Positive Constrained Control for
Continuous-time Fractional Positive Systems

F. Mesquine, A. Benzaouia, A. Hmamed, M. Benhayoun and F. Tadeo

Abstract

The stabilization problem for fractional linear continuous-time systems having bounded positive control is solved, with the additional condition of non negativity of the states. Thus, a methodology to develop state-feedback controllers is proposed, based on Linear Programming conditions. An illustrative example is provided to show the usefulness of the results.

Keywords: Fractional systems, positive systems, stabilization, bounded controls, linear programming.

I. INTRODUCTION

The first introduction of fractional derivative was proposed by Liouville and Riemann at the XIX century as pointed out in [20]. Background works on mathematical fractional calculus can be found in [24], [20], [22]. The first work on control of fractional systems was given in [22]. Recent works on stability and robust stability were studied extensively in [23], [15], [16], [21]. Most of these papers use algebraic stability conditions. A few papers use the Lyapunov concept to deal with the problem of analysis stability of fractional systems, for example [17]. We can emphasize [25], that presents an interesting discussion about the meaning of Lyapunov function concept for fractional systems. Besides, a condition of stability similar to ordinary systems is obtained for a quadratic Lyapunov function candidate. New developments based on Lyapunov equations can be found in [8].

This paper concentrates on a specific class of fractional systems: those with nonnegative states. In the literature, systems with non negative states are referred as positive systems (see [14] for general references). Nonnegative states appear in many practical problems, when the states represent
physical quantities that have an intrinsically constant sign (Absolute temperatures, levels, heights, concentrations, etc).

Some recent works in positive systems [1], [2], [11], [12], [13] provide new treatments for the stabilization of positive linear systems where all the proposed conditions are necessary and sufficient, and expressed in terms of Linear Programming (LP). It was shown in these works that the constraints on the control can be handled easily. This topic is of continuing interest (see [3], [4], [5], [18], [6], [7] and references therein), and is applied here for fractional systems. More precisely, the approach for stabilization of MIMO positive fractional linear systems by state feedback control presented in [8] is applied here to derive stabilizing conditions in the presence of constraints. This result enables to synthesize stabilizing controllers, that respect all requirements: bounds on the control and the state are taken into account while maintaining positiveness and asymptotic stability. The synthesis problem is solved by using linear programming approach, with the results derived from the work of [17] for direct Lyapunov function and [15], [16] for positive fractional systems.

The remainder of the paper is structured as follows: Section 2 deals with the problem statement while some preliminary results are recalled in Section 3. Section 4 is devoted to the main results concerning stabilizability and boundedness of the state and the control. Besides, an example is given in this section to illustrate the proposed method. Some remarks conclude the paper.

A. Notation and definitions

• $\mathbb{R}_+^n$ denotes the non-negative orthant of the $n$-dimensional real space $\mathbb{R}^n$.
• $M^T$ denotes the transpose of the real matrix $M$.
• A matrix $M \in \mathbb{R}^{n \times n}$ is called a Metzler matrix if its off-diagonal elements are nonnegative. That is, for $M = \{m_{ij}\}_{i,j=1}^n$, $M$ is Metzler if $m_{ij} \geq 0$ when $i \neq j$.
• A matrix $M$ (or a vector) is said to be nonnegative if all its components are nonnegative (by notation $M \geq 0$). It is said to be positive if all its components are positive ($M > 0$).

II. Problem statement

Consider the continuous-time commensurate fractional order linear system:

$$
\begin{cases}
D^\alpha x(t) = Ax(t) + Bu(t) \\
0 < \alpha \leq 1 \\
x(0) = x_0 \geq 0
\end{cases}
$$

(1)

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control and $D^\alpha x(t)$ represents the Riemann-Liouville fractional derivative of $x(t)$ defined by:
\[ D^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left( \int_0^t \frac{x(\tau)}{(t-\tau)^\alpha} d\tau \right) \]

\[ 0 < \alpha \leq 1 \]

and \( \alpha \in \mathbb{R} \) is the order of the fractional derivative. The Gamma function is defined by:

\[ \Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, z \in \mathbb{R}. \]

The Caputo fractional derivative is defined as follows:

\[ ^cD^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{dx(\tau)}{(t-\tau)^\alpha} d\tau \]

\[ 0 < \alpha \leq 1, \]

Note that the Caputo and Riemann-Liouville fractional derivatives are linked as follows:

\[ D^\alpha x(t) = ^cD^\alpha x(t) + \frac{t^{-\alpha}}{\Gamma(1-\alpha)} x_0 \]

The problem we are addressing below is to use conditions given in [8] on matrices \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times m} \), such that there exists a state feedback matrix \( K \in \mathbb{R}^{m \times n} \) satisfying:

- The fractional system is positive in closed-loop.
- The fractional closed-loop system is asymptotically stable (\( \lim_{t \to \infty} x(t) = 0 \)) while satisfying positive bounded constraints.

### III. Preliminaries

Consider the continuous-time fractional linear autonomous system:

\[
\begin{cases}
D^\alpha x(t) = Ax(t) \\
0 < \alpha \leq 1 \\
x_0 \geq 0
\end{cases}
\]  

The solution of this fractional system is given by:

\[ x(t) = \mathcal{E}_\alpha (At^\alpha) x_0, \]

where \( \mathcal{E}_\alpha (\cdot) \) represents the Mittag–Leffler function defined by:

\[ \mathcal{E}_\alpha (At^\alpha) = \sum_{k=0}^{\infty} \frac{(At^\alpha)^k}{\Gamma(k\alpha + 1)}. \]

**Definition 3.1:** Given any positive initial condition \( x_0 \in \mathbb{R}_+^n \), the system (6) is said to be positive if the corresponding trajectory is never negative: \( x(t) \in \mathbb{R}_+^n \) for all \( t \geq 0 \).
According to this definition, a condition for the system (6) to be positive is given by the following result.

**Lemma 3.1:** [15] The continuous-time fractional system (6) is positive if and only if the matrix $A$ is a Metzler matrix.

**Theorem 3.1:** [17] Assume that there exists a Lyapunov function $V(t, x(t))$ and class-$K$ functions $\beta_i, i = 1, 2, 3$ satisfying:

\[
\begin{cases}
\beta_1(\|x(t)\|) \leq V(t, x(t)) \leq \beta_2(\|x(t)\|) \\
cD^\alpha V(t, x(t)) \leq -\beta_3(\|x(t)\|)
\end{cases}
\]  

(9)

where $\alpha \in (0, 1)$, then the system (6) is asymptotically stable.

It is worth noting that it was also proven in [17] that the result of Theorem 3.1 remains valid when replacing the Caputo fractional derivative by the Riemann-Liouville fractional derivative.

Recall now a fundamental result of stability provided by [19] for the autonomous commensurate fractional order system (6).

**Theorem 3.2:** System (6) is bounded input bounded output stable if and only if

\[
|\arg(\lambda_i(A))| > \frac{\alpha \pi}{2}, i = 1, \ldots, n,
\]

(10)

where $\lambda_i(A)$ stands for the $i^{th}$ eigenvalue of matrix $A$.

**Remark 3.1:** The above stability condition corresponds to the system having eigenvalues inside a large region, which can include a sector of the right half complex plane. However, for fractional positive systems, it was shown in [16] that the eigenvalues of the system belong only to the left half complex plane.

Consider the continuous-time fractional linear system given by (1). If one uses a state feedback control

\[
u(t) = Kx(t)
\]

(11)

the closed-loop system becomes

\[
\begin{cases}
D^\alpha x(t) = (A + BK)x(t) \\
0 < \alpha \leq 1 \\
x_0 \geq 0
\end{cases}
\]

(12)

The problem we are dealing with is to design the controller which ensures asymptotic stability while maintaining the state positive even the open-loop system is not positive at all. In this situation, the system is called controlled positive [11].
Theorem 3.3: \[8\] If there exist a positive vector $\lambda \in \mathbb{R}^n$ and vectors $y_1, y_2, \ldots, y_n \in \mathbb{R}^m$ such that
\[
A\lambda + B \sum_{i=1}^{n} y_i < 0
\] (13)
\[
a_{ij}\lambda_j + b_i y_j \geq 0, i \neq j
\] (14)
then the closed-loop fractional system (12) is asymptotically stable when maintaining the state positive for all $x_o \geq 0$. Further, the stabilizing state feedback gain is given by
\[
K = \begin{bmatrix}
y_1 \\
y_2 \\
\vdots \\
y_n
\end{bmatrix},
\] (15)
where $a_{ij}$ represents the element $(ij)$ of matrix $A$ and $b_i$ are the row vectors of $B$.

In order to design controller that satisfies the asymptotic stability while maintaining positivity in closed-loop, linear programming may be used as follows:

Corollary 3.1: \[8\] If there exists a feasible solution to the following LP problem in the variables $\lambda \in \mathbb{R}^n$ and $y_1, \ldots, y_n \in \mathbb{R}^m$:
\[
\begin{align*}
A\lambda + B \sum_{i=1}^{n} y_i &< 0, \\
\lambda &> 0, \\
a_{ij}\lambda_j + b_i y_j &\geq 0, \ i \neq j = 1, \ldots, n,
\end{align*}
\] (16)
Then, the closed-loop system (12) is positive and asymptotically stable for any initial condition $x_o > 0$, under the state-feedback control law $u = K x$, with
\[
K = \begin{bmatrix}
y_1 \\
y_2 \\
\vdots \\
y_n
\end{bmatrix}.
\]

For the sequel development, the following lemma is necessary.

Lemma 3.2: \[10\] For matrix $A \in \mathbb{R}^{n \times n}$, one has
\[
E_\alpha(At^\alpha) - I = D^{-\alpha}(E_\alpha(At^\alpha)A)
\] (17)
Remark 3.2: The equality (17) generalizes the expression
\[
e^{At} - I = \int_0^t e^{A\tau} A d\tau,
\]
to the case of the general Mittag-Leffler exponential.

IV. Main Result

This section studies the problem of closed-loop stabilization and positiveness for fractional bounded positive controls. Consider the following constrained fractional continuous-time system:
\[
\begin{align*}
D^\alpha x(t) &= Ax(t) + Bu(t) \\
0 &< \alpha \leq 1 \\
x(0) &= x_o \geq 0 \\
0 &\leq u(t) \leq \bar{u}
\end{align*}
\] (18)
That is, the trajectory of the closed-loop fractional system is positive and the input is constrained to be bounded by a given value \( \bar{u} \). The aim here is to address the following problem:

Given \( \bar{u} > 0 \) find a positive vector \( \bar{x} \) and a state feedback matrix \( K \) giving a nonnegative and bounded control law \( 0 \leq u = Kx(t) \leq \bar{u} \) such that the following closed-loop system is positive and asymptotically stable:

\[
D^\alpha x(t) = (A + BK)x(t) \\
0 < \alpha \leq 1 \\
x(0) = x_o \geq 0 \\
0 \leq u(t) \leq \bar{u}
\]

(19)

**Lemma 4.1:** [10] Consider the autonomous system (6), for a given \( \bar{x} > 0 \) we have 

\[
0 \leq x(t) \leq \bar{x},
\]

for any initial condition satisfying \( 0 \leq x_o \leq \bar{x} \) if and only if \( Ax \leq 0 \).

Now, we state the main result of this section:

**Theorem 4.1:** For system (18), consider the following LP problem in the variables \( \bar{x} = [\bar{x}_1 \ldots \bar{x}_n]^T \in \mathbb{R}^n \) and \( y_1, \ldots, y_n \in \mathbb{R}^m \):

\[
\begin{align*}
\text{(LP2)} \quad & A\bar{x} + B\sum_{i=1}^n y_i < 0, \\
& \bar{x} > 0, \\
& y_i \geq 0, \quad i = 1, \ldots, n, \\
& \sum_{i=1}^n y_i \leq \bar{u}, \\
& a_{ij}\bar{x}_j + b_iy_j \geq 0, \quad i \neq j = 1, \ldots, n.
\end{align*}
\]

(20)

Then, the closed-loop system (19) is positive and asymptotically stable for any initial condition \( 0 < x_o \leq \bar{x} \), under the state-feedback bounded control law \( 0 \leq u = Kx \leq \bar{u} \), with \( K = [\bar{x}_1^{-1}y_1 \ldots \bar{x}_n^{-1}y_n] \).

**Proof 1:** : Take any \( \bar{x} = [\bar{x}_1 \ldots \bar{x}_n]^T \) and \( y_1, \ldots, y_n \) that solve (20) and define \( K = [\bar{x}_1^{-1}y_1 \ldots \bar{x}_n^{-1}y_n] \). Then, since for \( i \neq j = 1, \ldots, n \),

\[
a_{ij} + b_i\bar{x}_j^{-1}y_j = a_{ij} + b_iK_j = (A + BK)_{ij} \geq 0,
\]

we have that matrix \( A + BK \) is Metzler. The inequality \( A\bar{x} + B\sum_{i=1}^n y_i < 0 \) is equivalent to \( (A + BK)\bar{x} < 0 \). Since \( \bar{x} > 0 \), then by using Theorem 3.3, we can conclude that \( A + BK \) is a Metzler matrix. Further, by Lemma 4.1, the trajectory of the system (18) is such that \( 0 \leq x(t) \leq \bar{x} \) from any initial condition satisfying \( 0 \leq x(0) \leq \bar{x} \). Using this fact and recalling the inequalities \( \sum_{i=1}^n y_i \leq \bar{u}, y_i \geq 0 \) for \( i = 1, \ldots, n \), (or, equivalently, \( K \geq 0 \) and \( K\bar{x} \leq \bar{u} \)), it is easy to see that the state-feedback control \( u = Kx \) is such that \( 0 \leq u(t) \leq K\bar{x} \leq \bar{u} \) for any initial state satisfying \( 0 \leq x(0) \leq \bar{x} \). \[\square\]
A. Example: Positive Bounded System

Consider a fractional continuous-time system with $\alpha = 0.5$ described by (1), with the following system matrices:

$$A = \begin{bmatrix} -1 & -0.5 \\ -0.3 & -0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 0.4 \\ 0.2 \end{bmatrix}$$

Using a state feedback control, we want to impose for the controller to stabilize the system, that the closed-loop system is positive, and that the control signal is non-negative and with a value always smaller than $\bar{u} = 8$. Thus, the conditions of Theorem 4.1 must be fulfilled. One feasible solution to the LP problem (20) provides $K = \begin{bmatrix} 1.8944 & 1.4578 \end{bmatrix}$, with $\bar{x} = \begin{bmatrix} 1.3629 \\ 2.9090 \end{bmatrix}$.

It can be seen that the closed-loop system is positive, that the states are bounded by $\bar{x}$ and the control signal is always non-negative and smaller than $\bar{u} = 8$. For different initial positive conditions (smaller than $\bar{x}$), it can be seen in Figure 1 that the trajectories converge to 0 in the positive orthant. One of the corresponding controls is shown in Figure 2 where it can be seen that the imposed bounds on $u$ are fulfilled. The simulation was achieved with a time of 70s with a step of 0.1s and $k = 400$ in the exponential Mittag-Leffler expression (7).

![State Trajectories from different initial values](image)

Fig. 1. State Trajectories from different initial values

V. CONCLUSIONS

In this paper, the stabilization problem with bounded controls for fractional linear continuous-time systems is addressed, while imposing non-negativeness of the states. These results are obtained using direct Lyapunov function given in [8] leading to synthesis of controllers for the first time. In addition, the synthesis of state-feedback controllers is obtained by expressing the required conditions in terms of simple Linear Programming Problems. The extension of the synthesis method to fractional systems
with bounded control and state is presented. Finally, a numerical example has illustrated the usefulness of the obtained results and their feasibility and simplicity.

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