_r - 1}}{\tilde{V}_1^{\beta_r - 1 + 1}} \right) \text{sign} (z_r - v_{r-1}),$$  \hspace{1cm} (34)

and one checks that $\frac{\partial V_1}{\partial z_r}$ is homogeneous of degree zero with respect the family of dilations $\delta_e$ associated with $(p_1, \ldots, p_r)$.

### 4 Simulations

In this section, we verify the effectiveness of our design through simulations. We deal with a chain of integrator of order four and we show the robusteness with respect to perturbations.

Consider the fourth order integrator system given by

$$\dot{z}_1 = z_2,$$

$$\dot{z}_2 = z_3,$$

$$\dot{z}_3 = z_4,$$

$$\dot{z}_4 = u + \varphi(t).$$  \hspace{1cm} (35)
We study in the following subsections the different cases where \( \varphi \equiv 0 \), and \( \varphi \neq 0 \), for the continuous and the quasi-continuous HOST controllers. For all subsequent simulations, the control parameters are tuned as follows:

\[
l_1 = 1, \quad l_2 = 1, \quad l_3 = 4, \quad l_4 = 8, \quad \kappa = -1/5,
\]

with the initial condition

\[
z_1(0) = -5, \quad z_2(0) = 2, \quad z_3(0) = z_4(0) = 4.
\]

### 4.1 Simulation of pure integrator chain for \( u = u_0 \)

We start by stabilizing the pure integrator chain, (i.e., with \( \varphi \equiv 0 \)) by the controller \( u = u_0 \) where \( u_0 \) represents Hong’s Controller or the modified quasi-continuous version given Sections 3.4.1 and 3.4.2 respectively. Figures 1(a) and 2(a) represents the continuous and quasi-continuous controllers presented in Sections 3.4.1 and 3.4.2 respectively. These controllers force the state \((z_1, z_2, z_3, z_4)\) to converge to zero in finite time, as shown in Figure 1(b) and Figure 2(a).

![Figure 1](image1.png)  
**Figure 1**: Pure integrator chain without integration action (Continuous Control)

![Figure 2](image2.png)  
**Figure 2**: Pure integrator chain without integration action (Quasi-Continuous Control)
4.2 Stabilisation of pure integrator chain by HOST - $\varphi \equiv 0$

In this subsection, we show the performance of HOST for the pure integrator chain for $u = k_P u_0 - k_I \int \frac{\partial V_1}{\partial z_4} dt$. The simulation parameters related to $u_0$ with the initial condition are tuned as in the previous subsection. The gains $k_P$ and $k_I$ are given as follows

$$k_P = 1, \quad k_I = 1.$$  \(38\)

The state convergence is presented in Figure 3(b) and 4(b) for continuous and quasi-continuous controller given in Figure 3(a) and Figure 1(a) respectively. Figures 3(c) and 4(c) show the continuous integrator action which vanishes to zero as there is no perturbation to compensate.

![Figure 3: Pure integrator chain with integral action (Continuous Control)](image)

4.3 Stabilisation of pure integrator chain by HOST - $\varphi \neq 0$

We consider now the case of a perturbed system with a perturbation $\varphi$ defined as

$$\varphi(t) = \sin(t), \quad |\dot{\varphi}| \leq 1.$$  

Note that $\varphi$ is bounded as well. The result is similar to the previous cases. However the controller acts in order to compensate the perturbation and we can see clearly in Figure 5(a) and Figure 5(c) for the continuous controller that $u(t) = -\varphi(t)$ after convergence to zero of the state. Similar results are obtained in the case of the quasi-continuous controller in Figure 6(a) and Figure 6(c).
Figure 4: Pure integrator chain with integral action (Quasi-Continuous Control)

Figure 5: Perturbed integrator chain with integral action (Continuous Control)
Figure 6: Perturbed integrator chain with integral action (Quasi-Continuous Control)

5 Conclusion

In this paper we propose a general approach to design a continuous controller for a perturbed chain of integrators of arbitrary order generalizing the well-known supertwisting algorithm given for first and second order integrators, cf. [KCM+14]. We have first designed a controller for the pure chain of integrators verifying a geometric condition inspired from [LHC15, HLC]. The corresponding argument for getting convergence in finite time relies on the use of a weak Lyapunov function. As for the perturbed chain of integrators, we partially solve the general problem by using homogeneity arguments applied to an extended differential inclusion. Possible future work consists in addressing the general case of a perturbed chain of integrators and to design quasi-continuous controllers having better stabilization performances. Moreover, in order to choose efficiently the coefficients of the controllers in terms of the problem bounds and to get an estimate of the convergence time, it would also be of great help to have a global strict Lyapunov function in the state variable and the supertwisting variable.

References


