Stabilization of positive constrained T–S fuzzy systems: Application to a Buck converter

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Abstract

This paper deals with the problem of constrained stability and tracking of Takagi–Sugeno (T–S) fuzzy positive systems. Linear programming (LP) is used to insert the constraints in the design phase while imposing positivity in closed-loop. The theoretical results are applied to the buck DC–DC converter which is widely used in the photovoltaic generators. Based on the simulation results success of the method is shown for this application.

1. Introduction

In this paper, we are concerned with the control of the buck DC–DC converter which is widely used in photovoltaic energy transformation. The modelization step of this system gives rise to three aspects: the system is non-linear, the system is a positive one since its commonly used state variables must always be positive and the control is a duty ratio that must be constrained between 0 and 1. Hence, Takagi–Sugeno (T–S) fuzzy techniques can be used to design a control for the non-linear system [20]. Further, during the design step, asymmetrical constraints on the control, while imposing positivity in...
closed-loop, have to be taken into account. In consequence, the problem studied thereafter concerns
a special class of systems called constrained positive T–S models. From the history of the T–S
approach, this class can be interpreted as a collection of linear models interconnected by nonlinear
functions, called membership functions, which are time dependent variables. The most delicate problem
is the choice of premise variables that partition the space [21].

Positive systems have been of great interest to researchers in recent years [9,1,11,15,16,6]. The
class of positive T–S fuzzy systems was considered for the first time in [3]. The obtained results
were presented using LMI.s.

The buck converter has attracted the interest of many researchers [17,14,13,10,19] and
references therein. Almost all the available methods in the literature do not pay attention to
neither the positive aspect of the system nor the asymmetrical constraints on the control of the
buck converter. Based on the model given in [14], the tracking of the reference is realized upon
an error model using LMI tools in [13]. The constraints on the control were not considered nor
shown in simulation despite the main interest to obtain a duty ratio between 0 and 1. In [17], by
using backstepping and sliding mode techniques, the authors succeed to obtain an output voltage
following a given reference with a variable load resistor and a variable source voltage. However,
the duty ratio is shown by simulation to be restricted between 0 and 1 without taking this
constraint into account in the design phase. In [19,10], a robust output tracking approach was
proposed using LMI tool without taking into account the constraint aspect of the duty ratio. Note
that [10] used a different model of the buck converter.

In this paper, from a practical point of view, the real process consisting of a buck DC–DC
converter taken from [14] is studied. As this continuous-time system is nonlinear, the T–S fuzzy
representation is followed. Since, this system is positive and its control (duty ratio) is
asymmetrically constrained between 0 and 1 and no available control theory exists in the
literature to handle this problem, new theoretical developments are needed.

Hence, from a theoretical point of view, the conditions for stabilization of continuous-time T–S
systems with constrained control, not only for the buck converter but also in general case, are
presented. The constraints on the control are taken into account during the design phase while
imposing positivity of the state. The results are obtained by using linear programming (LP) for
continuous-time T–S systems which are different from those given in [8,7] concerning only
discrete-time T–S fuzzy systems. As an important consequence of this result, a corollary is
deduced for the stabilization of T–S fuzzy systems with positive control (0 ≤ u(t) ≤ u). The
obtained results are successfully applied to the buck converter system. In fact, the simulations
show that all requirements are achieved, namely, the tracking of a reference signal for the
capacity voltage, the positivity of the closed-loop states and finally the limitation of the duty ratio
between 0 and 1.

The rest of this paper is organized as follows: In Section 2, the model of the buck converter is
presented while Section 3 deals with the description of the T–S fuzzy model and a fuzzy control
law based on PDC structure is given for the buck converter. New stabilization conditions are
established for positive systems with an asymmetrically constrained control in Section 4. In the
same section, the application to the buck converter is given to show the need of such controllers
for such systems. Some conclusions are given in Section 5.

Notation:

• \( A^T \) denotes the transpose of a real matrix \( A \).
• A matrix \( A \in \mathbb{R}^{n \times n} \) is called a Metzler matrix if its off-diagonal elements are non-negative.

That is, if \( A = \{a_{ij}\}_{i,j=1}^n \), \( A \) is Metzler if \( a_{ij} \geq 0 \) whenever \( i \neq j \).
2. Problem formulation

The T–S model of a DC–DC buck converter, according to [13], is given by

\[
\dot{x}(t) = Ax(t) + B(x)u(t) + D
\]

\[0 \leq u(t) \leq 1\]

(1)

where

\[
x(t) = \begin{bmatrix} i(t) \\ v(t) \end{bmatrix} \in \mathbb{R}^2
\]

(3)

\[
A = \begin{bmatrix}
-\frac{R_L + \frac{R_{M}E}{CR+C}}{L} & -\frac{R}{CR+LR_C} \\
\frac{R}{CR+C} & -\frac{1}{CR+CR_C}
\end{bmatrix}
\]

(4)

\[
B(x) = \begin{bmatrix}
-\frac{R_L}{L}E - V_C \\
\frac{V_C}{L}
\end{bmatrix}
\]

(5)

\[
D = \begin{bmatrix}
-\frac{V_C}{L} \\
0
\end{bmatrix}
\]

(6)

with \(i(t), v(t)\) representing the inductance current and the capacity voltage respectively. \(u(t)\) is the duty ratio of the MOSFET which must be within the interval \([0, 1]\). \(R_C, R_L, R_M, E,\) and \(V_C\) are specified by Fig. 1.

The objective of this work is to design a control law limited between 0 and 1, representing the duty ratio, allowing the system to switch between two topologies in the continuous conduction mode (CCM) by achieving a tracking reference \(v_{ref}\). Further, since the system is nonlinear, the followed approach thereafter is the so-called Takagi–Sugeno (T–S) technique.
3. Preliminaries

In order to build a suitable control law, we use the T–S modelization. For this we denote \( M_1 = \min(i(t)) \), \( M_m = \max(i(t)) \) and we divide the interval \([M_1, M_m]\) into \( r \) subintervals \( I_s = [M_{s-1}, M_{s+1}] \), \( s = 2, \ldots, r - 1 \). Consider the following fuzzy rules:

IF \( i(t) \in S_s \) THEN
\[
\dot{x}(t) = Ax(t) + B_s u(t) + D_s, \quad s = 1, \ldots, r
\] (7)
\[0 \leq u(t) \leq 1\] (8)

where
\[
B_s = \begin{bmatrix}
-\frac{R_m M_s - E - V_f}{L} \\
0
\end{bmatrix}
\]
and \( S_s \) is the fuzzy set of the fuzzy region \( I_s \), \( r \) denotes the number of rules. The global TS fuzzy model of the buck converter is given by
\[
\dot{x}(t) = \sum_{s=1}^{r} h_s(x(t))(Ax(t) + B_s u(t)) + D
\] (9)

where
\[
h_s(x(t)) = \frac{\mu_s(x(t))}{\sum_{s=1}^{r} \mu_s(x(t))};
\]
\[
\mu_s(x(t)) = \prod_{j=1}^{r} S_j
\]

It is worth noting that according to the circuit of the buck converter given by Fig. 1, the state space vector is positive. Hence, one has to maintain this state positive during all the evolution of the system.

Make the following change of variable to transform the affine system to a non-affine one: \( B_s u(t) + D = B_s (u(t) + d(t)) = B_s w(t) \). For the case of our system, one has: \( d(t) = V_f / (R_m i(t) - E - V_f) \). Note that the new variable \( d(t) \) is varying time and is not suitable for the T–S fuzzy modeling. As \( i(t) \) belongs to \([0, i_{\text{max}}]\), one can choose the worst case \( d = -V_f / (E + V_f) \). The system becomes
\[
\dot{x}(t) = \sum_{s=1}^{r} h_s(x(t))(Ax(t) + B_s w(t)),
\] (10)
\[-d \leq w(t) \leq 1 + d\] (11)

To achieve our objective of the tracking of a given reference of the voltage capacity, we introduce the following idea. Let us consider the new control given by the following PDC control:
\[
w(t) = \sum_{s=1}^{r} h_s(x(t))(K_s x(t) + F_s x_{\text{ref}})
\] (12)
where \( x_{\text{ref}} \) is the state to be followed by the closed-loop system and the gains \( F_j \) are designed to achieve the tracking problem. Hence, the closed-loop system becomes

\[
\dot{x}(t) = \sum_{s=1}^{r} \sum_{j=1}^{r} h_s(x(t))h_j(x(t))((A + B_sK_j)x(t) + B_sF_jx_{\text{ref}})
\]

with \( G_{sj} = A + B_sK_j \). Similar notation is used for \( K_j \) and \( F_j \). Taking the steady state of each local closed-loop system, one can obtain \( 0 = G_{ss}x(\infty) + B_sF_sx_{\text{ref}} \). Since the objective is that \( x(\infty) = x_{\text{ref}} \), one can choose \( F_s = -B_s^*G_{ss} \), with \( B_s^* \) representing the pseudo-inverse of matrix \( B_s \), i.e., \( B_s^* = (B_s^T B_s)^{-1}B_s^T \).

Note that the above new system (10) admits unsymmetrical constraints. Consequently, one needs to take into account the unsymmetrical constraints during the design step. It is worth noting that, to the best of our knowledge, no theoretical approach exists in the literature to solve this problem for T–S fuzzy positive systems except the work of [3,4] where the problem is solved with LMIs but only for symmetrical constraints. A different Linear Programming based approach was proposed in [8] for T–S fuzzy discrete-time systems with delays and unsymmetrical constraints. This paper extends this new Linear Programming based technique to T–S fuzzy continuous-time systems.

4. Main result

In this section, we present the control law that realizes the tracking objective while respecting the constraints on the control \( u(t) \) in the first part. In the second part, the application of the obtained results to the buck converter is presented.

4.1. Controller synthesis

Consider the general T–S fuzzy system given by

\[
\dot{x}(t) = \sum_{s=1}^{r} h_s(x(t))(Ax(t) + B_s u(t))
\]

\[ -\bar{u} \leq u(t) \leq \bar{u}, \quad x_0 \geq 0 \]

with \( x \in \mathbb{R}^n, u \in \mathbb{R}^m \). For the sake of brevity, \( h(x(t)) \) will be noted by \( h(t) \). The T–S fuzzy system in closed-loop with a PDC control is given by

\[
\dot{x}(t) = \sum_{s=1}^{r} \sum_{j=1}^{r} h_s(x(t))h_j(x(t))(A + B_sK_j)x(t)
\]

\[ = \sum_{s=1}^{r} \sum_{j=1}^{r} h_s(x(t))h_j(x(t))G_{sj}x(t), \quad x_0 \geq 0 \]

Lemma 4.1. For the initial condition \( 0 \leq x_0 \leq \lambda \) and Metzler matrices \( G_{sj} \), the following statements are equivalent:

1. The state of the T–S fuzzy closed-loop system (14) is confined \([0, \lambda] \).
2. \( G_{sj}\lambda < 0, s,j = 1, \ldots, r \).
Theorem 4.1. If there exist a positive vector $\lambda \in \mathbb{R}^n$ and positive vectors $y^1_j, \ldots, y^n_j \in \mathbb{R}^m$, $j = 1, \ldots, r$, $z^1_j, \ldots, z^n_j \in \mathbb{R}^n$, $j = 1, \ldots, r$, such that the following Linear Programming is feasible:

\[
(LP)\begin{cases}
A\lambda + \sum_{l=1}^n B_l(y^l_j - z^l_j) < 0 \\
a_{kl}\lambda_l + b^l_k(y^l_j - z^l_j) \geq 0, \quad k \neq l \\
\sum_{l=1}^n y^l_j \leq \bar{u}, \\
\sum_{l=1}^n z^l_j \leq u, \\
s, j = 1, \ldots, r, \\
k, l = 1, \ldots, n,
\end{cases}
\]

then the closed-loop $T$–$S$ fuzzy system (14) is asymptotically stable at the origin while the state remains positive and the control respects $-\underline{u} \leq u(t) \leq \bar{u}$.

The gain matrix is given by

\[
K_j = \begin{bmatrix}
\frac{y^1_j - z^1_j}{\lambda_1} & \frac{y^2_j - z^2_j}{\lambda_2} & \cdots & \frac{y^n_j - z^n_j}{\lambda_n}
\end{bmatrix}
\]

(15)

with

\[
B_s = \begin{bmatrix}
b^1_s \\
b^2_s \\
\vdots \\
b^n_s
\end{bmatrix}
\]

(16)

Proof. Consider the Lyapunov function candidate

\[
V(x(t)) = x(t)^T \lambda, \quad \lambda \in \mathbb{R}^n_+
\]

(17)

Let us compute its derivative along the dual system trajectories given by

\[
\dot{x}(t) = \sum_{s=1}^r \sum_{j=1}^r h_s(x(t))h_j(x(t))G_{sj}^T x(t)
\]

(18)
According to (LP) conditions, let

\[ K \lambda = \sum_{l=1}^{n} (y_j^l - z_j^l), \]

one has \( G_{sj} \lambda = A \lambda + \sum_{l=1}^{n} B_s (y_j^l - z_j^l) < 0, \ s, j = 1, \ldots, r. \) To ensure that the state will always be positive, we have to satisfy that the closed-loop matrices are Metzler: let \( a_{kl} \lambda_l + b_k^l (y_j^l - z_j^l) \geq 0, k \neq l, 1, \ldots, n, \) is equivalent to \( a_{kl} + b_k^l (y_j^l - z_j^l)/\lambda_l \geq 0, \) note that \( K_j^l = (y_j^l - z_j^l)/\lambda_l, \) then one has \( a_{kl} + b_k^l K_j^l \geq 0 \) which implies \( (A + B_s K_j)(k, l) \geq 0, k \neq l, 1, \ldots, n; \) that is, matrices \( A + B_s K_j \) are Metzler. Using Lemma 4.1, condition \( A \lambda + \sum_{l=1}^{n} B_s (y_j^l - z_j^l) < 0 \) is necessary and sufficient to have \( 0 \leq x(t) \leq \lambda. \)

On the other hand, using \( K_j = [(y_j^1 - z_j^1)/\lambda_1 (y_j^2 - z_j^2)/\lambda_2 \ldots (y_j^n - z_j^n)/\lambda_n], \) by letting \( K_j^+ \lambda = \sum_{l=1}^{n} y_j^l \) and \( K_j^- \lambda = \sum_{l=1}^{n} z_j^l \) where \( K_j^+ \) and \( K_j^- \) are positive matrices, and \( K_j \lambda = (K_j^+ - K_j^-) \lambda. \) Now to have a constrained control, we also add the following conditions \( y_j^l > 0, z_j^l > 0, \) while imposing that \( \sum_{l=1}^{n} y_j^l \leq \overline{u} \) and \( \sum_{l=1}^{n} z_j^l \leq \underline{u}. \) In fact, for \( 0 \leq x(t) \leq \lambda, \) one has the following developments for positive gains \( K_j^+ \) and \( K_j^- : \)

\[ 0 \leq K_j^+ x(t) \leq K_j^+ \lambda, \quad -K_j^- \lambda \leq -K_j^- x(t) \leq 0 \]

Summing up these two inequalities, one has

\[ -K_j^- \lambda \leq (K_j^+ - K_j^-) x(t) \leq K_j^+ \lambda, \quad -K_j^- \lambda \leq K_j x(t) \leq K_j^+ \lambda \]

By virtue of (15), it follows that

\[ -\sum_{l=1}^{n} z_j^l = -K_j^- \lambda \leq K_j x(t) \leq K_j^+ \lambda = \sum_{l=1}^{n} y_j^l \]

According to (LP) conditions,

\[ -\underline{u} \leq K_j x(t) \leq \overline{u} \]

Summing up on \( j \) and bearing in mind that \( h_j(t) \geq 0 \) and \( \sum_{j=1}^{r} h_j(t) = 1, \) one has

\[ -\sum_{j=1}^{r} h_j(t) \underline{u} \leq \sum_{j=1}^{r} h_j(t) K_j x(t) \leq \sum_{j=1}^{r} h_j(t) \overline{u} \]

Consequently, \( -\underline{u} \leq u(t) \leq \overline{u}. \) \( \square \)
Corollary 4.1. If there exist a positive vector $\lambda \in \mathbb{R}^n$ and positive vectors $y_1^j, \ldots, y_n^j \in \mathbb{R}^m$, $j = 1, \ldots, r$, such that the following Linear Programming is feasible:

$$(LP) \quad \begin{align*}
A\lambda + \sum_{l=1}^n B_{3l} y_j^l &< 0 \\
\sum_{l=1}^n a_{kl}\lambda_l + b_{kl} y_j^l &\geq 0, \quad k \neq l \\
\sum_{l=1}^n y_j^l &\leq \pi, \\
s, j = 1, \ldots, r, \\
k, l = 1, \ldots, n,
\end{align*}$$

then the closed-loop T–S fuzzy system (14) is asymptotically stable at the origin while the state remains positive and the control respects $0 \leq u(t) \leq \pi$.

The gain matrix is given by

$$K_j = \begin{bmatrix}
y_1^j \\
\lambda_1 \\
y_2^j \\
\lambda_2 \\
\vdots \\
\vdots \\
y_n^j \\
\lambda_n
\end{bmatrix}$$

(21)

Proof. The proof is directly deduced from Theorem 4.1 by letting $u = 0$. □

The result of Corollary 4.1 can also be applied to positive systems in open-loop with $A$ Metzler and $B$ positive.

Comment 4.1. The results above open up new possibilities of taking into account a null constraint on the left side of the control. In fact, this null constraint cannot be handled with methods that already exist in the literature. Symmetrically saturating controls are centered at the origin and hence null constraints have no sense in these LMI based approaches and similar ones in [12,18,2]. However, in the positive invariance approach, null constraints can be handled, but only with algebraic conditions [6].

4.2. Application to buck converter

In this section, the results of the previous section are used. However, the actual problem is a tracking one while the results of Theorem 4.1 concern only stabilization problems. This explains the transformations leading to the control law (12). To obtain a simple T–S fuzzy model of the buck converter, we choose one interval with $r = 2$ rules, as follows:

IF $i(t) = M_1 = i_{\text{min}}$ THEN,

$$B_1 = \begin{bmatrix}
-R_{\text{oh}} + E + V_i \\
L \\
0
\end{bmatrix}$$

IF $i(t) = M_2 = i_{\text{max}}$ THEN,

$$B_2 = \begin{bmatrix}
-R_{\text{oh}} + E + V_i \\
L \\
0
\end{bmatrix}$$
the membership functions are given by

\[ h_1(t) = \frac{i_{\text{max}} - i(t)}{M_2 - M_1} \]

\[ h_2(t) = \frac{i(t) - i_{\text{min}}}{M_2 - M_1} \]

Data of the studied buck converter are given in Table 1 [13], with \( i_{\text{max}} = 3\text{A} \) and \( i_{\text{min}} = 0\text{A} \). The (LP) program applied to the buck converter model is feasible for the following values:

\[ \lambda = \begin{bmatrix} 0.2254 \\ 1.7859 \end{bmatrix}, \quad d = -0.0266; \]

\[ y_1^1 = y_1^2 = 0.0036; \quad y_2^1 = y_2^2 = 0.0634; \]

\[ z_1^1 = z_1^2 = 0.0206; \quad z_2^1 = z_2^2 = 0.0024; \]

\[ K_1 = [-0.05756 \quad 0.0342], \quad K_2 = K_1; \]

**Table 1**

Data of the buck converter.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R )</td>
<td>6 Ω</td>
</tr>
<tr>
<td>( R_L )</td>
<td>48.8 mΩ</td>
</tr>
<tr>
<td>( E )</td>
<td>30 V</td>
</tr>
<tr>
<td>( L )</td>
<td>98.58 mH</td>
</tr>
<tr>
<td>( R_c )</td>
<td>0.162 Ω</td>
</tr>
<tr>
<td>( R_M )</td>
<td>0.27 Ω</td>
</tr>
<tr>
<td>( V_f )</td>
<td>0.82 V</td>
</tr>
<tr>
<td>( C )</td>
<td>202.5 μF</td>
</tr>
</tbody>
</table>

![Fig. 2. Membership functions.](image)
\[ Ac_1 = A + B_1 K_1 = 10^3 \begin{bmatrix} -0.0257 & 0.0008 \\ 4.8084 & -0.8014 \end{bmatrix}, \]
\[ Ac_2 = A + B_1 K_2 = 10^3 \begin{bmatrix} -0.0251 & 0.0005 \\ 4.8084 & -0.8014 \end{bmatrix}; \]
\[ Ac_3 = A + B_2 K_1 = Ac_1 \]
\[ Ac_4 = A + B_2 K_2 = Ac_2 \]

The simulation results are shown in Figs. 3 and 4. Membership functions are depicted in Fig. 2. Fig. 3 presents the trajectories of the capacity voltage and the inductance current while Fig. 4 plots the duty ratio \( u(t) \). One can notice the important fact that the evolution of the duty ratio is effectively between 0 and 1. Furthermore, the tracking of the desired trajectories is achieved as specified.
The simulation results obtained by LMIs are shown in Figs. 5 and 6. One can notice the important fact that the evolution of the duty ratio, which must be between 0 and 1, violates these limitations at each change of the followed reference. Besides, the tracking of the specified trajectories is achieved but with a steady error.

**Comment 4.2.**

- To achieve the tracking goal, gains $F_j$ were introduced. However, for the constraints, the presented theoretical results do not take into account these gains. Nevertheless, as $F_j$ are small for the buck converter example, the constraints on the gains $K_j$ are sufficient to compensate the variation of the tracking gains $F_j$. This drawback is still unsolved from a theoretical point of view.
- It is worth noting that as no existing results in the literature deal neither with positive system nor with asymmetrical constraints (even null constraint), no constructive comparison with our
result can be done. However, using standard stabilization LMIs given by

\[ AX + B_i Y_j + X A^T + Y_j^T B_i^T < 0, \quad i = j = 1, \ldots, r \]  

(22)

The results obtained with these LMIs are shown in Figs. 5 and 6 following the same reasoning for computing tracking gains as presented above. One can notice that the evolution of the duty ratio exceeds the physical limits at each change of the reference to be followed while the tracking is achieved with a static error.

5. Conclusion

The conditions for stabilization of T–S fuzzy systems with constrained control while imposing positivity are presented. The constraints are taken into account during the design phase using an LP approach. The obtained solution to the LP program enables one to deduce the controller gains for each linear model. Similar membership functions are used to design the global controller that respects the control constraints and to achieve the tracking of the reference signal without violating the positivity of the system. The obtained results are applied to the buck DC–DC converter. Simulation results show the effectiveness of the approach. Particularly, the duty ratio respects the physical limitation between 0 and 1 which was shown in Fig. 4. Further, a comparison with LMI based results is achieved showing the usefulness of our result with respect to positivity and saturation.

References


