Published in International Journal of Plasticity 04/2010; 26(4):507-515

An alternative treatment of phenomenological higher-order strain-gradient plasticity theory

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ABSTRACT

Phenomenological higher-order strain-gradient plasticity is here presented through a formulation inspired by previous work for strain-gradient crystal plasticity. A physical interpretation of the phenomenological yield condition that involves an effect of second gradient of the equivalent plastic strain is discussed, applying a dislocation theory-based consideration. Then, a differential equation for the equivalent plastic strain-gradient is introduced as an additional governing equation. Its weak form makes it possible to deduce and impose extra boundary conditions for the equivalent plastic strain. A connection between the present treatment and strain-gradient theories based on an extended virtual work principle is discussed. Furthermore, a numerical implementation and analysis of constrained simple shear of a thin strip are presented.

Keywords: Strain-gradient plasticity; Size effects; Dislocations; Constitutive behavior

1. Introduction

Metals exhibit strongly size dependent mechanical behavior at the micron or submicron scales. A number of ideas for generalizing plasticity theories to account for the size-effects have been proposed so far. One of the approaches to incorporation of the size- effects is to formulate plastic strain-gradient-dependent work-hardening rules (Acharya and Bassani, 2000; Bassani, 2001; Huang et al., 2004; Brinckmann et al., 2006; Zhang et al., 2007). In this class of theories, only scalar quantities that represent a strain-hardened state are modified with plastic strain-gradients. Thus, the formulation of the boundary-value problem remains the same as the conventional model, i.e. it is sufficient to consider the conventional surface traction or prescribed displacement conditions at the boundaries. This class is called “lower-order” theories.

The second type of approach, which is of particular interest in the present study, is "higher-order" extension of the conventional plasticity theory (Aifantis, 1984; Mühlhaus and Aifantis, 1991). This class of theories makes it possible to treat additional boundary conditions for the plastic strain and/or for its gradient: for example, one can impose a zero-plastic strain condition at a hard interface. Most of recent higher-order strain-gradient plasticity theories have introduced an extended virtual work statement as a starting premise, which postulates existence of unconventional (microscopic or higher-order) stress quantities work-conjugate to the plastic strain rate and to the plastic strain rate gradient within the body, as well as an unconventional traction work-conjugate to the plastic strain rate on the boundaries (Fleck and Hutchinson, 2001; Gudmundson, 2004; Gurtin and Anand, 2005; Bardella, 2006; Bardella, 2007; Abu Al-Rub et al., 2007; Abu Al-Rub, 2008; Gurtin and Anand, 2009; Fleck and Willis, 2009; Polizzotto, 2009). This virtual work statement has been used to derive an additional force balance law and a corresponding extra traction condition. The former is equivalent to the strain-gradient-dependent yield condition proposed in the pioneering work of Aifantis (1984). In the work of Aifantis (1984) and Aifantis and Mühlhaus (1991), the higher-order stress quantities did not appear explicitly. In fact, Aifantis and Mühlhaus introduced an incremental variational principle that was followed by Fleck and Hutchinson (2001) in a similar form. But, the purpose of introducing the variational principle was to *deduce* essential and natural boundary conditions for the plastic strain.

In the context of crystal plasticity, the authors (Kuroda and Tvergaard, 2006; 2008) have investigated two types of formulations for the higher-order theories. One is the *work-conjugate* type, and the other is the *non-work-conjugate* type. The former is based on an extended virtual work statement (Gurtin, 2000; Gurtin, 2002; Borg, 2007) that is very similar to that for the phenomenological gradient plasticity mentioned above (e.g. Fleck and Hutchinson, 2001). Meanwhile, in the non-work-conjugate type of theories, there is no introduction of the higher-order stress quantities (Yefimov et al., 2004; Evers et al., 2004; Arsenlis et al., 2004; Bayley et al., 2006). Instead, definition equations for the geometrically necessary dislocation (GND) density that corresponds to the spatial gradient of the crystallographic slip are considered as additional partial differential governing equations. Their weak forms make it possible to consider extra boundary conditions for the crystallographic slips, which are not involved in the conventional theories. Kuroda and Tvergaard (2006; 2008) have shown that there exists an equivalency between the two formulations and they give fundamentally the same predictions for the same boundary value problems, although the formulations have different backgrounds and unlike mathematical expressions.

In the present study, an alternative treatment of the phenomenological higher-order strain-gradient plasticity is addressed along the line of the previous crystal plasticity studies (Kuroda and Tvergaard, 2006; 2008). We discuss the physical interpretation of the second strain-gradient-dependent yield condition (Aifantis, 1984), applying a dislocation theory-based consideration. Then, a differential equation for the equivalent plastic strain-gradient was introduced as an additional governing equation, which is an analogy to the equations for the GND densities in crystal plasticity. Then, its connection to the extended virtual work statement-based theories is discussed. Furthermore, we present a numerical implementation of the present formulation and perform analysis of a boundary-value problem: constrained simple shear of thin strips.

It is noted that Fleck and Hutchinson (1997) proposed an extension of the Toupin–Mindlin higher-order elasticity theory into the plasticity regime, which can be regarded as the third type of strain-gradient plasticity theory. This type of theory, in which a strain-gradient dependence is introduced even in the elastic range, will not be featured in the present paper. Some comparisons between the second and third-types can be found in Fleck and Hutchinson (2001) and Engelen et al. (2006).

2. Alternative treatment of higher-order strain-gradient plasticity: Formulation

The theory considered here does not differ much from the conventional plasticity theory. We confine attention to small strain conditions, where geometry changes are neglected. An additive decomposition of the total strain rate $\dot{\mathbf{E}}$ is considered as

$$\dot{\mathbf{E}} \equiv (\dot{\mathbf{u}} \otimes \nabla)_{\mathrm{sym}} = \dot{\mathbf{E}}^{\mathrm{e}} + \dot{\mathbf{E}}^{\mathrm{p}} , \tag{1}$$

where superscripts e and p denote elastic and plastic parts, $\dot{\mathbf{u}}$ is the displacement rate, ∇ is the gradient operator, and $\otimes$ is the tensor product operator, while $(\bullet)_{\mathrm{sym}}$ denotes the symmetric part of the tensor, and a superposed dot denotes the material-time derivative. A standard Hooke's law is adopted for elasticity, and a coaxial flow rule is used for plasticity:

$$\dot{\mathbf{E}}^{\mathrm{e}} = \mathbf{C}^{-1} : \dot{\boldsymbol{\sigma}}; \qquad \dot{\mathbf{E}}^{\mathrm{p}} = \dot{\phi}\mathbf{N}^{\mathrm{p}}; \qquad \mathbf{N}^{\mathrm{p}} = \frac{\boldsymbol{\sigma}'}{|\boldsymbol{\sigma}'|} , \tag{2}$$

where $\boldsymbol{\sigma}$ is the standard symmetric stress tensor, $\mathbf{C}$ is a fourth-order elasticity tensor (isotropic), $\boldsymbol{\sigma}'$ is the deviatoric part of the stress, $|(\bullet)| = \sqrt{(\bullet):(\bullet)}$, and $\dot{\phi}$ is a plastic multiplier.

2.1. Rate-dependent case: an elasto-viscoplasticity version

We first consider a rate-dependent, viscoplastic model, and then take its rate-independent limit. Here we introduce the following conventional power law:

$$\dot{\phi} = \dot{\phi}_0 \left(\frac{\sigma_{\mathrm{e}}}{S(\varepsilon^{\mathrm{p}})} \right)^{1/m}, \tag{3}$$

where $\sigma_{\mathrm{e}} = \sqrt{\tfrac{3}{2}} |\boldsymbol{\sigma}'|$ (the von Mises type of effective stress), $\dot{\phi}_0$ is a reference strain rate, m is a rate-sensitivity parameter, and S represents a strain hardening state that is assumed as a function of an equivalent plastic strain

$$\varepsilon^{\mathrm{p}} = \int_0^t \dot{\varepsilon}^{\mathrm{p}} \mathrm{d}t; \qquad \dot{\varepsilon}^{\mathrm{p}} = \sqrt{\tfrac{2}{3}}\, \dot{\phi} \tag{4}$$

with time t. A simple gradient-enhanced model is introduced as

$$\dot{\phi} = \begin{cases} \dot{\phi}_0 \left(\dfrac{\sigma_{\mathrm{e}} + \beta \nabla^2 \varepsilon^{\mathrm{p}}}{S(\varepsilon^{\mathrm{p}})} \right)^{1/m} & \text{for } \sigma_{\mathrm{e}} + \beta \nabla^2 \varepsilon^{\mathrm{p}} > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

where β is a positive scalar coefficient and $\nabla^2 \ (= \nabla \cdot \nabla)$ is the Laplace operator. Although adding the term $\beta \nabla^2 \varepsilon^{\mathrm{p}}$ was first introduced by Aifantis (1984), this also might be motivated by an argument based on the dislocation theory as follows. Within a uniform field of dislocations with the same sign, dislocations oppositely equidistant from the material point would give opposite stress values to that point (Evers et al., 2004; Bayley et al., 2006), according to the classical elastic solution for the stress field caused by an isolated dislocation (e.g. Cottrell, 1952). Namely, a resulting internal stress at the material point, which is caused by uniformly distributed GNDs around that point, completely cancels out. Therefore, a dislocation-induced internal stress should arise in response to *spatial gradients* of the GND density, not to the GND density itself (Groma et al., 2003; Evers et al., 2004; Bayley et al., 2006; Geers et al., 2007; Kuroda and Tvergaard, 2006; 2008; Suzuki et al., 2009)[1]. The density of the GNDs corresponds to the spatial gradient of crystallographic slip (Ashby, 1970). Therefore, the internal stresses shall develop in response to the second gradient of the slip. The introduction of the term $\beta \nabla^2 \varepsilon^{\mathrm{p}}$ is consistent with this argument. The coefficient β involves a material length scale. The physical interpretation of the material length scale is still an open question. One may be able to find an appropriate value of β for a specific material by fitting to experimental results such as Fleck et al. (1994), Stölken and Evans (1998), and Suzuki et al. (2009). In a context of crystal plasticity, Groma et al. (2003) and Geers et al. (2007) have discussed a physical approach to material length scales, which are set by

[1] The detailed equations for this idea can be found in Evers et al. (2004), Bayley et al. (2006) and Suzuki et al. (2009).

dislocation densities.

It is noted that the present phenomenological model uses the effective stress σ_e and the equivalent plastic strain ε^p to describe the external and internal stress effects, respectively. Consequently, the positive or negative direction of the resolved shear stress and the slip, which would be appropriately accounted for in the context of crystal plasticity, is not considered. So, the Bauschinger-like effect caused by pile-ups of single-signed dislocations cannot be represented within the present simplified theory.

As shown above, the terms $\sigma_\mathrm{e} + \beta\nabla^2\varepsilon^\mathrm{p}$ in Eq. (5) represent a net stress at the material point, i.e. the superposition of the stress caused by external forces and the internal stress due to nonuniform distribution of nonredundant dislocations. This net stress activates plastic straining, i.e. the generation and movement of dislocations. Thus, the plastic dissipation may be evaluated by $\mathcal{D} = (\sigma_\mathrm{e} + \beta\nabla^2\varepsilon^\mathrm{p})\dot{\varepsilon}^\mathrm{p} \geq 0$. This consequence is consistent with a recent discussion in Gurtin and Anand (2009) that the nonlocal term $\beta\nabla^2\varepsilon^\mathrm{p}$ in the Aifantis' original theory should be *energetic*.

With neglect of the body force effect, the standard equilibrium and boundary conditions are

$$\left.\begin{array}{ll} \nabla\cdot\boldsymbol{\sigma} = \mathbf{0}, & \\ \boldsymbol{\sigma}\cdot\mathbf{n} = \underline{\mathbf{t}} & \text{on } s_t, \\ \dot{\mathbf{u}} = \dot{\underline{\mathbf{u}}} & \text{on } s_u, \end{array}\right\} \tag{6}$$

where $\mathbf{n}$ is the normal to the surface, $\underline{\mathbf{t}}$ denotes a prescribed traction on the surface s_t, $\dot{\underline{\mathbf{u}}}$ is the prescribed displacement rate on the surface s_u, while $s\ (= s_t + s_u)$ is the surface of the body, and the underbars, $(\,\_\,)$, mean that the values of the quantity are specified on the boundary. Eqs. (6) are equivalent to the incremental virtual work principle

$$\int_v \dot{\boldsymbol{\sigma}} : \delta\dot{\mathbf{E}}\,\mathrm{d}v = \int_{s_t} \dot{\underline{\mathbf{t}}}\cdot\delta\dot{\mathbf{u}}\,\mathrm{d}s\,, \tag{7}$$

where v is the volume (region) of the body, $\delta\dot{\mathbf{u}}$ is an arbitrary virtual velocity satisfying $\delta\dot{\mathbf{u}} = 0$ on s_u, and $\delta\dot{\mathbf{E}}$ is the corresponding virtual strain rate. The constitutive relation (2) with Eq. (5), i.e.

$$\dot{\boldsymbol{\sigma}} = \mathbf{C} : (\dot{\mathbf{E}} - \dot{\phi}\mathbf{N}^\mathrm{p}) \tag{8}$$

is used together with Eq. (7).

With only the relations introduced above, extra boundary conditions for ε^p cannot be imposed[2]. Here, we introduce a quantity $\mathbf{g}^{\mathrm{p}}$ defined formally as a solution of the differential equation

$$\mathbf{g}^{\mathrm{p}} + \nabla \varepsilon^{\mathrm{p}} = \mathbf{0}\,, \tag{9}$$

which is subjected to specific boundary conditions. The $\mathbf{g}^{\mathrm{p}}$ may be interpreted as a measure of the GND density in a phenomenological sense. Now, we consider a weak form of Eq. (9),

$$\int_v (\mathbf{g}^{\mathrm{p}} + \nabla \varepsilon^{\mathrm{p}}) \cdot \check{\mathbf{g}}^{\mathrm{p}} \mathrm{d}v = 0\,, \tag{10}$$

where $\check{\mathbf{g}}^{\mathrm{p}}$ is an arbitrary weighting function. Using a relation $\nabla \cdot (\varepsilon^{\mathrm{p}} \check{\mathbf{g}}^{\mathrm{p}}) = \nabla \varepsilon^{\mathrm{p}} \cdot \check{\mathbf{g}}^{\mathrm{p}} + (\nabla \cdot \check{\mathbf{g}}^{\mathrm{p}}) \varepsilon^{\mathrm{p}}$ and the divergence theorem, we obtain

$$\int_v \mathbf{g}^{\mathrm{p}} \cdot \check{\mathbf{g}}^{\mathrm{p}} \mathrm{d}v = \int_v \varepsilon^{\mathrm{p}} (\nabla \cdot \check{\mathbf{g}}^{\mathrm{p}}) \mathrm{d}v - \int_s \varepsilon^{\mathrm{p}} \mathbf{n} \cdot \check{\mathbf{g}}^{\mathrm{p}} \mathrm{d}s\,. \tag{11}$$

With the surface integration term in Eq. (11), the extra boundary conditions for ε^{p} can be specified.

In the present treatment of strain-gradient plasticity, the standard incremental virtual work principle (Eq. (7)) and Eq. (11) are the governing equations to be solved simultaneously. These equations themselves always hold independent of constitutive assumptions, but they are connected by the constitutive relations involving the higher-order gradient term, as in Eq. (5).

2.2. Rate-independent limit: an elasto-plastic version

The power law function (5) may be inverted to give a rate-dependent dynamic yield function (using $\mathbf{g}^{\mathrm{p}}$ that has been introduced in Eq. (9)):

$$\sigma_{\mathrm{e}} - S(\varepsilon^{\mathrm{p}})(\dot{\phi}/\dot{\phi}_0)^m - \beta \nabla \cdot \mathbf{g}^{\mathrm{p}} = 0\,. \tag{12}$$

Taking the rate-independent limit, i.e. $m \to 0$, the yield function becomes

$$\sigma_{\mathrm{e}} - S(\varepsilon^{\mathrm{p}}) - \beta \nabla \cdot \mathbf{g}^{\mathrm{p}} = 0\,. \tag{13}$$

Using the consistency condition, the multiplier is now determined as

$$\dot{\phi} = \frac{(\partial \sigma_{\mathrm{e}} / \partial \boldsymbol{\sigma}) : \mathbf{C} : \dot{\mathbf{E}} - \beta \nabla \cdot \dot{\mathbf{g}}^{\mathrm{p}}}{A} > 0 \quad \text{for plastic loading,} \tag{14}$$

with

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[2] In several studies (Tomita, 1992; Kuroda, 1996; Lele and Anand, 2009), constitutive equations that involve higher-order strain gradient term(s) have been treated without considerations for extra boundary conditions as in the lower-order theories: i.e. the gradient term(s) is evaluated from strain values of neighboring material points at each stage of (finite element) computation, and it is simply returned to the constitutive equation at the next time step.

$$A \equiv \frac{\partial \sigma_{\rm e}}{\partial \boldsymbol{\sigma}} : \mathbf{C} : \mathbf{N}^{\rm p} + \sqrt{\frac{2}{3}} \left(\frac{{\rm d}S}{{\rm d}\varepsilon^{\rm p}} + \frac{{\rm d}\beta}{{\rm d}\varepsilon^{\rm p}} \nabla \cdot \mathbf{g}^{\rm p} \right). \tag{15}$$

Here, a possibility that β is a function of $\varepsilon^{\rm p}$ has been considered. Substituting Eq. (14) into Eq. (8) gives an expression for the elasto-plastic constitutive equation:

$$\dot{\boldsymbol{\sigma}} = \left[\mathbf{C} - \frac{\left(\mathbf{C}:\mathbf{N}^{\rm p}\right) \otimes \left\{ (\partial \sigma_{\rm e} / \partial \boldsymbol{\sigma}) : \mathbf{C} \right\}}{A} \right] : \dot{\mathbf{E}} + \frac{\beta \nabla \cdot \dot{\mathbf{g}}^{\rm p}}{A} \mathbf{C} : \mathbf{N}^{\rm p} = \mathbf{C}^{\rm ep} : \dot{\mathbf{E}} + \frac{\beta \nabla \cdot \dot{\mathbf{g}}^{\rm p}}{A} \mathbf{P} \tag{16}$$

with $\mathbf{P} = \mathbf{C}:\mathbf{N}^{\rm p}$. As is seen above, in the rate-independent case, $\dot{\mathbf{g}}^{\rm p}$ has appeared in Eqs. (14) and (16). Thus, the rate form of Eq. (9) is needed as a counterpart of the system of governing equations, i.e.

$$\dot{\mathbf{g}}^{\rm p} + \nabla \dot{\varepsilon}^{\rm p} = \mathbf{0}, \tag{17}$$

and its weak form

$$\int_v \dot{\mathbf{g}}^{\rm p} \cdot \check{\mathbf{g}}^{\rm p} {\rm d}v = \int_v \dot{\varepsilon}^{\rm p} (\nabla \cdot \check{\mathbf{g}}^{\rm p}) {\rm d}v - \int_s \underline{\dot{\varepsilon}}^{\rm p}\, \mathbf{n} \cdot \check{\mathbf{g}}^{\rm p} {\rm d}s \tag{18}$$

replaces Eq. (11).

3. **Connection to existing higher-order theories**

Defining a vector quantity, $\boldsymbol{\tau} = -\beta \mathbf{g}^{\rm p}$ (with a constant $\beta (>0)$), and a scalar quantity,

$$Q = \begin{cases} S(\varepsilon^{\rm p})(\dot{\phi}/\dot{\phi}_0)^m & \text{for rate-dependent case,} \\ S(\varepsilon^{\rm p}) & \text{for rate-independent case,} \end{cases} \tag{19}$$

the yield functions, (12) and (13), may be written as

$$\sigma_{\rm e} - Q + \nabla \cdot \boldsymbol{\tau} = 0. \tag{20}$$

It is noted that Eq. (20) for the rate independent case (this is the same as Eq. (13)) is identical to the original proposition of Aifantis (1984),

$$\sigma_{\rm e} = S(\varepsilon^{\rm p}) - \beta \nabla^2 \varepsilon^{\rm p}. \tag{21}$$

The form of Eq. (20) itself is the same as the Fleck–Hutchinson's microforce balance, but in their theory, different constitutive models have been used for Q and $\boldsymbol{\tau}$. This will be discussed later.

Considering Eq. (1) and a relation $\boldsymbol{\sigma} : \dot{\mathbf{E}}^{\rm p} = \sigma_{\rm e} \dot{\varepsilon}^{\rm p}$, the standard virtual work relation is rewritten as

$$\int_v (\boldsymbol{\sigma} : \delta \dot{\mathbf{E}}^{\rm e} + \sigma_{\rm e}\ \delta \dot{\varepsilon}^{\rm p})\, {\rm d}v = \int_{s_t} \underline{\mathbf{t}} \cdot \delta \dot{\mathbf{u}}\, {\rm d}s \tag{22}$$

Furthermore, substituting Eq. (20) into Eq. (22) and using a relation $\nabla\cdot(\boldsymbol{\tau}\delta\dot{\varepsilon}^{\mathrm{p}}) = (\nabla\cdot\boldsymbol{\tau})\delta\dot{\varepsilon}^{\mathrm{p}} + \boldsymbol{\tau}\cdot\nabla\delta\dot{\varepsilon}^{\mathrm{p}}$ together with the divergence theorem, one can reach the relation

$$\int_v (\boldsymbol{\sigma} : \delta\dot{\mathbf{E}}^{\mathrm{e}} + Q\ \delta\dot{\varepsilon}^{\mathrm{p}} + \boldsymbol{\tau}\cdot\nabla\delta\dot{\varepsilon}^{\mathrm{p}})\mathrm{d}v = \int_{s_t} \underline{\mathbf{t}}\cdot\delta\dot{\mathbf{u}}\mathrm{d}s + \int_s \underline{\chi}\ \delta\dot{\varepsilon}^{\mathrm{p}}\mathrm{d}s \tag{23}$$

with a definition of $\underline{\chi} = \boldsymbol{\tau}\cdot\mathbf{n}$. This expression is the same as the extended virtual work statement of Fleck and Hutchinson (2001), which was introduced as a major premise of their theory. It is noted that only when the relation $\nabla\cdot\boldsymbol{\tau} = -\beta\nabla\cdot\mathbf{g}^{\mathrm{p}}$ holds the present formulation can be rewritten into the form of Eq. (23) and this identification holds only if β is constant.

Gudmundson (2004) and Gurtin and Anand (2005) developed a more general virtual-work principle in which a second-order plastic microstress is work-conjugate to $\dot{\mathbf{E}}^{\mathrm{p}}$ and a third-order plastic microstress is work-conjugate to $\nabla\dot{\mathbf{E}}^{\mathrm{p}}$. The connection between the general principle and the simpler one (Eq. (23)) has been shown in Gurtin and Anand (2009). Further, Gurtin and Anand (2009) have insisted that the Fleck–Hutchinson's microforce balance with their own constitutive relations for $\boldsymbol{\tau}$ should reduce to the Aifantis' yield criterion of Eq. (21) with a constant β in order to satisfy thermodynamic conditions extensively discussed by Gurtin and co-workers.

In the one-parameter version of the Fleck and Hutchinson (2001) theory, the rates of $\boldsymbol{\tau}$ and Q are given by

$$\dot{\boldsymbol{\tau}} = \ell_*{}^2 h(E^{\mathrm{p}})\nabla\dot{\varepsilon}^{\mathrm{p}} \tag{24}$$

$$\dot{Q} = h(E^{\mathrm{p}})\dot{\varepsilon}^{\mathrm{p}} \tag{25}$$

with

$$E^{\mathrm{p}} = \int_0^t \dot{E}^{\mathrm{p}}\,\mathrm{d}t; \qquad \dot{E}^{\mathrm{p}} = \sqrt{(\dot{\varepsilon}^{\mathrm{p}})^2 + \ell_*{}^2\nabla\dot{\varepsilon}^{\mathrm{p}}\cdot\nabla\dot{\varepsilon}^{\mathrm{p}}}\ . \tag{26}$$

The Fleck and Hutchinson (2001) theory can be recast in a form similar to that discussed in the previous section. But, we cannot directly apply Eq. (17) in the formulation, because the spatial variation of the modulus $h(E^{\mathrm{p}})$ enters the consistency condition of the yield function. We use

$$\dot{\mathbf{g}}^{\mathrm{p}}_{\mathrm{FH}} + \ell_*{}^2 h(E^{\mathrm{p}})\nabla\dot{\varepsilon}^{\mathrm{p}} = \mathbf{0} \tag{27}$$

and its weak form

$$\int_v \dot{\mathbf{g}}^{\mathrm{p}}_{\mathrm{FH}}\cdot\check{\mathbf{g}}^{\mathrm{p}}\,\mathrm{d}v = \int_v \ell_*{}^2 h\dot{\varepsilon}^{\mathrm{p}}\,\nabla\cdot\check{\mathbf{g}}^{\mathrm{p}}\,\mathrm{d}v + \int_v \ell_*{}^2\,\dot{\varepsilon}^{\mathrm{p}}\,\nabla h\cdot\check{\mathbf{g}}^{\mathrm{p}}\,\mathrm{d}v - \int_s \ell_*{}^2 h\underline{\dot{\varepsilon}}^{\mathrm{p}}\,\mathbf{n}\cdot\check{\mathbf{g}}^{\mathrm{p}}\,\mathrm{d}s\ , \tag{28}$$

instead of Eqs. (17) and (18). Then, Eq. (28) is the additional equation to be solved together with the Eq. (7). The multiplier $\dot{\phi}$ is determined through the consistency condition, $\dot{\sigma}_{\mathrm{e}} - \dot{Q} + \nabla\cdot\dot{\boldsymbol{\tau}} = 0$, as

$$\dot{\phi} = \frac{(\partial\sigma_e / \partial\boldsymbol{\sigma}) : \mathbf{C} : \dot{\mathbf{E}} - \nabla \cdot \dot{\mathbf{g}}^{p}_{FH}}{(\partial\sigma_e / \partial\boldsymbol{\sigma}) : \mathbf{C} : \mathbf{N}^{p} + \sqrt{\tfrac{2}{3}} h(E^{p})} > 0 \quad \text{for plastic loading.} \tag{29}$$

If h is taken to be constant, the one-parameter version of Fleck and Hutchinson (2001) theory coincides with the present model, as described in the subsection 2.2, and with the Aifantis theory (Eq. (21)), understanding that $\beta = \ell_*{}^2 h$ and that the second term on the right-hand side of Eq. (28) vanishes.

4. Analysis

In this section, the formulation presented in section 2 is implemented in a finite element analysis. In the rate-dependent case, the standard analysis of the displacement rate ($\dot{\mathbf{u}}$) field and the unconventional analysis for the $\mathbf{g}^{p}$ field (based on Eq. (11)) can be decoupled. In the latter, $\mathbf{g}^{p}$ is taken to be extra nodal degrees of freedom. Meanwhile, in the rate-independent case, a coupled analysis for $\dot{\mathbf{u}}$ and $\dot{\mathbf{g}}^{p}$ is required. The detailed finite element implementation of the rate-independent model is presented in the Appendix.

A strip with height H in the x_2-direction is subjected to simple shear. The macroscopic boundary conditions are

$$\left.\begin{array}{lll} \dot{u}_1 = 0, \quad \dot{u}_2 = 0 & & \text{along } x_2 = 0, \\ \dot{u}_1 = \dot{U} = \sqrt{2} H \dot{\phi}_0, & \dot{u}_2 = 0 & \text{along } x_2 = H, \end{array}\right\} \tag{30}$$

and the macroscopic shear strain Γ is defined by

$$\Gamma = U / H . \tag{31}$$

Top and bottom surfaces of the strip are assumed to be bounded by hard materials impenetrable to dislocations, and extra boundary conditions are set to be

$$\left.\begin{array}{ll} \varepsilon^{p} = 0 & \text{at} \quad x_2 = 0 \\ \varepsilon^{p} = 0 & \text{at} \quad x_2 = H \end{array}\right\}. \tag{32}$$

Since the strip is assumed to be extended infinitely in the x_1-direction, all field quantities are required to be periodic in the x_1-direction with a periodic length W that can be chosen arbitrarily,

$$\dot{u}_i(0, x_2) = \dot{u}_i(W, x_2) . \tag{33}$$

First, we derive an analytical solution for this problem under rate-independent conditions. A linear strain hardening material with $S(\varepsilon^{p}) = \sigma_0 + h_0 \varepsilon^{p}$ is considered, where σ_0 is an initial yield stress, and h_0 is a strain hardening modulus. The length scale coefficient is taken to be $\beta = L^2 \sigma_0$. Thus, the yield function (13) (equivalently Eq. (20) or (21)) is specified to

$$L^2\sigma_0 \frac{\mathrm{d}^2\varepsilon^{\mathrm{p}}}{\mathrm{d}y^2} + \sqrt{3}\sigma_{12} = \sigma_0 + h_0\varepsilon^{\mathrm{p}}, \tag{34}$$

where y stands for x_2. The corresponding boundary conditions are given in Eq. (32). The equilibrium condition is

$$\frac{\mathrm{d}\sigma_{12}}{\mathrm{d}y} = 0 . \tag{35}$$

Then, the solution for ε^{p} is obtained as

$$\varepsilon^{\mathrm{p}} = \frac{F}{\lambda^2}\left[1 - \cosh(\lambda y) - \sinh(\lambda y)\frac{1-\cosh(\lambda H)}{\sinh(\lambda H)}\right] \tag{36}$$

with

$$F = \frac{\sqrt{3}\sigma_{12} - \sigma_0}{L^2\sigma_0}; \qquad \lambda^2 = \frac{h_0}{L^2\sigma_0}. \tag{37}$$

Defining a local shear strain γ and plastic shear strain $\gamma^{\mathrm{p}} = \sqrt{3}\varepsilon^{\mathrm{p}}$, the elasto-plastic constitutive equation is written as

$$\sigma_{12} = \mu(\gamma - \gamma^{\mathrm{p}}) \quad \text{or} \quad \gamma = \frac{\sigma_{12} + \mu\gamma^{\mathrm{p}}}{\mu}, \tag{38}$$

where μ is the shear modulus. The macroscopic shear strain Γ is obtained as

$$\Gamma = \frac{\int_0^H \gamma \mathrm{d}y}{H} = \frac{\sigma_{12}}{\mu} + \frac{\sqrt{3}F}{\lambda^2 H}\left[H - \frac{\sinh(\lambda H)}{\lambda} + \frac{(\cosh(\lambda H)-1)^2}{\lambda\sinh(\lambda H)}\right]. \tag{39}$$

For the finite element analysis, the sample with height *H* is discretized by a column of 40 four-node elements with 2×2 full integration both for the analyses of $\dot{\mathbf{u}}$ and $\dot{\mathbf{g}}^{\mathrm{p}}$. The reason for choosing the same order of interpolation both for $\dot{\mathbf{u}}$ and $\dot{\mathbf{g}}^{\mathrm{p}}$ is that the spatial gradients of $\mathbf{g}^{\mathrm{p}}$ correspond to a stress quantity that should be evaluated at the integration points where the strain rate is computed.

Finite element and analytical solutions for the rate-independent case with the length scales of *L*/*H* = 0.3 and 1.0 are shown in Fig.1. The material parameter values have been taken as $\sigma_0/\mu =$ 0.0078, $\sigma_0/h_0 = 0.05$. In the analytical solutions, the extra boundary conditions have been directly applied to the gradient-dependent yield criterion that is viewed as an additional balance law rather than a mere constitutive relation. Meanwhile, in the present mathematical treatment, the extra boundary conditions have been applied to Eq. (17) that is viewed as a balance law being independent of any constitutive assumption. The two methods give identical solution.

A computational result for a viscoplastic material with *m* = 0.2 and 0.05 is depicted in

Fig. 2 together with the elasto-plastic solution. The length scale is $L/H=1$. The solution for viscoplastic material with $m = 0.05$, a rate-sensitivity that could be realistic for metals at room temperature, is rather close to the elasto-plastic solution. For a rather large value of viscosity, $m = 0.2$, typical viscoplastic effects can be seen: a delay in the development of 'layers' and a smooth transition from elastic to inelastic deformations. The same viscoplastic effect was found in the context of crystal plasticity (Kuroda and Tvergaard, 2006).

5. Discussion

The present treatment started from the introduction of the second-order strain-gradient term into the conventional plasticity model as the internal stress effect based on dislocation theory. Then, a differential equation for the equivalent plastic strain-gradient was introduced as an additional governing equation. Its weak form allowed us to deduce the extra boundary conditions.

In the formulations presented in Gurtin and Anand (2009), one may derive specific constitutive equations formally in a thermodynamically consistent manner. This procedure is totally based on the extended virtual work principle introduced as a major premise, Eq. (23), which emphasizes the existence of higher-order stresses and higher-order tractions. The statement that the higher-order traction exists in balance with the higher-order stress on a boundary as $\underline{\chi} = \boldsymbol{\tau} \cdot \mathbf{n}$ is seen from the mathematical derivation, but it is not easy to see the physical background, whereas the standard traction is related to the stress through the Cauchy's formula $\underline{\mathbf{t}} = \boldsymbol{\sigma} \cdot \mathbf{n}$.

The formulations based on the extended virtual work statement and the present treatment have been developed from quite different starting points. In the former, we are able to see through the mathematics a thermodynamic consistency using the concept of higher-order stress quantities based on the work-conjugate considerations. By contrast, in the present treatment, all the development has been expressed by the conventional stress quantities, and the origin of the gradient term is interpreted in terms of dislocation theory. The two formulations give the same solutions in the case of a constant length scale coefficient, as shown in the sections 3 and 4.

Appendix

For the finite element implementation of the rate-independent model, it may be convenient to write the elasto-plastic constitutive equation (16) in a matrix form

$$\{\dot{\sigma}\} = [C^{ep}]\{\dot{E}\} + \frac{\beta \nabla \cdot \dot{\mathbf{g}}^{p}}{A}\{P\} , \tag{40}$$

where $\{\dot{\sigma}\}^{T} = \{\dot{\sigma}_{11},\ \dot{\sigma}_{22},\ \dot{\sigma}_{12}\}$, $\{\dot{E}\}^{T} = \{\dot{E}_{11},\ \dot{E}_{22},\ 2\dot{E}_{12}\}$, $\{P\}^{T} = \{P_{11}, P_{22}, P_{12}\}$ for plane strain problems. Defining nodal displacement rates $\{\dot{U}\}$, and nodal plastic strain rate gradients $\{\dot{G}^{p}\}$, the displacement rates $\{\dot{u}\}$ and plastic strain rate gradients $\{\dot{g}^{p}\}$ and $\nabla \cdot \dot{\mathbf{g}}^{p}$ within an element can be represented as

$$\{\dot{u}\} = \begin{bmatrix} N^{(1)} & 0 & \cdots \\ 0 & N^{(1)} & \cdots \end{bmatrix} \begin{Bmatrix} \dot{U}_1^{(1)} \\ \dot{U}_2^{(1)} \\ \vdots \end{Bmatrix} \equiv [N]\{\dot{U}\}; \quad \{\dot{g}^{p}\} = \begin{bmatrix} N^{(1)} & 0 & \cdots \\ 0 & N^{(1)} & \cdots \end{bmatrix} \begin{Bmatrix} \dot{G}_1^{p(1)} \\ \dot{G}_2^{p(1)} \\ \vdots \end{Bmatrix} \equiv [N]\{\dot{G}^{p}\} \tag{41}$$

$$\nabla \cdot \dot{\mathbf{g}}^{p} = \begin{bmatrix} \frac{\partial N^{(1)}}{\partial x_1} & \frac{\partial N^{(1)}}{\partial x_2} & \cdots \end{bmatrix} \begin{Bmatrix} \dot{G}_1^{p(1)} \\ \dot{G}_2^{p(1)} \\ \vdots \end{Bmatrix} \equiv [\mathrm{div}B]\{\dot{G}^{p}\} , \tag{42}$$

where $N^{(I)}$ are finite element shape functions and the superscript (I) denotes the node number within the element. As we choose $\dot{\mathbf{g}}^{p}$ as nodal degrees of freedom, we only need continuity of $\dot{\mathbf{g}}^{p}$ on element boundaries, not its gradient. A substitution of Eq. (40) into the incremental virtual work principle (7) gives the following finite element equation,

$$\int_v [B]^{T}[C^{ep}][B]\mathrm{d}v\ \{\dot{U}\} + \int_v \tfrac{\beta}{A}[B]^{T}\{P\}[\mathrm{div}B]\mathrm{d}v\ \{\dot{G}^{p}\} = \int_s [N]^{T}\{\dot{\underline{t}}\}\mathrm{d}s , \tag{43}$$

where $[B]$ is the standard displacement rate–strain rate matrix that contains spatial derivatives of $N^{(I)}$.

An additional finite element equation corresponding to Eq. (18) takes the form

$$\int_v [N]^{T}[N]\mathrm{d}v\ \{\dot{G}^{p}\} = \int_v [\mathrm{div}B]^{T}\dot{\varepsilon}^{p}\mathrm{d}v - \int_s [N]^{T}\{n\}\dot{\underline{\varepsilon}}^{p}\mathrm{d}s . \tag{44}$$

Using (14), a finite element approximation for $\dot{\varepsilon}^{p}$ can be written as

$$\dot{\varepsilon}^{p} = \sqrt{\tfrac{2}{3}}\dot{\phi} = \sqrt{\tfrac{2}{3}}\tfrac{1}{A}\{R\}^{T}[B]\{\dot{U}\} - \sqrt{\tfrac{2}{3}}\tfrac{\beta}{A}[\mathrm{div}B]\{\dot{G}^{p}\} . \tag{45}$$

Substituting Eq. (45) into Eq. (44) gives

$$-\int_v \sqrt{\tfrac{2}{3}}\tfrac{1}{A}[\mathrm{div}B]^{T}\{R\}^{T}[B]\mathrm{d}v\ \{\dot{U}\} + \int_v \left([N]^{T}[N] + \sqrt{\tfrac{2}{3}}\tfrac{\beta}{A}[\mathrm{div}B]^{T}[\mathrm{div}B]\right)\mathrm{d}v\ \{\dot{G}^{p}\} = -\int_s [N]^{T}\{n\}\dot{\underline{\varepsilon}}^{p}\mathrm{d}s, \tag{46}$$

where $\{R\}$ contains components of $(\partial\sigma_{\mathrm{e}} / \partial\boldsymbol{\sigma}):\mathbf{C}$.

Consequently, using Eqs. (43) and (46), the final algebraic system of equations can be written in the form

$$\begin{bmatrix} K_{(UU)} & K_{(UG)} \\ K_{(GU)} & K_{(GG)} \end{bmatrix} \begin{Bmatrix} \dot{U} \\ \dot{G}^{\mathrm{p}} \end{Bmatrix} = \begin{Bmatrix} \dot{F}_1 \\ \dot{F}_2 \end{Bmatrix}. \tag{47}$$

For the rate-dependent cases ($m > 0$), a total form of Eq. (44) (without superposed dots) is not coupled with the standard displacement rate field analysis in a linear incremental method. Solutions may be obtained through a stagger type of method.

Although we do not show results here, Eqs. (24)-(29) have also been implemented numerically, and the corresponding finite element procedure along the line of Eqs. (41)-(47) has been used to analyze the constrained simple shear problem (defined in Eqs. (30)-(33)) with a nonlinear hardening as in Fleck and Hutchinson (2001). This led to good agreement with the curves presented by these authors.

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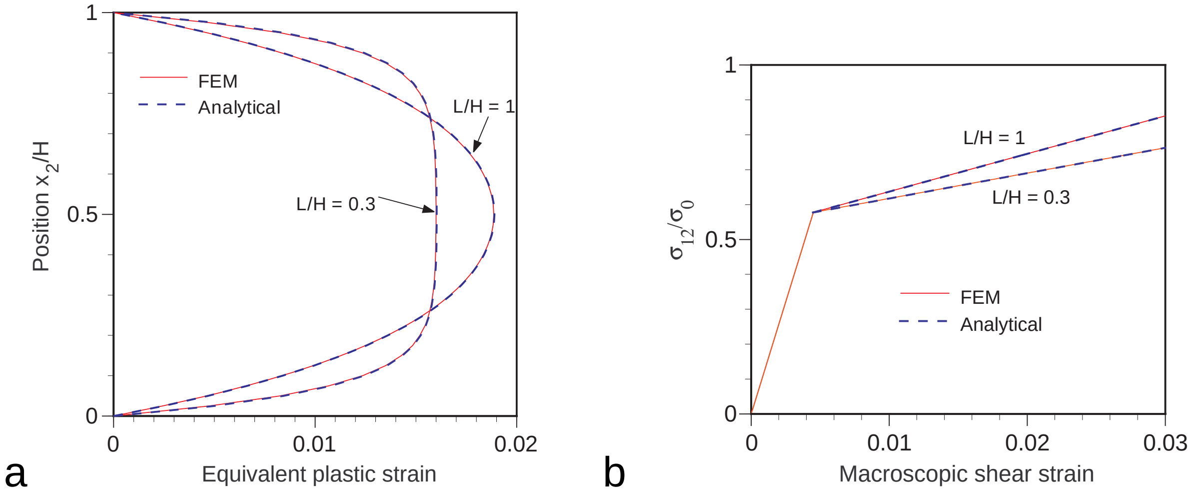
Published in International Journal of Plasticity 04/2010; 26(4):507-515

Figure captions

Fig. 1. Comparison of finite element and analytical solutions for constrained simple shear problem of elastoplastic thin strip (rate-independent material) with linear hardening. (a) Distribution of equivalent plastic strain ε^{p} across the thickness at a macroscopic strain Γ of 0.03; (b) Shear stress σ_{12} versus macroscopic shear strain Γ.

Fig. 2. Comparison of viscoplastic (*m* = 0.2, 0.05) and elastoplastic solutions for constrained simple shear problem of thin strip with linear hardening (finite element solutions for *L*/*H* = 1). (a) Distribution of equivalent plastic strain ε^{p} across the thickness at various stages of the macroscopic strain Γ; (b) Shear stress σ_{12} versus macroscopic shear strain Γ.



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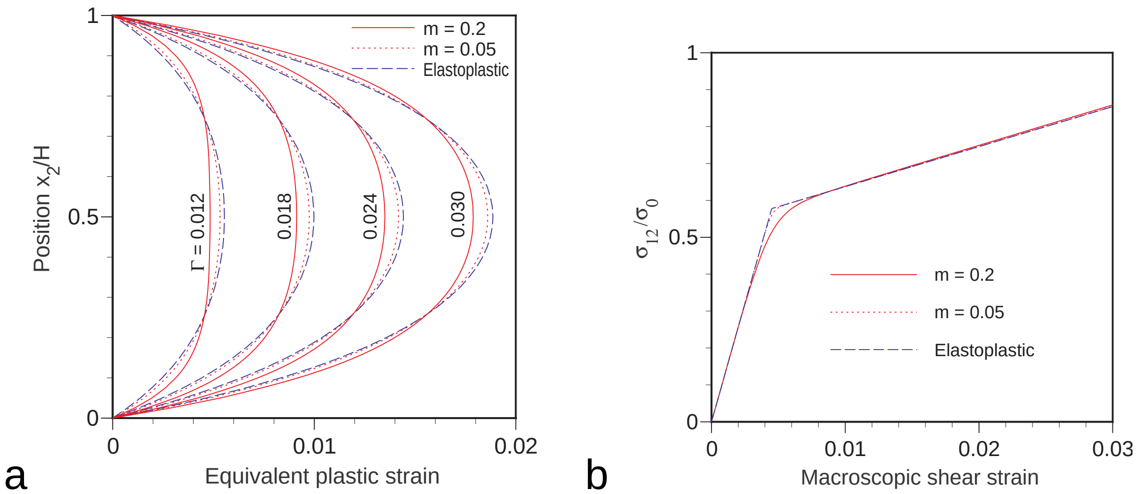

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