Sharing Authority in Proof But Not All at Once

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It is important to give students authority over the proving process, but this shared authority can be more manageable and productive if proof is broken down into three phases — proof initiation, proof construction, and proof validation — where authority can be shared gradually and strategically.

As teachers and researchers of proof in high school geometry, we have often seen whole-class proving episodes where the teacher is the one doing the intellectual work even though they want students to take a more active role. Active involvement is important for all mathematical practices (Koestler et al. 2013) but especially for proving because it is the students who ultimately have to be convinced by the argument and understand the conclusion. It is also the students who we hope will become better at making conjectures, providing reasons, raising doubts, and establishing mathematical truth that makes sense to them. Therefore, it is important to share mathematical authority with students throughout the proving process.

But proving is complex, and teaching proving is even more so, thus it can be extremely challenging to turn the authority entirely over to students. For example, the students may not be accustomed to having this authority and they may resist by asking the teacher to tell them what they should prove and how. Or students may be willing to take the reins but things may deviate from the intended learning goals of the lesson or may take too long, squandering valuable instructional time. One way to gradually turn over mathematical authority to students while also maintaining an appropriate amount of teacher control over the direction of the lesson is to strategically share proving authority one piece at a time. We break the proving process into three stages—proof initiation, proof construction, and proof validation (Otten et al. 2017)—and we describe how to share partial authority with students during these stages.

What do we mean by authority and proving?

Throughout students' mathematical learning experiences, there are always dynamics with regard to who is leading the learning activities (e.g., teacher directed or student directed) and who determines whether the mathematics is complete, correct, and sufficiently clear. These dynamics reveal the primary author of the learning experiences, and in mathematics classrooms, it is typically the teacher who is the mathematical authority (Muis 2004). Yet there can be substantial benefits from allowing students to share mathematical authority, such as increased engagement and understanding (e.g., Webel 2010) and inclusion of typically-marginalized students (Esmonde and Langer-Osuna 2013). In this article, we focus on gradually sharing mathematical authority but we note that this is separate from the teacher's managerial authority.

To think about authority in relation to teaching proof, we focus on whole-class proving when a classroom is working together to create and understand a proof. We adopt the breakdown of whole-class proving into three phases: proof initiation, proof construction, and proof validation (Otten et al. 2017). According to Otten and colleagues, authority over proof initiation is attributed to whoever (or whatever) articulates or identifies a claim and determines that it should be proved. Authority over proof construction is at-

Proof Initiation	Proof Construction	Proof Validation
● How was the proof initiated and by whom? ● In what ways (if any) was the claim refined or modified? ● Who provided initial ideas about the truth-value of the claim?	● Who structured the proof? ● Who requested components of the argument? ● Who provided components of the argument? ● Who clarified components of the argument or offered alternative arguments?	● What general statements were made about proof and who made them? ● Who critiqued or confirmed the argument? ● Who stated that the proof was complete?

Figure 1. Three phases of the proving process with authority in each phase.

tributed to whoever (or whatever) leads the discourse as the argument is being formed, which may proceed straightforwardly or may involve some clarifications and corrections. Finally, authority over proof validation is attributed to whoever (or whatever) confirms that the proof is complete and correct. It is possible that a teacher may serve as the authority in all of these phases, or perhaps it is the teacher together with the textbook. But it is also possible for students to be invited into shared authority, even if it is just for one of the phases.

Phases of Whole-Class Proving

Sharing Authority During Proof Initiation

Formal proving typically begins in a high school Geometry course (Herbst 2002) and, because it was not emphasized earlier, students often feel proving is an arbitrary requirement imposed on them by an external authority (McCrone and Martin 2006). Within geometry lessons, it is also usually an external authority initiating specific proofs because the teacher or textbook provides the mathematical claim and directs students to prove it (Herbst and Brach 2006). The proof initiation phase, however, provides two opportunities to share authority with students: conjecturing, where students can help generate the mathematical claim to be proved, or choosing, where students can help decide if a given mathematical claim would be worthwhile trying to prove (do we predict that the claim is true or false? do we need some convincing?).

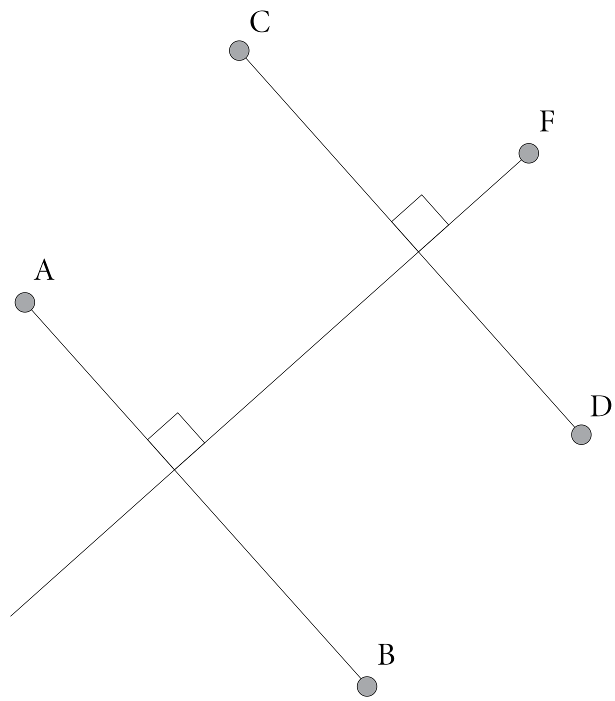

Figure 2. An opportunity for students to contribute to the formulation of a claim about perpendicular and parallel lines (see interactive GeoGebra applet at https://www.geogebra.org/m/sndsrxhh).

With regard to conjecturing, Otten and colleagues (2014) found that textbooks overwhelmingly present the important theorems in the narrative portion of lessons; students do not have many opportunities to formulate conjectures themselves. An example of sharing authority with students through conjecturing would be to provide a mathematical situation to be explored but give the students time to notice the key phenomena or try to express a general claim in the form of a conjecture. Take for example, the theorem that two lines perpendicular to the same line must themselves be parallel. Rather than providing students with the statement of the theorem, we could invite the students to examine several trios of lines with two that are perpendicular to the third, or some trios where the first two lines are not parallel but one is perpendicular to the third line and they inspect whether the other line is perpendicular or not. Students, then, could play an active part in conjecturing the claim to be proved.

Even in lessons where there is not enough time for an exploration to generate the claim, authority could still be shared with students in regard to the phrasing of the claim. For example, a teacher may introduce the situation and provide three different phrasings of the claim to be proved: "Two perpendiculars guarantees parallel," "If two lines are perpendicular to a third, then those two lines are parallel to each other," and "Right angles on one line means the other lines are parallel." The students can then be invited to comment on the different phrasings and they can have the authority to choose which one to use as the basis for a proof. Even if they choose unwisely, it can be a learning experience and they may have greater buy-in because of making the choice at the start.

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Another way to share authority in the proof initiation phase is to have students choose which claim they would most like to prove. In this case, rather than three phrasings of the same mathematical result, a teacher could present three different claims and ask students to determine which they are unsure about or which they would like to understand better. During proof initiation, students can also be invited to predict whether the mathematical claim is true or false. A simple way to do this is to give students a claim, such as "a line parallel to one side of a triangle divides the other two proportionally" (CCSS.MATH.CONTENT.HSG.SRT.B) and ask them to "Prove or Disprove-and-Salvage" the given claim. By modifying the direction to students from "Prove" to "Prove or Disprove-and-Salvage," students are drawn in with some choice right at the beginning (Bleiler-Baxter and Pair 2017).

These ways of sharing authority in the proof initiation phase may seem fairly simple but they have several key benefits. By involving the students in "discovering" the claim they are likely to be more interested in finding out whether it is true. Then, if they do see later that it is true in all possible cases (even if teachers led that phase of the proving process), students can be affirmed for their discoveries and can feel more like mathematicians. One way to capitalize on these accomplishments in the classroom space (or in a virtual space such as Trello; https://trello.com/en-us) is to have a Conjecture Wall for generalizations or potential truths that students notice; then, after the proof validation phase (see below), the conjectures could be moved over to a Theorem Wall. You can also assign a student's name to the given mathematical conjecture, for example, calling the conjectures on the wall "Jerome's conjecture," "Maggie's conjecture," and "Weston and Jasmine's conjecture." Research has shown that naming conjectures after students causes the other students in the class to "legitimately consider the veracity of the conjecture" (Bleiler-Baxter and Pair 2017).

Moreover, sharing authority during proof initiation opens a door for different sorts of mathematical competencies beyond doing correct work. Some students who may have difficulties in constructing an airtight proof may nevertheless be adept at noticing a pattern or intuiting an underlying truth. Other students may have strengths with regard to attending to precision, pointing out ambiguities in phrasing, or helping to clarify the issue in question. These competencies can be affirmed during proof initiation, and then the rest of the class and the teacher can work together on proof construction and proof validation. In many instances, the teacher may wish to reclaim authority for that construction and validation, but in other instances, those later phases might provide other opportunities for shared authority.

Sharing Authority During Proof Construction

Teachers often maintain mathematical authority during whole-class constructions of proofs (Otten et al. 2017), which makes some sense because students are usually new to proof and may benefit from teachers demonstrating this complex process. Proof constructions are also often the centerpiece of proving lessons, so teachers who use an "I do, we do, you do" instructional model are accustomed to being the authority. Some drawbacks to always having teacher authority during proof construction, however, are that students may passively copy the steps and may come to view proof construction as a rigid procedural process that can only be learned directly from the teacher or the textbook (Boyle et al. 2015).

The proof construction phase, however, provides multiple opportunities for teachers to share their authority—they may invite students to help set the overall plan for the proof or to provide the steps and justifications within the argument. With regard to helping set the overall plan, consider the claim, "If both pairs of opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram." The teacher may start by inviting students to draw what is given and write down the target to be deduced (see Figure 3). Then, rather than the teacher proceeding to lead the proof construction, s/he could ask the students for any strategic ideas about how to proceed. Perhaps one student will bring up the idea that it is often helpful to create triangles and look for congruent triangles, which could bridge from the given information about sides to some helpful information about angles. Another student might have the idea of listing definitions or theorems that involve parallelograms and looking for something that might serve as a linchpin between the start and the end. The teacher could then allow the students to work on one or more of the seemingly-viable proof plans, or the teacher might even reclaim authority in leading the proof construction, but at least in this way the teacher has shared authority for the planning even if not for the entire proof construction.

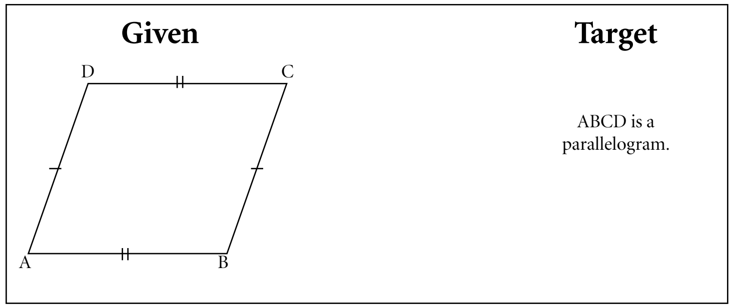

Figure 3. Given information and a space for student proof planning directed toward a goal.

Another way to share authority during proof construction is to relinquish some control to students in generating the steps and justifications in the argument. This may mean being more flexible with students in terms of the format of the proof (e.g., not just a strict two-column proof) and also being open to proofs with extraneous steps or imprecise justifications. In these cases, we would encourage teachers to view the imprecisions as part of the long-term learning process. A particular strategy that we have found useful for sharing authority in proof construction is called the "group proof activity," described in Bleiler-Baxter, Pair, and Reed (2020). In this activity, students are given a claim to prove independently for homework. They bring in their independent ideas and then work in groups of three or four. Within groups, students share their independent proofs, pausing to ask questions and clarify each student's argument. Then, as a team, they are tasked with creating an even stronger group proof. This group proof activity places the authority over proof construction firmly in the hands of the students, as they negotiate with one another the best ways to craft their new and improved group argument.

There are several benefits of sharing authority in proof construction. First, students become actively engaged in the process rather than passively following the teacher. Active involvement in proof construction can deepen the students' understanding of the purpose of proof and positions students as mathematical thinkers, giving them autonomy over their own learning. When teachers accept both correct and incorrect answers, allowing the classroom community to provide feedback, it provides an avenue for a productive classroom community where students assess their peers and learn collaboratively.

A challenge that may arise as a result of sharing authority in proof construction, however, is how to decide when the proof is complete. The teacher may reclaim the authority and validate the argument at the end or may involve the students in the proof validation phase.

Sharing Authority During Proof Validation

If authority has been shared during the proof initiation and the proof construction, then it may feel natural to extend this shared authority to the proof validation by asking students if they are fully convinced. You may direct their attention to the original claim and the original purpose of the proof process and ask them if this has been achieved. But you can share authority during proof validation even if you did not share authority earlier in the process. A simple way to do this, if you are wrapping up the proof with a class discussion, is to use talk moves such as "is everyone convinced?" or "if a skeptic walked in, what part of the proof would they most likely attack?" You might also ask, "Do you think we could still find a counterexample if we looked hard enough?"

Another way to share authority with students during proof validation is to give them several sample arguments for the same mathematical claim—some that are considered valid by the teacher and some not. Then, ask the students to determine which sample arguments they believe are valid proofs and which are not valid proofs, and why. This task allows students to insert their own ideas related to what should count as proof and why, while sparking interesting negotiations amongst peers in class when disagreements occur.

For example, while discussing five different arguments for the Growing S-Pattern Task (see the top portion of Figure 4), we observed a group of four students coming to a consensus that one argument should not count as a valid proof because it used only examples and did not prove the given pattern for all possible cases, and it did not specify what possible numbers the pattern should hold for (e.g., integers, natural numbers). When discussing the argument in Figure 4, however, this same group of students could not agree about whether the argument should count as proof. One student argued that the expression was correct but that the reason-

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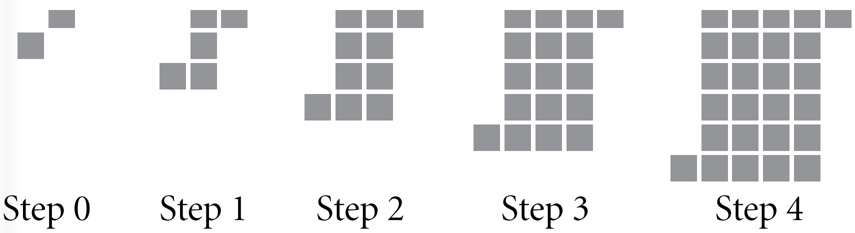
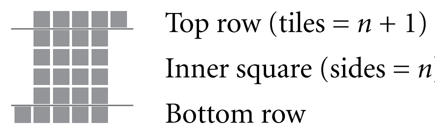
ing, which involves an "inner square," does not hold for Step 0 because there is no square in the center — the reasoning is incomplete. Another student believed it should count as proof since it could be inferred that there is a 0×0 square in Step 0. These discussions allowed the whole class to meaningfully consider the generality of reasoning and what is acceptable to the community in terms of implicit assumptions.

As an extension, a teacher might ask students to combine their individual thoughts related to what should count as a proof (and why) and work together to develop a class rubric for proof writing. By developing criteria for what should count as proof, students then have a reference or standard to refer back to when they make future judgements or validations about proofs that have already been initiated and constructed. Moreover, their voice and their understandings are made evident in the class-developed rubric for proof writing. So, while this gives the students a way to share authority related to proof validation, it also gives the teacher insight into what students perceive as valid mathematical argumentation (Ko, Yee, Bleiler-Baxter, and Boyle 2016). Table 1 shows a proof rubric created by students after reviewing sample arguments from their peers.

Task: Write an expression for the total number of tiles in the figure at an arbitreary step (n) in the pattern. Prove that your expression is a valid representation or the number of tiles. You may use drawings, words, number and/or symbols for your proof.



You can view this by slicing the figure to yield an "inner square" along with a top and bottom row of tiles that are one more than the step number. Using Step 4 as an exampls:



The inner square is the same on each side as the step number. The top/bottom row contains one more tile than the step number. $2(n + 1) + n^2$ can represent these features. Therefore, $2(n + 1)$ accounts for top and bottom row and n^2 accounts for the inner square.

Figure 4. An argument that may or may not be a proof for the Growing S-Pattern task.

Reasoning: Explain your thought process as to why the statement is correct (or incorrect)

Structure/Scaffolding/Organization: Essay sandwich — beginning, middle, and end; state what will be proved/disproved; have a concluding statement; step by step; starts from beginning and builds up

Representational support: Providing multiple necessary representations (visual, algebraic, table, real-world application), if applicable

Validity: Consistent, correct/true, variables must make sense, valid, using definitions or theorems correctly; eliminate the need for the reader to make assumptions; define necessary components

Generalization: Proof needs to be generalized; uses variables to encompass any case; not specific to a certain case

Table 1. A rubric that students can use to review proof arguments.

Sharing authority in proof validation has numerous benefits. It promotes an authentic view of proof among the students where proof is a human endeavor of removing doubt from a community, which aligns with the convincing and establishing-truth purposes of proof. Shared authority during proof validation also encourages students to critique reasoning and press one another for precise thinking, while they learn to do so in respectful ways.

Conclusion

Recognizing that it is not always feasible or appropriate to share full authority with students, we have broken the proving process down into three phases and identified ways to gradually and strategically share certain aspects of authority. Figure 5 encapsulates these recommendations.

As a teacher who is best positioned to know the needs and appropriate levels of challenge for your own students, there are a variety of ways you may wish to enact these ideas for shared authority. For instance, you may decide to first share authority during proof initiation to get student buy-in but then you still provide modeling for the proof construction and validation, so that students can learn the expectations of mathematical argumentation. Gradually, then, you may release authority over the construction and validation. Another way to approach it would be for you to provide the claim and the validation, because you know the curriculum and the rigor required, but to start by sharing authority with students for the proof construction. Only later, after students have gained some experience, do they gradually take the reins over the initiation and validation. In any case, we hope that this framework and the idea of sharing authority one piece at a time is helpful and, regardless of the starting point or the path, we can all move toward the goal of students being able to make claims, determine the veracity of claims, develop the proof argument, and agree as a community that it is valid and complete.

Proof Initiation	Proof Construction	Proof Validation
● Have students formulate the conjecture ● Have students phrase (or rephrase) the claim ● Have students choose which claim to prove ● Have students "prove or disprove-and-salvage" given claims	● Have students prepare the plan or outline for the proof ● Have students generate steps and justifications for their arguments ● Use a "group proof activity" where students bring proof ideas to a group and then formulate a new proof together	● Lead a discussion for the class to decide collectively if a proof is complete ● Give students several proofs to choose which ones are valid/ complete or invalid ● Have the class devise a rubric for determining if an argument is a proof

Figure 5. Three phases of the proving process with strategies for sharing authority in each phase.

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