**Abstract**—A novel approximation of linear phase almost equiripple low-pass finite impulse response filter is introduced. Its frequency response closely approaches the frequency response of an optimal equiripple low-pass finite impulse response filter. The presented approximation is based on the generating polynomial of the almost equiripple LP FIR filter. Here, we present the differential equation for the generating polynomial of the almost equiripple low-pass finite impulse response filter. The frequency response of such filter closely approaches the frequency response of an optimal equiripple low-pass finite impulse response filter. The approximation problem of an equiripple LP FIR filter remains still unresolved. This paper is seen as an extension of our previous activity focused on almost equiripple half-band FIR filters [2]. Here we are primarily concerned with the approximation of a linear phase almost equiripple LP FIR filter. We have derived the linear differential equation for the generating polynomial of the almost equiripple LP FIR filter. Based on the differential equation we have developed an efficient algorithm for algebraic evaluating the impulse response of the filter.

**II. BASIC TERMS**

We assume an impulse response $h(k)$ of type I, i.e. with odd length of $N = 2n+1$ coefficients and with even symmetry

$$a(0) = h(n), \quad a(k) = 2h(n + k) = 2h(n - k), k = 1 \ldots n.$$  

The transfer function of the filter is

$$H(z) = \sum_{k=0}^{2n} h(k) z^{-k} = z^{-n} \left[ a(0) + \sum_{k=1}^{n} a(k) T_k(w) \right] = z^{-n} Q(w)$$  

where $T_k(w)$ is Chebyshev polynomial of first kind. The function $Q(w)$ represents a polynomial in the variable $w = 0.5(z + z^{-1})$ which on the unit circle $z = e^{j\omega T}$ reduces to the real valued zero phase transfer function $Q(w)$ of the real argument $w = \cos(\omega T)$. Consequently we use the standard interval $w \in [-1, 1]$ for polynomial approximation.

**III. GENERATING POLYNOMIAL**

The generating polynomial for the low-pass FIR filter is related to the Zolotarev polynomial [4]. The extremal values of the Zolotarev polynomial $Z_{p,q}(w, \kappa)$ of degree $n = p + q$ alternate between -1 and +1 $(p+1)$-times in the interval $|w_p, 1|$, and $(q + 1)$-times in the interval $[-1, w_s]$. By inspection, the Zolotarev polynomial satisfies the differential equation

$$
(1 - w^2)(w - w_p)(w - w_s) \left( \frac{dZ_{p,q}(w, \kappa)}{dw} \right)^2 = n^2(w - w_m)^2 (1 - Z_{p,q}^2(w, \kappa)). \tag{3}
$$

This nonlinear differential equation expresses the fact that the first derivative $Z_{p,q}'(w, \kappa)$ does not vanish at the points $w = \pm 1, w_p, w_s$ where $Z_{p,q}(w, \kappa) = \pm 1$ for which the right hand side of eq. (3) vanishes, and that $w = w_m$ is a turning point corresponding to the local extrema at which $Z_{p,q}(w, \kappa) \neq \pm 1$. Elliptic modulus $\kappa$ is the driving parameter of all elliptic functions [4] appearing in the parametric solution of (3). We have derived the linear differential equation of the second order

$$
(w-w_p)(w-w_s)(w-w_m) \left[(1-w^2) \frac{d^2}{dw^2} - \frac{d}{dw} \right] Z_{p,q}(w, \kappa) \\
+ \left[(w-w_m)(w-w_p)+w_s\right] (w-w_p)(w-w_s)(1-w^2) \\
\times \frac{d}{dw} Z_{p,q}(w, \kappa) \\
+ n^2(w-w_m)^3(w) Z_{p,q}(w, \kappa) = 0. \tag{4}
$$

Assuming the second solution of (4) in form
fraction of quarter-period, namely and for Jacobi’s Zeta function $Z_\kappa$ complete elliptic integral of the first kind $K$. In both algorithms we have used the standard notation [3] for Jacobi’s elliptic functions $sn(u_0|\kappa)$, $cn(u_0|\kappa)$, $dn(u_0|\kappa)$ and $cd(u_0|\kappa)$, for the complete elliptic integral of the first kind $K(\kappa)$ of modulus $\kappa$ and for Jacobi’s Zeta function $Z(u_0|\kappa)$. The value $u_0$ is a fraction of quarter-period, namely $u_0 = p \frac{K(\kappa)}{n}$.

$$\sqrt{1 - w^2} S_{p,q}(w, \kappa)$$ we obtain the differential equation

$$\begin{align*}
&(w - w_p)(w - w_s)(w - w_m)
\left[ (1 - w^2) \frac{d^2}{dw^2} - 3w \frac{d}{dw} \right] S_{p,q}(w, \kappa)
+ \left[ (w - w_m) \left( w - \frac{w_p + w_s}{2} \right) - (w - w_p)(w - w_s) \right]
\times (1 - w^2) \frac{d}{dw} S_{p,q}(w, \kappa)
+ \left[ n(n+2)(w-w_m)^3 - (w_m - \frac{w_p + w_s}{2})(w+w_m)(w+2w_m) \right]
+ \left[ w_m(w_m - \frac{w_p + w_s}{2}) + w_m(w_pw_s - \frac{(w_p + w_s)^2}{2}) \right]
\times S_{p,q}(w, \kappa) = 0.
\end{align*}$$

(5)

The function $\sqrt{1 - w^2} S_{p,q}(w, \kappa)$ is the iso-extremal solution of the approximation equation (3) with the polynomial $S_{p,q}(w, \kappa)$ which forms the generating polynomial (Fig. 1) for the almost equiripple LP FIR filter specifications. With respect to the impulse response of the filter, it is advantageous to express the generating polynomial in the form of an expansion in Chebyshev polynomials of second kind $U_m(w)$

$$S_{p,q}(w, \kappa) = \sum_{m=0}^{n} a_+(m)U_m(w).$$

(6)

By inserting expansion (6) into equation (5) we can derive the recursive formula for the coefficients $a_+(m)$, see Tab. I. The recursion for the coefficient $a_+(m)$ is closely related to the algorithm for the Zolotarev polynomials [4]. In both algorithms we have used the standard notation [3] for Jacobi’s elliptic functions $sn(u_0|\kappa)$, $cn(u_0|\kappa)$, $dn(u_0|\kappa)$ and $cd(u_0|\kappa)$, for the complete elliptic integral of the first kind $K(\kappa)$ of modulus $\kappa$ and for Jacobi’s Zeta function $Z(u_0|\kappa)$. The value $u_0$ is a fraction of quarter-period, namely $u_0 = p \frac{K(\kappa)}{n}$.

IV. ZERO PHASE TRANSFER FUNCTION OF AN ALMOST EQUIRIPPLE LP FIR FILTER

The zero phase transfer function $Q(w)$ (Fig. 2) of an almost equiripple LP FIR filter is a normalized primitive function of the generating polynomial $S_{p,q}(w, \kappa)$

$$Q(w) = -N_1 + \frac{1}{N_2 - N_1} \int_{-N_1}^{w} S_{p,q}(w, \kappa) dw = -N_1 + \frac{1}{N_2 - N_1} S(w)$$

(7)

where the norming values $N_1$, $N_2$ are

$$N_1 = \left\{ \begin{array}{ll}
S(w=1) = |P(\omega T = \pi)| & \text{for } q \text{ even} \\
S(w=\cos(\frac{n\pi}{n+1})) = |P(\omega T = \frac{n\pi}{n+1})| & \text{for } q \text{ odd}
\end{array} \right. \quad (8)
$$

and $|P(e^{j\omega T})|$ is the amplitude frequency response corresponding to the polynomial $S(w)$. The amplitude frequency response $|H(e^{j\omega T})|$ corresponding to the zero phase transfer function $Q(w)$ from Fig. 2 is shown in Fig. 3. The algebraic

$$\begin{align*}
N_2 = \left\{ \begin{array}{ll}
S(w=1) = |P(\omega T = 0)| & \text{for } p \text{ even} \\
S(w=\cos(\frac{\pi}{n+1})) = |P(\omega T = \frac{\pi}{n+1})| & \text{for } p \text{ odd}
\end{array} \right. \quad (9)
\end{align*}$$

algorithm for the impulse response coefficients $h(k)$ is robust and provides easily polynomials with tens of thousands of coefficients. For illustration, the remarkable selectivity of the generating polynomial is shown in Fig. 4.

V. DESIGN PROCEDURE

The main objective in filter design is to evaluate the filter degree satisfying filter specification and also to evaluate the impulse response coefficients. Based on the differential equation (5) we have derived a simple procedure for evaluation of the impulse response $h(k)$ of the filter (Tab. I). Its length is $2(p+q) + 3$ coefficients. The design procedure consists of few steps:

1. Specify the minimum attenuation in the stop-band $a_{s,0}$ or the maximum attenuation in the pass-band $a_{p,0} = 20 \log(a_0)$. They are related by $a_p + a_s = 1$. Further, specify the pass-band frequency $\omega_p T$ and the stop-band frequency $\omega_s T$. The maximum width of the transition

Fig. 1. Generating polynomial $S_{19,11}(w, 0.55)$

Fig. 2. Zero phase transfer function $Q(w)$ for $p = 19$, $q = 11$ and $\kappa = 0.55$
band is \( \Delta \omega T = (\omega_s - \omega_p)T \) (Fig. 3). The degree equation relates the degree \( n \) of the generating polynomial \( S_{p,q}(w, \kappa) \), the maximum width of the transition band \( \Delta \omega T \) and the minimum attenuation in the stopband \( a_{\text{min}} \).

\( \xi_1 = -14.02925485 \), \( \xi_2 = -32.86119410 \), \( \xi_3 = -5.8017336 \), \( \xi_4 = 2.99564719 \), \( \xi_5 = -21.24188066 \) and \( \xi_6 = 0.28632078 \). Equation (10) approximates the empirical data with excellent accuracy.

\[
p = \text{round} \left[ n \frac{(\omega_s + \omega_p)T}{2\pi} \right], \quad q = n - p .
\]

(11)

\[
\kappa = \sqrt{1 - \left( \frac{1 - k}{1 + k} \right)^2}, \quad (12)
\]
where \( k \) is approximated by the formula

\[
  k = \left\{ \begin{array}{l}
  \chi_1 + \chi_2 \left( \frac{\chi_2}{p + \chi_3} \right)^4 \left( n + \chi_5 p + \chi_6 \right) w_p \\
  + \chi_5 + \left( \frac{\chi_8}{p + \chi_5} \right)^4 \left( n + \chi_9 p + \chi_{10} \right)^{\chi_7+\chi_9+\chi_14} \end{array} \right.
\]

(13)

for \( w_p = \cos \left( \frac{\pi - \Delta \omega T}{2} \right) \) and \( \chi_1 = -0.00452871 \), \( \chi_2 = 0.51350112 \), \( \chi_3 = 2.56407699 \), \( \chi_4 = 1.12297611 \), \( \chi_5 = 0.01473844 \), \( \chi_6 = 0.14824220 \), \( \chi_7 = 0.00245539 \), \( \chi_8 = 0.52499043 \), \( \chi_9 = 0.75104615 \), \( \chi_{10} = 1.29448910 \), \( \chi_{11} = -1.06038228 \), \( \chi_{12} = 0.64247743 \), \( \chi_{13} = -0.00932499 \), \( \chi_{14} = 1.88486768 \). Equation (13) was obtained by approximating the empirical values.

VI. EXAMPLE OF THE DESIGN

Let us design a pair of magnitude complementary FIR filters. The low-pass FIR filter is specified by the pass-band frequency \( \omega_p T = 0.27 \pi \), stop-band frequency \( \omega_s T = 0.3 \pi \) and by the minimal attenuation in the stop-band \( a_{s,\text{dB}} = -100 \text{ dB} \).

Based on the presented design procedure, we get the degree of the generating polynomial \( n = 223.0302777 \rightarrow 224 \) (10), integer indices \( p = 64 \), \( q = 160 \) (11) and elliptic modulus \( \kappa = 0.467363 \) (13). The length of the filter is \( N = 451 \) coefficients. The actual filter properties are \( \omega_p T = 0.271898 \pi \), \( \omega_s T = 0.301728 \pi \) and \( a_{s,\text{dB}} = -100.36 \text{ dB} \). The amplitude frequency responses \( 20 \log revisited equation for the generating polynomial. We have derived an algebraic procedure for an efficient evaluation of the impulse response of the filter. The practicality of the design procedure has been demonstrated.

VII. CONCLUSION

We have introduced an approximation of the almost equiripple low-pass FIR filter based on the generating polynomial which is related to the Zolotarev polynomial. We have presented the differential equation for the generating polynomial. Employing the differential equation we have derived an algebraic procedure for an efficient evaluation of the impulse response of the filter. The practicality of the design procedure has been demonstrated.

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