Sylwia Cichacz; Mirko Horňák
Decomposition of bipartite graphs into closed trails

_Czechoslovak Mathematical Journal_, Vol. 59 (2009), No. 1, 129--144

Persistent URL: [http://dml.cz/dmlcz/140468](http://dml.cz/dmlcz/140468)

Terms of use:

© Institute of Mathematics AS CR, 2009

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to
digitized documents strictly for personal use. Each copy of any part of this document must
contain these Terms of use.

This paper has been digitized, optimized for electronic delivery and stamped
with digital signature within the project _DML-CZ: The Czech Digital
Mathematics Library_ [http://project.dml.cz](http://project.dml.cz)
DECOMPOSITION OF BIPARTITE GRAPHS INTO CLOSED TRAILS

Sylwia Cichacz, Kraków, and Mirko Horňák, Košice

(Received February 6, 2007)

Abstract. Let Lct(G) denote the set of all lengths of closed trails that exist in an even graph G. A sequence \((t_1, \ldots, t_p)\) of elements of Lct(G) adding up to \(|E(G)|\) is G-realisable provided there is a sequence \((T_1, \ldots, T_p)\) of pairwise edge-disjoint closed trails in G such that \(T_i\) is of length \(t_i\) for \(i = 1, \ldots, p\). The graph G is arbitrarily decomposable into closed trails if all possible sequences are G-realisable. In the paper it is proved that if \(a \geq 1\) is an odd integer and \(M_{a,a}\) is a perfect matching in \(K_{a,a}\), then the graph \(K_{a,a} - M_{a,a}\) is arbitrarily decomposable into closed trails.

Keywords: even graph, closed trail, graph arbitrarily decomposable into closed trails, bipartite graph

MSC 2010: 05C70

All graphs we are dealing with in this paper are simple, finite and nonoriented. We use the standard terminology and notation of graph theory.

For \(p, q \in \mathbb{Z}\) let \([p, q]\) denote the integer interval bounded by \(p\) and \(q\), i.e. \([p, q] := \{z \in \mathbb{Z} : p \leq z \leq q\}\); similarly, let \([p, \infty) := \{z \in \mathbb{Z} : p \leq z\}\). The concatenation of finite sequences \(A = (a_1, \ldots, a_m)\) and \(B = (b_1, \ldots, b_n)\) is the sequence \(AB := (a_1, \ldots, a_m, b_1, \ldots, b_n)\). The concatenation is an associative operation on finite sequences; we use this fact in the notation \(\prod_{i=1}^{k} A_i\) representing the concatenation of finite sequences \(A_i, i \in [1, k]\), in the order given by the sequence \((A_1, \ldots, A_k)\). As usual, \(A^k\) denotes \(\prod_{i=1}^{k} A_i\) with \(A_i = A\) for any \(i \in [1, k]\), and \(A^0\) is the empty sequence ( ). A finite sequence \(A = (a_1, \ldots, a_m)\) is changeable to a finite sequence \(A' = (a'_1, \ldots, a'_m)\) of the same length (in symbols \(A \sim A'\)) if there is a bijection...
\[ \pi \subseteq [1, m] \times [1, m] \] such that \( a_i' = a_{\pi(i)} \) for any \( i \in [1, m] \). If \( I \subseteq [1, m] \), we denote by \( A(I) \) the subsequence of \( A \) formed by all \( a_i \)'s with \( i \in I \) (ordered in compliance with the natural ordering of \( I \)).

A **closed trail** of length \( n \in [3, \infty) \) (an \( n \)-trail for short) is a sequence \( \prod_{i=1}^{n+1} (x_i) \) of vertices of \( G \) such that \( x_1 = x_{n+1} \) and if \( i, j \in [1, n], i \neq j \), then \( \{x_i, x_{i+1}\} \in E(G) \) and \( \{x_j, x_{j+1}\} \neq \{x_i, x_{i+1}\} \). A graph \( G \) is Eulerian if it has a closed trail of length \( |E(G)| \). It is well known that a graph of order at least three is Eulerian if and only if it is connected and even (all its vertices are of even degrees). Thus, we may identify the notions of a closed trail in a graph \( G \) and a nontrivial connected even subgraph of \( G \). Let \( \text{Lct}(G) \) be the set of all lengths of closed trails existing in \( G \) and let \( \text{Sct}(G) \) be the set of all finite sequences consisting of elements of \( \text{Lct}(G) \) that add up to \( |E(G)| \). Deleting a closed trail from an even graph \( G \) yields an even subgraph of \( G \). Continuing this process until all edges of \( G \) are exhausted leads to a sequence \( \vec{T} := (\vec{T}_1, \ldots, \vec{T}_p) \) of pairwise edge-disjoint closed trails in \( G \) such that, for any \( i \in [1, p] \), \( \vec{t}_i := |E(\vec{T}_i)| \in \text{Lct}(G) \), and \( \vec{\tau} := (\vec{t}_1, \ldots, \vec{t}_p) \in \text{Sct}(G) \); the sequence \( \vec{\tau} \) is said to be \( G \)-**realisable** and the sequence \( \vec{T} \) is a \( G \)-**realisation** of the sequence \( \vec{\tau} \). An even graph \( G \) is arbitrarily decomposable into closed trails (ADCT) provided all sequences of \( \text{Sct}(G) \) are \( G \)-realisable.

There are some classes of even graphs that are known to be ADCT. Among these are complete graphs \( K_n \) for \( n \) odd, the graphs \( K_n - M_n \), where \( M_n \) is a perfect matching in \( K_n \), for \( n \) even (Balister [1]) and complete bipartite graphs \( K_{a,b} \) for \( a, b \) even (Horňák and Woźniak [8]). An even graph that is large and dense enough is necessarily ADCT. Namely, according to Balister [2], there are positive constants \( n \) and \( \varepsilon \) such that an even graph \( G \) is ADCT whenever \( |V(G)| \geq n \) and \( \delta(G) \geq (1 - \varepsilon)|V(G)| \). Horňák and Kocková [7] proved that if an even complete tripartite graph \( K_{p,q,r} \) with \( p \leq q \leq r \) is ADCT, then either \( (p, q, r) \in \{(1, 1, 3), (1, 1, 5)\} \) or \( p = q = r \); moreover, the graphs \( K_{1,1,3}, K_{1,1,5} \) and \( K_{p,p,p} \) with \( p = 5 \cdot 2^l \), \( l \in [0, \infty) \), are ADCT. The notion of an ADCT graph can be generalized in a natural way to oriented graphs (see Balister [3] and Cichacz [5]) and to pseudographs (Cichacz et al. [6]).

It may happen that an even graph is not ADCT though all its connected components are. For example, both \( C_8 \) (an 8-vertex cycle) and \( K_{2,4} \) are ADCT, but \( C_8 \cup K_{2,4} \) is not since the sequence \( (4)^4 \in \text{Sct}(C_8 \cup K_{2,4}) \) is not \( (C_8 \cup K_{2,4}) \)-realisable. On the other hand, if the graphs \( G^1, G^2 \) are ADCT and \( E(G^1) \cap E(G^2) = \emptyset \), but \( V(G^1) \cap V(G^2) \neq \emptyset \), when trying to prove that a sequence \( \tau \in \text{Sct}(G^1 \cup G^2) \) is \( (G^1 \cup G^2) \)-realisable, we have at our disposal not only closed trails of \( G^1 \) and \( G^2 \), but also closed trails \( T^1 \cup T^2 \), where \( T^1 \) is a closed trail of \( G^i \), \( i = 1, 2 \), and \( V(T^1) \cap V(T^2) \neq \emptyset \). Therefore, a potential general strategy for proving that a
graph $G$ is ADCT can be described as follows: Write $G$ as an edge-disjoint, but not vertex-disjoint, union of ADCT graphs $G^1$ and $G^2$, and require from $G^i$-realisations, $i = 1, 2$, to have an additional property that some of their chosen trails contain common vertices of $V(G^1) \cap V(G^2)$.

Clearly, when analyzing whether a nontrivial connected even graph $G$ is ADCT, it is sufficient to show that any sequence $(t_1, \ldots, t_p) \in \text{Sct}(G)$ of length $p \geq 2$ is $G$-realisable; indeed, the graph $G$ is Eulerian, and so the unique sequence $([E(G)])$ of length 1 in $\text{Sct}(G)$ is trivially $G$-realisable. We have also the following evident statement:

**Lemma 1.** If $G$ is an even graph, $\tau_1, \tau_2 \in \text{Sct}(G)$ and $\tau_1 \sim \tau_2$, then the sequence $\tau_1$ is $G$-realisable if and only if $\tau_2$ is.

Pick disjoint sets $X^j = \{x^j_i : i \in [1, \infty]\}$, $j = 1, 2$, and let $X^j_{p,q} := \{x^j_i : i \in [p,q]\}$ for $p,q \in [1,\infty)$. In this paper the complete bipartite graph $K_{a,b}$ will have the bipartition $\{X^1_{1,a}, X^2_{1,b}\}$ and $M_{a,a}$ will be the perfect matching in $K_{a,a}$ consisting of $\{x^1_i, x^2_i\}$ for $i \in [1,a]$. If $a$ is odd, then $K'_{a,a} := K_{a,a} - M_{a,a}$ is an even graph. The main aim of our paper is to show that the graph $K'_{a,a}$ is ADCT for any odd $a \in [1,\infty)$. We proceed by induction on $a$ and we use the above general strategy. For odd $a \geq 7$ consider the even subgraph $F_a \cong K'_{a-4,a-4}$ of $K'_{a,a}$ induced on the set $X^1_{5,a} \cup X^2_{5,a}$. The even graph $H_a := K'_{a,a} - F_a$ is an edge-disjoint union of the even graph $K'_{5,5}$ and two even subgraphs $G^1_a \cong G^2_a \cong K_{4,a-5}$ of $K'_{a,a}$ where $G^i_a$ is induced on the set $X^i_{4} \cup X^3_{6,a}$, $i = 1,2$. Thus putting $G_a := K'_{5,5} \cup G^1_a$ we obtain $H_a = G_a \cup G^2_a$. We shall show subsequently that the graphs $K'_{5,5}$ and $G_a, H_a$ are ADCT; furthermore, $G_a$-realisations and $H_a$-realisations can be chosen to have appropriate additional properties. Note that all the graphs mentioned are bipartite. The following assertion shows the maximal extent of the set $\text{Lct}(G)$ for an even bipartite graph $G$.

**Proposition 2.** If $G$ is an even bipartite graph, then $\text{Lct}(G) \subseteq \{2k : k \in [2,|E(G)|/2 - 2]\} \cup \{|E(G)|\}$.

**Proof.** All subgraphs of $G$ are bipartite, hence all closed trails in $G$ (as edge-disjoint unions of cycles) are of even lengths. A subgraph $T$ of $G$ with $|E(T)| = |E(G)| - 2$ is not even (and therefore not a closed trail) for $G - T$ has at least two vertices of degree one.

When proving that an even bipartite graph $G$ is ADCT we do not exhibit the structure of $\text{Lct}(G)$ explicitly, but we show implicitly that $\text{Lct}(G)$ is of maximal extent by finding all $G$-realisations that are theoretically possible from the point of view of Proposition 2.
Recall again the result on complete bipartite graphs:

**Theorem 3.** If \( a, b \) are even integers with \( 2 \leq a \leq b \), then the graph \( K_{a,b} \) is ADCT.

We know due to Chou et al. [4] that sequences of \( \text{Sct}(K_{a,b}) \) with small terms have \( K_{a,b} \)-realisations consisting of cycles:

**Theorem 4.** If \( a, b \) are even integers with \( a \geq 4, b \geq 6 \) and \( \tau = (t_1, \ldots, t_p) \in \text{Sct}(K_{a,b}) \) with \( t_i \in \{4, 6, 8\} \) for any \( i \in [1, p] \), then there is a \( K_{a,b} \)-realisation \((T_1, \ldots, T_p)\) of the sequence \( \tau \) such that \( T_i \) is a cycle for any \( i \in [1, p] \).

We start our analysis by dealing with \( a \leq 5 \).

**Proposition 5.** The graph \( K'_{a,a} \) with \( a \in \{1, 3, 5\} \) is ADCT.

**Proof.** We have \( K'_{1,1} \cong 2K_1 \), and so for \( a = 1 \) the result follows from \( \text{Sct}(K'_{1,1}) = \text{Lct}(K'_{1,1}) = \emptyset \).

Since \( K'_3,3 \cong C_6 \), the unique sequence \( (6) \in \text{Sct}(K'_3,3) \) is trivially \( K'_3,3 \)-realisable.

The sequences \((4)^5, (4)^2(6)^2\), and \((6)^2(8)\) are \( K'_5,5 \)-realisable, see Figure 1. Observe that any two 4-trails of the left \( K'_5,5 \)-realisation have a common vertex, hence every sequence in \( \text{Sct}(K'_5,5) \), whose all terms are divisible by 4, is \( K'_5,5 \)-realisable. Moreover, in the middle \( K'_5,5 \)-realisation any 4-trail has a common vertex with any 6-trail. Therefore, the remaining sequences \((4, 6, 10), (6, 14), (10)^2 \in \text{Sct}(K'_5,5) \) are \( K'_5,5 \)-realisable, too. \( \square \)

![Figure 1. \( K'_5,5 \)-realisations of three sequences](image)

We shall need also the following three simple statements:
Proposition 6. If $G$ is a complete bipartite graph with bipartition $\{X, Y\}$ and $\pi \subseteq X \times X$, $\varrho \subseteq Y \times Y$ are bijections, then the mapping $\alpha \subseteq V(G) \times V(G)$ with $\alpha|X = \pi$ and $\alpha|Y = \varrho$ is an automorphism of $G$.

Proposition 7. If $a \in [1, \infty)$ and $\pi \subseteq [1, a] \times [1, a]$ is a bijection, then the mappings $\pi, \tilde{\pi} \subseteq V(K_{a,a}') \times V(K_{a,a}')$, determined by $\pi(x_i^j) = x_{\pi(i)}^j$ and $\tilde{\pi}(x_i^j) = x_{3-j}^{3-i}$ for any $i \in [1, a]$ and $j \in [1, 2]$, are automorphisms of $K_{a,a}'$.

Lemma 8. If $T_1, T_2$ are edge-disjoint closed trails in $K_{5,5}'$ and $k \in [1, 2]$, then $|(V(T_1) \cup V(T_2)) \cap X_{1,5}^k| \geq 3$.

Proof. If $|E(T_1) \cup E(T_2)| \geq 10$, then the edges of $E(T_1) \cup E(T_2)$ must cover at least $\lceil \frac{10}{4} \rceil = 3$ vertices of $X_{1,5}^k$ (note that $\Delta(K_{5,5}') = 4$). The same is true if both $T_1$ and $T_2$ are 4-trails, since then the subgraph of $K_{5,5}'$ that is induced by the eight edges incident with $x_i^k$ or $x_j^k$, $i, j \in [1, 5]$, $i \neq j$, has two vertices of degree 1 (namely $x_i^{3-k}$ and $x_j^{3-k}$), and so it cannot be equal to $T_1 \cup T_2$. \hfill \Box

Theorem 9. The graph $G_a$ is ADCT for any odd integer $a \geq 7$. Moreover, given $s \in [4, 5]$, any sequence $\tau = (t_1, \ldots, t_p) \in \text{Sct}(G_a)$ of length $p \geq 2$ has a $G_a$-realisation $(T_1, \ldots, T_p)$ such that $T_1$ contains as a subgraph a 3-vertex path with endvertices $x^2_1$ and $x^2_s$ and $T_2$ contains the vertex $x^2_2$.

Proof. We use the general strategy with ADCT graphs $G^1 := K_{5,5}'$ (Proposition 5) and $G^2 := G^1_a$ (Theorem 3); the structure of the graph $G_a$ is presented in Figure 2.

![Figure 2. The graph $G_a$](image)

First we show how to proceed provided three special conditions are fulfilled.
(C1) If there is $I^1$ with $[1, 2] \subseteq I^1 \subseteq [1, p]$ and $\sum_{i \in I_1} t_i = |E(G^1)| = 20$, put $I^2 := [1, p] - I^1$ and $\tau_l := \tau(I^l), l = 1, 2$. There is a $G^1$-realisation $(T_1, T_2)T^1$ of the sequence $\tau^1$ and a $G^2$-realisation $T^2$ of the sequence $\tau^2$. Then $T := (T_1, T_2)T^1T^2$ is a $G_a$-realisation of the sequence $\tau^1|\tau^2 \sim \tau$. Any closed trail in a bipartite graph with bipartition $\{U, V\}$ is an alternating sequence of vertices of $U$ and $V$. Therefore, by Proposition 7 and Lemma 8, we may suppose without loss of generality that the trails $T_1$ and $T_2$ have the required properties.

(C2) If there are $I^1$ and $j \in [1, p] - I^1$ such that $[1, 2] \subseteq I^1 \cup \{j\}$, $\sum_{i \in I_1} t_i \leq 16$ and $\sum_{i \in I_1} t_i + t_j \geq 24$, put $I^2 := [1, p] - I^1 - \{j\}, t^1_j := 20 - \sum_{i \in I_1} t_i$ and $t^2_j := \sum_{i \in I_1} t_i + t_j - 20$. There is a $G^1$-realisation $(T^1_j)T^1$ of the sequence $(t^1_j)\tau(I^1) \in \text{Sct}(G^1), l = 1, 2$; for $i \in [1, 2] - \{j\} \subseteq I^1$ let $T_i$ be a $t_i$-trail of $T^1$. Using Propositions 6, 7 and Lemma 8 we may suppose without loss of generality that $T_1$ (or $T^1_1$ if $j = 1$) contains a subgraph a 3-vertex path with endvertices $x_{2i}^1$ and $x_{2j}^2$, $T_2$ (or $T^1_2$ if $j = 2$) contains the vertex $x_{2j}^2$ and $V(T^1_1) \cap V(T^2_2) \cap X_{1, 4} \neq \emptyset$. Then $T_j := T^1_j \cup T^2_j$ is a $t_j$-trail and $(T^1_j)T^1T^2$ is an appropriate $G_a$-realisation of the sequence $(t^1_j)\tau(I^1)\tau(I^2) \sim \tau$.

(C3) If there are $I^1$ and $\{j, k\} \subseteq [1, p] - I^1$ such that $[1, 2] \subseteq I^1 \cup \{j, k\}, \min\{t_j, t_k\} \geq 8, \sum_{i \in I_1} t_i \leq 12$ and $\sum_{i \in I_1} t_i + t_j + t_k \geq 28$, put $I^2 := [1, p] - I^1 - \{j, k\}$, $t^1_j := \min\{16 - \sum_{i \in I_1} t_i, t_j - 4\}, t^1_k := \max\{4, 24 - \sum_{i \in I_1} t_i - t_j\}, t^2_j := t_j - t^1_j$ and $t^2_k := t_k - t^1_k$. Then $t^1_j + t^1_k + \sum_{i \in I_1} t_i = |E(G^1)|$ and there is a $G^1$-realisation $(T^1_j, T^1_k)T^1$ of the sequence $(t^1_j, t^1_k)\tau(I^1), l = 1, 2$; for $i \in [1, 2] - \{j, k\} \subseteq I^1$ let $T_i$ be a $t_i$-trail of $T^1$. By Propositions 6, 7 and Lemma 8 we may suppose without loss of generality that $T_1$ (or $T^1_1$ if $1 \in \{j, k\}$) contains a subgraph a 3-vertex path with endvertices $x_{2i}^1$ and $x_{2m}^2$, $T_2$ (or $T^1_2$ if $2 \in \{j, k\}$) contains the vertex $x_{2m}^2$ and $V(T^1_m) \cap V(T^2_m) \cap X_{1, 4} \neq \emptyset$ for any $m \in \{j, k\}$. Then $T_m := T^1_m \cup T^2_m$ is a $m$-trail, $m = j, k$ and $(T^1_j, T_k)T^1T^2$ is a required $G_a$-realisation of the sequence $(t^1_j, t^1_k)\tau(I^1)\tau(I^2) \sim \tau$.

Let $i_1, i_2 \in [1, 2]$ be such that $i_1 \neq i_2$ and $t_{i_1} \leq t_{i_2}$. Since there are no additional requirements on $T_i$ with $i \in [3, p]$, having in mind Lemma 1, in our analysis we may suppose without loss of generality that $t_i \leq t_{i+1}$ for any $i \in [3, p - 1]$.

(1) $t_1 + t_2 \geq 24$.

(11) If $t_{i_1} \geq 18$, then $I^1 := \emptyset, j := 1, k := 2$ (C3), i.e. the condition (C3) is satisfied with the presented values of $I^1$, $j$ and $k$.

(12) If $t_{i_1} \leq 16$, then $I^1 := \{i_1\}, j := i_2$ (C2).

(2) If $t_1 + t_2 = 22$, then $t_{i_1} \leq 10, t_{i_2} \geq 12$ and $\sum_{i=3}^p t_i = 4a - 22 \equiv 2 \pmod{4}$, hence there is $l \in [3, p]$ with $t_l \equiv 2 \pmod{4}$.

(21) If $t_p \geq 8$, then $I^1 := \{i_1\}, j := i_2, k := p$ (C3).
(22) If \( t_p = t_l = 6 \), then \( I^1 := \{i_1, p\}, j := i_2 \rightarrow (C2) \).

(3) If \( t_1 + t_2 = 20 \), then \( I^1 := [1, 2] \rightarrow (C1) \).

(4) If \( t_1 + t_2 = 18 \), then \( t_1 \leq 8, t_2 \geq 10 \) and there is \( l \in [3, p] \) with \( t_l \equiv 2 \) (mod 4).

(4) If \( t_1 + t_2 = 18 \), then \( t_1 \leq 8, t_2 \geq 10 \) and there is \( l \in [3, p] \) with \( t_l \equiv 2 \) (mod 4).

(41) If \( t_1 \geq 10 \), then \( I^1 := \{i_1\}, j := i_2, k := l \rightarrow (C3) \).

(42) If \( t_1 = 6 \), then \( I^1 := \{i_1, l\}, j := i_2 \rightarrow (C2) \).

(5) If \( t_1 + t_2 \leq 16 \), let \( q \in [2, p - 1] \) be determined by the inequalities \( \sum_{i=1}^{q+1} t_i \leq 22 \) and \( \sum_{i=1}^{q+1} t_i \geq 24 \).

(51) If \( \sum_{i=1}^{q} t_i = 22 \), then \( q \geq 3 \) and there is \( l \in [q + 1, p] \) with \( t_l \equiv 2 \) (mod 4).

(511) \( t_q \geq 6 \).

(5111) If \( t_p \geq t_q + 2 \), then \( I^1 := [1, q - 1], j := p \rightarrow (C2) \).

(5112) If \( t_1 = t_q \) for any \( i \in [q + 1, p] \), then \( t_q = t_l \equiv 2 \) (mod 4).

(51121) If \( t_q \geq 10 \), then \( I^1 := [1, q - 1], j := q, k := q + 1 \rightarrow (C3) \).

(51122) If \( t_q = 6 \), put \( \tau^1 := (4) \prod_{i=1}^{q-1} (t_i) \in \text{Sct}(G^1), \tau^2 := (8)(6)^{p-1}q \in \text{Sct}(G^2) \) and consider a \( G^1 \)-realisation \( (T^1_q) \prod_{i=1}^{q+1} (T_i) \) of the sequence \( \tau^1 \) and a \( G^2 \)-realisation \( (T^2_{q+1}) \prod_{i=q+2}^{p} (T_i) \) of the sequence \( \tau^2 \) yielded by Theorem 4. Let \( T^1_q = \prod_{i=1}^{5} (b_i) \) with \( b_1 = b_5 \in X_{1,5}^1 \) and let \( T^2_{q+1} = \prod_{i=1}^{9} (c_i) \) with \( c_1 = c_9 \in X_{1,4}^1 \). Since \( T^2_{q+1} \) is a cycle, we have \( V(T^2_{q+1}) \cap X_{1,4}^1 = X_{1,4}^1 \). By Proposition 7 we may suppose without loss of generality that \( b_1 = c_1 \) and \( b_3 = c_5 \). With \( T_q := (c_1, b_2) \prod_{i=5}^{9} (c_i) \) and \( T_{q+1} := (c_1, b_4) \prod_{i=1}^{5} (c_{i-1}) \) then \( (T_1, \ldots, T_p) \) is a \( G_a \)-realisation of the sequence \( \tau \). Since \( q \geq 3 \), by Proposition 7 and Lemma 8 we may suppose without loss of generality that the additional requirements on \( T_1 \) and \( T_2 \) are fulfilled.

(512) If \( t_q = 4 \), then \( t_1 + t_2 \equiv 2 \) (mod 4), and so \( q \geq 4 \) and \( \sum_{i=1}^{q-2} t_i = 14 \).

(5121) If \( t_p \geq 10 \), then \( I^1 := [1, q - 2], j := p \rightarrow (C2) \).

(5122) If \( t_p \leq 8 \), then \( t_1 = 6 \) and \( I^1 := [1, q - 2] \cup \{l\} \rightarrow (C1) \).

(52) If \( \sum_{i=1}^{q} t_i = 20 \), then \( I^1 := [1, q] \rightarrow (C1) \).

(53) If \( \sum_{i=1}^{q} t_i = 18 \), then \( q \geq 3 \) and there is \( l \in [q + 1, p] \) with \( t_l \equiv 2 \) (mod 4).

(531) If \( t_q \geq 6 \), then \( \sum_{i=1}^{q-1} t_i \leq 12 \).

(5311) If \( t_p \geq t_q + 6 \), then \( I^1 := [1, q - 1], j := p \rightarrow (C2) \).
If there is \( m \in [q+1, p] \) with \( t_m = t_q + 2 \), then \( I^1 := [1, q-1] \cup \{m\} \to (C1) \).
\[(5313)\] If \( t_i \in \{t_q, t_q + 4\} \) for any \( i \in [q+1, p] \), then \( t_q \equiv t_i \equiv 2 \) (mod 4), hence \( t_q \leq 10 \).
\[(53131)\] If \( t_q = 10 \), then \( q = 3 \), \( I^1 := [1, q-1] \), \( j := q \), \( k := q + 1 \) \to (C3) .
\[(53132)\] If \( t_q = 6 \), put \( \tau^1 := (8) \prod_{i=1}^{q-1} (t_i) \in \text{Sct}(G^1) \) and \( \tau^2 := (t_p - 2) \prod_{i=q+1}^{p-1} (t_i) \in \text{Sct}(G^2) \). Consider a \( G^1 \)-realisation \( (T^1_q) \prod_{i=1}^{q-1} (T_i) \) of the sequence \( \tau^1 \) and a \( G^2 \)-realisation \( (T^2_p) \prod_{i=q+1}^{p-1} (T_i) \) of the sequence \( \tau^2 \). Let \( T^1_q = \prod_{i=1}^{9} b_i \) with \( b_1 = b_2 \in X_{1,5}^1 \) and let \( T^2_p = \prod_{i=1}^{t_p-1} (c_i) \) with \( c_1 = c_{t_p-1} \in X_{1,4}^1 \). We have \( |V(T^1_q) \cap X_{1,5}^1| \geq 3 \) (if \( T^1_q \) is not a cycle, it is a union of two edge-disjoint 4-trails and then it suffices to use Lemma 8). Therefore, we may suppose without loss of generality that \( b_5 \neq b_1 \). Moreover, by Proposition 6, the assumption \( c_1 = b_1 \) and \( c_3 = b_5 \) also does not cause a loss of generality. With \( T_q := (b_1, c_2) \prod_{i=1}^{5} (b_{6-i}) \) and \( T_p := (c_1, b_8, b_7, b_6) \prod_{i=3}^{t_p-1} (c_i) \) then, using Proposition 7 and Lemma 8, we may suppose without loss of generality that \( (T_1, \ldots, T_p) \) is an appropriate \( G_a \)-realisation of the sequence \( \tau \).
\[(532)\] \( t_q = 4 \).
\[(5321)\] If \( t_l \geq 10 \), then \( I^1 := [1, q-1] \), \( j := l \) \to (C2) .
\[(5322)\] If \( t_l = 6 \), then \( I^1 := [1, q-1] \cup \{l\} \to (C1) .
\[(54)\] If \( \sum_{i=1}^{q} t_i \leq 16 \), then \( I^1 := [1, q] \), \( j := q + 1 \) \to (C2) .

**Theorem 10.** The graph \( H_a \) is ADCT for any odd integer \( a \geq 7 \). Moreover, any sequence \( \tau = (t_1, \ldots, t_p) \in \text{Sct}(H_a) \) of length \( p \geq 2 \) has an \( H_a \)-realisation \( (T_1, \ldots, T_p) \) such that there are \( (i_r, j_r) \in [5, a] \times [1, 2] \) with \( x_{i_r}^{j_r} \in V(T_r) \), \( r = 1, 2 \), and \( i_1 \neq i_2 \).

**Proof.** We proceed similarly as in the proof of Theorem 9 with ADCT graphs \( G^1 := G^2_a \) (Theorem 3) and \( G^2 := G_a \) (Theorem 9). The graph \( H_a \) is depicted in Figure 3.

(C4) If there is \( I^1 \subseteq [1, p] \) such that \( |[1, 2] \cap I^1| \geq 1 \) and \( \sum_{i \in I^1} t_i = |E(G^1)| = 4a - 20 \), put \( I^2 := [1, p] - I^1 \) and \( \tau^l := \tau(I^l) \), \( l = 1, 2 \). Let \( T^l \) be a \( G^l \)-realisation of the sequence \( \tau^l \), \( l = 1, 2 \), and let \( T_i \) be a \( t_i \)-trail of \( T^1 T^2 \), \( i = 1, 2 \). If \( [1, 2] \subseteq I^l \), by Proposition 6 we may suppose without loss of generality that \( x_{i}^{j} \in V(T_i) \), \( i = 1, 2 \); in such a case we are done with \( (i_1, j_1) := (6, 1) \) and \( (i_2, j_2) := (7, 1) \). If there is \( m \in [1, 2] \) such that \( m \in I^l \) and \( 3 - m \in I^2 \), then, by Proposition 6 and Theorem 9, we may suppose without loss of generality that \( (i_m, j_m) := (6, 1) \) and \( (i_{3-m}, j_{3-m}) := (5, 2) \) are appropriate pairs.
(C5) If there are $I^1$ and $j \in [1, p] - I^1$ such that $|[1, 2] \cap (I^1 \cup \{j\})| \geq 1$, $\sum_{i \in I^1} t_i \leq 4a - 24$ and $\sum_{i \in I^1} t_i + t_j \geq 4a - 16$, put $I^2 := [1, p] - I^1 - \{j\}$, $t^1_j := 4a - 20 - \sum_{i \in I^1} t_i$, $t^2_j := \sum_{i \in I^1} t_i + t_j + 20 - 4a$ and $m := \min(\{0\} \cup I^2)$. Consider a $G^1$-realisation $(T^1_j)T^1$ of the sequence $(t^1_j)\tau(I^1) \in \text{Sct}(G^1)$ and let $T_i$ be a $t_i$-trail of $T^1$ with $i \in ([1, 2] - \{j\}) \cap I^1$. By Proposition 6 we may suppose without loss of generality that $x^1_2 \in V(T^1_j)$, $j \in [1, 2] \Rightarrow x^1_{5+j} \in V(T^1_j)$ and $x^1_{5+i} \in V(T_i)$ for any $i \in ([1, 2] - \{j\}) \cap I^1$.

If $I^2 = \emptyset$ (so that $m \geq 1$), by Theorem 9 there is a $G^2$-realisation $(T_m, T^2_j)T^2$ of the sequence $(t^2_m)\tau(I^2 - \{m\}) \in \text{Sct}(G^2)$ such that $\{x^2_1, x^2_2\} \subseteq V(T^2)$ and $x^2_2 \in V(T^2_j)$. Then $T_j := T^1_j \cup T^2_j$ is a $t_j$-trail and $(T_j, T^1_j)T^1T^2$ is a required $H_a$-realisation of the sequence $(t_j, t_m)\tau(I^1)\tau(I^2 - \{m\}) \sim \tau$. Appropriate pairs are as follows: if $m \in [1, 2]$, then $(i_m, j_m) := (5, 2)$ and $(i_{3-m}, j_{3-m}) := (8 - m, 1)$; if $m \notin [1, 2]$, then $(i_r, j_r) := (5 + r, 1)$, $r = 1, 2$.

If $I^2 = \emptyset$ (and $m = 0$), then $T_j := T^1_j \cup G^2$ is a $t_j$-trail and $(T^1_j)T^1$ is an appropriate $H_a$-realisation of the sequence $(t_j)\tau(I^1) \sim \tau$.

(C6) If there are $I^1$ and $\{j, k\} \subseteq [1, p] - I^1$ such that $[1, 2] \subseteq I^1 \cup \{j, k\}$, $\min\{t_j, t_k\} \geq 8$, $\sum_{i \in I^1} t_i \leq 4a - 28$ and $\sum_{i \in I^1} t_i + t_j + t_k \geq 4a - 12$ (we may suppose without loss of generality that $j < k$), then with $I^2 := [1, p] - I^1 - \{j, k\}$, $t^1_j := \min\{4a - 24 - \sum_{i \in I^1} t_i, t_j - 4\}$, $t^1_k := \max\{4, 4a - 16 - \sum_{i \in I^1} t_i - t_j\}$, $t^2_j := t_j - t^1_j$ and $t^2_k := t_k - t^1_k$ we have $t^1_j + t^1_k + \sum_{i \in I^1} t_i = |E(G^1)|$ and $\tau^1 := (t^1_j, t^1_k)\tau(I^1) \in \text{Sct}(G^1)$, $l = 1, 2$. Consider a $G^1$-realisation $(T^1_j, T^1_k)T^1$ of the sequence $\tau^1$ and let $T_i$ be a $t_i$-trail of $T^1$ with $i \in [1, 2] - \{j, k\} \subseteq I^1$. Because of Proposition 6 we may suppose without loss of generality that $x^1_2 \in V(T^1_j)$, $x^2_2 \in V(T^1_k)$, $m \in [1, 2] \cap \{j, k\} \Rightarrow x^1_{5+m} \in V(T^1_m)$ and $x^1_{5+i} \in V(T_i)$ for any $i \in [1, 2] - \{j, k\}$. By Theorem 9 there is a $G^2$-realisation $(T^2_j, T^2_k)T^2$ of the sequence $\tau^2$ such that $x^2_1 \in V(T^2_j)$ and $x^2_2 \in V(T^2_k)$.
Then $T_m := T^1_m \cup T^2_m$ is a $t_m$-trail, $m = j, k$ and $(T_j, T_k)T^1T^2$ is an $H_a$-realisation of the sequence $(t_j, t_k)\tau(I^1)\tau(I^2) \sim \tau$ with required properties; appropriate pairs are $(i_r, j_r) := (5 + r, 1), r = 1, 2$.

The additional requirements on $T_1$ and $T_2$ are symmetrical and there are no additional requirements on $T_i$ with $i \in [3, p]$; therefore, in our analysis we may suppose without loss of generality that $t_1 \leq t_2$ and $t_i \leq t_{i+1}$ for any $i \in [3, p - 1]$.

1) $t_1 + t_2 \geq 4a - 16$.

2) If $t_1 \leq 4a - 24$, then $I^1 := \{1\}, j := 2 \rightarrow (C5)$.

3) If $t_1 \geq 4a - 22$, then $t_1 \geq 6$.

4) If $a \geq 9$, then $t_1 + t_2 \geq 8a - 44 \geq 4a - 12$, $t_1 \geq 14$ and $I^1 := \emptyset, j := 1, k := 2 \rightarrow (C6)$.

5) If $a = 7$, then $|E(G^1)| = 8$.

6) If $t_1 \geq 8$, then $t_1 + t_2 \geq 4a - 12$ and $I^1 := \emptyset, j := 1, k := 2 \rightarrow (C6)$.

7) If $t_1 = 6$, by Theorem 9 there is a $G^2$-realisation $(T^2_2) \prod_{i=3}^p (T_i)$ of the sequence $(t_2 - 2) \prod_{i=3}^p (t_i) \in \text{Sct}(G^2)$ such that $T^2_2$ contains as a subgraph a 3-vertex path with endvertices $x^2_1$ and $x^3_1$. Thus, we may suppose without loss of generality that $T^2_2 = t^3_1 - 1 \prod_{i=1} (c_i)$ where $c_1 = c_{t_2-1} = x^2_1$ and $c_3 = x^2_4$. With $T_1 := (x^1_1, c_2, x^2_3, x^1_7, x^2_3, x^1_9, x^1_7)$ and $T_2 := (c_1, x^1_7, x^2_3, x^1_9) \prod_{i=3} (c_i)$ then $(T_1, \ldots, T_p)$ is a required $H_a$-realisation of the sequence $\tau$; appropriate pairs are $(i_r, j_r) := (5 + r, 1), r = 1, 2$.

2) If $t_1 + t_2 = 4a - 18$, then $\sum_{i=3}^p t_i = 4a - 2 \equiv 2 \pmod{4}$ and there is $l \in [3, p]$ satisfying $t_l \equiv 2 \pmod{4}$.

3) If $t_1 \leq 4a - 28$, then $t_2 \geq 10$.

4) If $t_1 \geq 4a - 30$, then $I^1 := \{1, p\}, j := 2 \rightarrow (C5)$.

5) If $t_1 = 4a - 28$, then $t_2 = 10, a \leq 9, t_1 = 8, a = 9$ and $I^1 := \{2, p\} \rightarrow (C4)$.

6) If $t_1 \geq 4a - 26$, then $t_2 \leq 8, a = 7, t_1 = 4$ and $t_2 = 6$.

7) If $t_1 \geq 4a - 26$, then $t_2 \leq 8, a = 7, t_1 = 4$ and $t_2 = 6$.

8) If $t_2 = 6$, then from $\sum_{i=3}^p t_i = 26$ it follows that $t_3 = 4$, and so $I^1 := \{1, 3\} \rightarrow (C4)$.

9) If $t_1 + t_2 = 4a - 20$, then $I^1 := [1, 2] \rightarrow (C4)$.

10) If $t_1 + t_2 = 4a - 22$, then $a \geq 9, t_2 \geq 8$ and there is $l \in [3, p]$ with $t_l \equiv 2 \pmod{4}$.

11) If $t_1 \leq 4a - 34$, then $t_2 \geq 12$. 

12) If $t_1 \leq 4a - 34$, then $t_2 \geq 12$. 

138
(411) If $t_l \geq 10$, then $I^1 := \{1\}$, $j := 2$, $k := l \rightarrow (C6)$.

(412) If $t_l = 6$, then $I^1 := \{1, l\}$, $j := 2 \rightarrow (C5)$.

(42) If $t_j \geq 4a - 32$, then $a = 9$ and $t_2 \in \{8, 10\}$.

(421) If $t_l \geq 10$, then $I^1 := \{1\}$, $j := 2$, $k := l \rightarrow (C6)$.

(422) If $t_l = 6$, then $t_i \in \{4, 6\}$ for any $i \in [3, p]$, $\sum_{i=3}^p t_i = 38$ and the sequence $\prod_{i=3}^p (t_i)$ contains at least two 4’s and at least one 6. Thus, there is $I^1 \subseteq [2, p]$ such that $2 \in I^1$, $\sum_{i \in I^1} t_i = 16$ and the condition (C4) is satisfied.

(5) If $t_1 + t_2 \leq 4a - 24$, let $q \in [2, p - 1]$ be determined by the inequalities

\[
\sum_{i=1}^q t_i \leq 4a - 18 \quad \text{and} \quad \sum_{i=q+1}^p t_i \geq 4a - 16.
\]

(51) If $\sum_{i=1}^q t_i = 4a - 18$, then $q \geq 3$ and there is $l \in [q + 1, p]$ with $t_l \equiv 2 \pmod{4}$.

(511) If $t_p \geq t_q + 2$, then $I^1 := [1, q - 1]$, $j := p \rightarrow (C5)$.

(5112) If $t_i = t_q$ for any $i \in [q + 1, p]$, then $t_q = t_l \equiv 2 \pmod{4}$.

(51121) If $t_q \geq 10$, then $I^1 := [1, q - 1]$, $j := k := q + 1 \rightarrow (C6)$.

(51122) If $t_q = 6$, then $6|4a - 2 = 6(p - q)$, hence $a \equiv 5 \pmod{6}$ and $p - q \geq 7$.

(511221) If $t_2 \geq 12$, then $I^1 := \{1\} \cup [3, q + 1]$, $j := 2 \rightarrow (C5)$.

(511222) $t_2 \leq 10$.

(5112221) If $t_2 = 10$, then $I^1 := [q + 5, p]$, $j := 2 \rightarrow (C5)$.

(5112222) If $t_2 = 8$, then $I^1 := \{1\} \cup [3, q + 1] \rightarrow (C4)$.

(5112223) If $t_2 = 6$, then $I^1 := \{2\} \cup [q + 5, p] \rightarrow (C4)$.

(5112224) $t_2 = 4$.

(51122241) If $t_3 = 4$, then $I^1 := [1, 3] \cup [q + 6, p] \rightarrow (C4)$.

(51122242) If $t_3 = 6$, then $\tau = (4)^2 (6)^{p-2}$, $6p - 4 = |E(H_0)| = 8a - 20$ and $p \equiv 0 \pmod{2}$. Put $\tau_1 := (8)(6)^2$, $\tau_2 := (6)^{\frac{p-4}{2}} =: \tau_3$ and consider a $K_{5,5}$-realisation $(T_{1,2}, T_{3,4})$ of the sequence $\tau_1$ presented in Figure 1, a $G_a$-realisation $(\bar{T}_5) \prod_{i=6}^p (T_i)$ of the sequence $\tau_2$ and a $G_{a'}$-realisation $\prod_{i=6}^p (T_i)$ of the sequence $\tau_3$. The closed trail $T_{1,2}$ is an 8-cycle, hence by Proposition 7 we may suppose without loss of generality that $V(T_{1,2}) \cap X_{1,5} = X_{1,4}$ and $T_{1,2} = \prod_{i=1}^9 (b_i)$ with $b_1 = b_9 \in X_{1,4}$. By Proposition 6 we may suppose without loss of generality that $\bar{T}_5 = \prod_{i=1}^7 (c_i)$ with $c_1 = c_7 = b_1$, $c_3 = b_3$, $c_5 = b_7$, $c_2 = x_6^2$ and $c_6 = x_7^2$. Then $(T_1, \ldots, T_p)$ with $T_1 := (b_1, b_2, b_3, c_2, b_1)$, $T_2 := (b_9, b_8, b_7, c_6, b_9)$ and $T_5 := (b_3, c_4, b_7, b_6, b_5, b_4, b_3)$ is a required $H_a$-realisation of the sequence $\tau$; appropriate pairs are $(i_r, j_r) := (5 + r, 2)$, $r = 1, 2$. 

139
(512) If \( t_q = 4 \), then \( q \geq 4 \) and \( \sum_{i=1}^{q-2} t_i = 4a - 26 \).

(5121) If \( t_p \geq 10 \), then \( I^1 := [1, q - 2] \), \( j := p \to (C5) \).

(5122) If \( t_p \leq 8 \), then \( t_i = 6 \) and \( I^1 := [1, q - 2] \cup \{l\} \to (C4) \).

(52) If \( \sum_{i=1}^{q} t_i = 4a - 20 \), then \( I^1 := [1, q] \to (C4) \).

(53) If \( \sum_{i=1}^{q} t_i = 4a - 22 \), then \( q \geq 3 \).

(531) \( t_q \geq 6 \).

(5311) If \( t_p \geq t_q + 6 \), then \( I^1 := [1, q - 1] \), \( j := p \to (C5) \).

(5312) If there is \( m \in [q+1, p] \) such that \( t_m = t_q + 2 \), then \( I^1 := [1, q - 1] \cup \{m\} \to (C4) \).

(5313) If \( t_i \in \{t_q, t_q + 4\} \) for any \( i \in [q + 1, p] \), then \( t_q \equiv t_i \equiv 2 \) (mod 4), \( (p - q)(t_q + 4) \geq 4a + 2 = \sum_{i=1}^{q} t_i + 24 \geq t_q + 24, p - q \geq \frac{t_q + 24}{t_q + 4} > 1 \) and \( p - q \geq 2 \).

(53131) If \( t_{p-1} \geq 10 \), then \( I^1 := [1, q - 1] \), \( j := p - 1, k := p \to (C6) \).

(53132) If \( t_{p-1} = 6 \), then \( t_q = 6 \).

(531321) If \( t_2 \geq 8 \), then \( I^1 := \{1\} \cup [3, q + 1] \), \( j := 2 \to (C5) \).

(531322) If \( t_2 \leq 6 \), then by Theorem 4 there exists a \( G^1 \)-realisation \( T^1 := (T^1_q)_{i=1}^{q-1} \prod_{i=1}^{p} (T_i) \) of the sequence \( (8)_{i=1}^{q-1} \prod_{i=1}^{p} (t_i) \) such that all trails of \( T^1 \) are cycles. Therefore, by Proposition 6 we may suppose without loss of generality that \( x^1_{5+i} \in V(T_i) \), \( i = 1, 2 \), and \( T^1_q = \prod_{i=1}^{9} (b_i) \) with \( b_1 = b_9 = x^2_1 \) and \( b_5 = x^2_3 \). By Theorem 9 there is a \( G^2 \)-realisation \( (T^2_{q+1})_{i=q+2}^{p} \prod_{i=q+2}^{p} (T_i) \) of the sequence \( (4)_{i=q+2}^{p} \prod_{i=q+2}^{p} (t_i) \) such that \( T^2_{q+1} \) contains as a subgraph a 3-vertex path with endvertices \( x^2_1 \) and \( x^3_3 \). Thus, we may suppose without loss of generality that \( T^2_{q+1} = \prod_{i=1}^{5} (c_i) \) where \( c_1 = c_5 = x^2_1 \) and \( c_3 = x^3_3 \). Then \( (T_1, \ldots, T_p) \) with \( T_{q+1} := (b_5, c_4) \prod_{i=1}^{5} (b_i) \) and \( T_{q+2} := (b_9, c_2) \prod_{i=5}^{g} (b_i) \) is a required \( H_a \)-realisation of the sequence \( \tau \); appropriate pairs are \( (i_r, j_r) := (5 + r, 1), r = 1, 2 \).

(532) \( t_q = 4 \).

(5321) If \( t_p \geq 10 \), then \( I^1 := [1, q - 1] \), \( j := p \to (C5) \).

(5322) If \( t_p \leq 8 \), then \( t_i = 6 \) and \( I^1 := [1, q - 1] \cup \{l\} \to (C4) \).

(54) If \( \sum_{i=1}^{q} t_i \leq 4a - 24 \), then \( I^1 := [1, q] \), \( j := q + 1 \to (C5) \).

\[ \square \]

**Theorem 11.** If \( a \) is an odd integer, \( a \geq 3 \), then the graph \( K'_a, a \) is ADCT. Moreover, if \( r = \frac{1}{6}(a(a - 1) - 2) \in \mathbb{Z} \), there is a \( K'_a \)-realisation \( (T^1_1, \ldots, T^1_r) \) of the sequence \( (6)^{r-1}(8) \in \text{Sct}(K'_a, a) \) such that \( T^1_r \) has as a subgraph a 5-vertex path.
Proof. We proceed by induction on $a$. The graphs $K'_{a,a}$ with $a \leq 5$ are ADCT by Proposition 5. Further, the 8-trail of the $K'_{5,5}$-realisation of the sequence $(6)^2(8) \in \text{Sct}(K'_{5,5})$ presented in Figure 1 is a cycle, and so trivially it has as a subgraph a 5-vertex path.

So, suppose that $a \geq 7$, the graph $K'_{a-4,a-4}$ is ADCT and, provided $s := \frac{1}{6}((a - 4)(a - 5) - 2) \in \mathbb{Z}$, there is a $G^1$-realisation $\prod_{i=1}^{s} (T^1_i)$ of the sequence $(6)^{s-1}(8) \in \text{Sct}(G^1)$ such that $T^1_s$ has as a subgraph a 5-vertex path. We can use again the general strategy, since the graph $K'_{a,a}$ (see Figure 4) is an edge-disjoint union of ADCT graphs $G^1 := F_a$ (the induction hypothesis) and $G^2 := H_a$ (Theorem 10). Consider a sequence $\tau = (t_1, \ldots, t_p) \in \text{Sct}(K'_{a,a})$.

![Figure 4. The graph $K'_{a,a}$](image)

(C7) If there is $I^1 \subseteq [1, p]$ such that $\sum_{i \in I^1} t_i = a^2 - 9a + 20 = |E(G^1)|$, put $I^2 := [1, p] - I^1$, $\tau^l := \tau(I^l) \in \text{Sct}(G^l)$ and consider a $G^l$-realisation $T^l$ of the sequence $\tau^l$, $l = 1, 2$. Then $T^1T^2$ is a $K'_{a,a}$-realisation of the sequence $\tau^1\tau^2 \sim \tau$.

(C8) If there are $I^1$ and $j \in [1, p] - I^1$ such that $\sum_{i \in I^1} t_i \leq a^2 - 9a + 16$ and $\sum_{i \in I_1} t_i + t_j \geq a^2 - 9a + 24$, put $I^2 := [1, p] - I^1 - \{j\}$, $t^1_j := a^2 - 9a + 20 - \sum_{i \in I_1} t_i$, $t^2_j := \sum_{i \in I^1} t_i + t_j - a^2 + 9a - 20$. Then $\tau^l := (t^1_j)\tau(I^l) \in \text{Sct}(G^l)$, $l = 1, 2$. By Theorem 10 there is a $G^2$-realisation $(T^2_j)T^2$ of the sequence $\tau^2$ such that there is $(i_1, j_1) \in [5, a] \times [1, 2]$ with $x^{i_1}_{j_1} \in V(T^2_j)$. By the induction hypothesis there is a $G^1$-realisation $(T^1_j)T^1$ of the sequence $\tau^1$; by Proposition 7 we may suppose without loss of generality that $x^{i_1}_{j_1} \in V(T^1_j)$. Then $T_j := T^1_j \cup T^2_j$ is a $t_j$-trail and $(T_j)T^1T^2$ is a $K'_{a,a}$-realisation of the sequence $(t_j)\tau(I^1)\tau(I^2) \sim \tau$.  

141
(C9) If there are \( I^1 \) and \( \{j, k\} \subseteq [1, p] - I^1 \) such that \( \min\{t_j, t_k\} \geq 8 \), \( \sum_{i \in I^1} t_i \leq a^2 - 9a + 12 \) and \( \sum_{i \in I^1} t_i + t_j + t_k \geq a^2 - 9a + 28 \), then with \( I^2 := [1, p] - I^1 - \{j, k\} \), \( t_j^1 := \min\{a^2 - 9a + 16 - \sum_{i \in I^1} t_i, t_j - 4\} \), \( t_j^1 := \max\{4, a^2 - 9a + 24 - \sum_{i \in I^1} t_i - t_j\} \), \( t_j^2 := t_j - t_j^1 \) and \( t_k^2 := t_k - t_k^1 \) we have \( t_j^1 + t_k^1 + \sum_{i \in I^1} t_i = |E(G^l)| \) and \( \tau^l := (t_j^1, t_k^1)\tau(I^1) \in \text{Sct}(G^l) \), \( l = 1, 2 \). Theorem 10 yields a \( G^l \)-realisation \( (T^2_j, T^2_k)\tau^2 \) of the sequence \( \tau^2 \) such that there are \( (i_r, j_r) \in [5, a] \times [1, 2] \), \( r = 1, 2 \), with \( x^j_{i_1} \in V(T^2_j) \), \( x^j_{i_2} \in V(T^2_k) \) and \( i_1 \neq i_2 \).

By the induction hypothesis there is a \( G^1 \)-realisation \( (T^1_j, T^1_k)\tau^1 \) of the sequence \( \tau^1 \);

by Proposition 7 we may suppose without loss of generality that \( x^j_{i_1} \in V(T^1_j) \) and \( x^j_{i_2} \in V(T^1_k) \) (note that both \( T^1_j \) and \( T^1_k \) have at least two vertices in both \( X^2_{5,a} \) and \( X^2_{5,a} ) \). Then \( T_m := T^1_m \cup T^2_m \) is a \( m \)-trail, \( m = j, k \), and \( (T_j, T_k)I^1T^2 \) is a \( K'_{a,a} \)-realisation of the sequence \( (t_j, t_k)\tau(I^1)\tau(I^2) \sim \tau \).

Because of Lemma 1 we may suppose without loss of generality that \( \tau \) is a non-decreasing sequence. Let \( q \in [0, p - 1] \) be determined by the inequalities \( \sum_{i=1}^{q} t_i \leq a^2 - 9a + 22 \) and \( \sum_{i=q+1}^{q+1} t_i \geq a^2 - 9a + 24 \).

(1) If \( \sum_{i=1}^{q} t_i = a^2 - 9a + 22 \), then \( \sum_{i=q+1}^{p} t_i = 8a - 22 \) and there is \( l \in [q + 1, p] \) such that \( t_i \equiv 2 \mod 4 \).

(11) \( t_q \geq 6 \).

(111) If \( t_p \geq t_q + 2 \), then \( I^1 := [1, q - 1], j := p \rightarrow (C8) \).

(112) If \( t_i = t_q \) for any \( i \in [q + 1, p] \), then \( t_q = t_i \equiv 2 \mod 4 \).

(1121) If \( t_q \geq 10 \), then \( I^1 := [1, q - 1], j := q, k := q + 1 \rightarrow (C9) \).

(1122) If \( t_q = 6 \), then \( 6q \geq \sum_{i=1}^{q} t_i \geq 8, q \geq 2, 8a - 22 = 6(p - q) \), \( 4a - 11 \equiv 0 \mod 3 \), \( a \equiv 5 \mod 6 \), \( a(a - 1) \equiv 2 \mod 6 \), the sequence \( \tau \) must contain at least two \( 4's \) and \( I^1 := [3, q + 1] \rightarrow (C7) \).

(12) If \( t_q = 4 \), then \( 4q \geq 8 \) and \( q \geq 2 \).

(121) If \( t_l \geq 10 \), then \( I^1 := [1, q - 2], j := l \rightarrow (C8) \).

(122) If \( t_l = 6 \), then \( I^1 := [1, q - 2] \cup \{l\} \rightarrow (C7) \).

(2) If \( \sum_{i=1}^{q} t_i = a^2 - 9a + 20 \), then \( I^1 := [1, q] \rightarrow (C7) \). Note that if the \( r \) defined in the statement of our Theorem is an integer, then \( a(a - 1) \equiv 2 \mod 6 \), \( a \equiv 5 \mod 6 \), \( a^2 - 9a + 20 \equiv 0 \mod 6 \), \( 4a - 20 \equiv 0 \mod 6 \), and so \( \tau = (6)^{p-1}(8) \) yields \( 8a - 20 = 6(p - q - 1) + 8, 6(p - q - 1) \geq 60, p - q - 1 \geq 10, 6(p - q - 1) \equiv 0 \mod 4 \) and \( p - q - 1 \equiv 0 \mod 2 \). The graph \( G^2 \) is an edge-disjoint union of ADCT graphs \( G^2_1 := G^2_a, G^2_2 := G^2_a \) and \( G^2_3 := K^4_5 \). Put \( \tau^1 := (6)^9, \tau^2 := (6)^{\frac{p-2}{2}} =: \tau^2, \tau^3 := (6)^2(8) \) and let \( T^1 \) be a \( G^1 \)-realisation of the sequence \( \tau^1 \) and let \( T^2_m \) be

142
a $G_m^2$-realisation of the sequence $\tau_m^2$, $m = 1, 2, 3$, where $T_m^2 = (T_{p-2}, T_{p-1}, T_p)$ is that presented in Figure 1. Then $T_1^1 T_2^2 T_3^2$ is a $K'_{a,a}$-realisation of the sequence $(6)^{p-1}(8)$ and the 8-trail $T_p$ (which is a cycle) has trivially as a subgraph a 5-vertex path.

(3) If $\sum_{i=1}^q t_i = a^2 - 9a + 18$, there is $l \in [q + 1, p]$ such that $t_l \equiv 2 \pmod{4}$. 

(31) $t_q \geq 6$.

(311) If $t_p \geq t_q + 6$, then $I^1 := [1, q - 1], j := p \rightarrow (C8)$.

(312) If there is $m \in [q + 1, p]$ such that $t_m = t_q + 2$, then $I^1 := [1, q - 1] \cup \{m\} \rightarrow (C7)$.

(313) If $t_i \in \{t_q, t_q + 4\}$ for any $i \in [q + 1, p]$, then $t_q \equiv t_l \equiv 2 \pmod{4}$.

(3131) $p \geq q + 2$.

(31311) If $t_{p-1} \geq 10$, then $I^1 := [1, q - 1], j := p - 1, k := p \rightarrow (C9)$.

(31312) $t_{p-1} = 6$.

(313121) If $t_1 = 4$, then $I^1 := [2, q + 1] \rightarrow (C7)$.

(313122) If $t_1 = 6$, then $a^2 - 9a + 18 = 6q, a \equiv 3 \pmod{6}$, $\sum_{i=q+1}^p t_i = 8a - 18 \equiv 0 \pmod{6}$, $t_p = 6, \tau = (6)^p$, $8a - 18 = 6(p - q), p - q \geq 9, 6(p - q) \equiv 6 \pmod{48}$ and $p - q - 1 \equiv 0 \pmod{8}$. The graph $G^2$ is an edge-disjoint union of ADCT graphs $G_1^2 := G_a$ and $G_2^2 := G_b$. Put $\tau^1 := (8)(6)^{q-1}, \tau_1^2 := (6)^{\frac{q+5}{2}}$ and $\tau_2^2 := (4)(6)^{\frac{q-5}{2}}$. By the induction hypothesis and by Lemma 1 there is a $G^1$-realisation $(T_q^1)T^1_1$ of the sequence $\tau^1$ such that $T^1_q$ has as a subgraph a 5-vertex path. By Proposition 7 we may suppose without loss of generality that $T^1_q = \prod_{i=1}^9 (b_i)$ where $b_1 = b_9 \in X_{5,a}^1$ and $\prod_{i=1}^5 (b_i)$ is a path. By Theorem 10 there is a $G^2_1$-realisation $T^2_2$ of the sequence $\tau_1^2$. Further, by Theorem 3 there is a $G^2_2$-realisation $(T^2_{q+1})T^2_2$ of the sequence $\tau_2^2$; by Proposition 6 we may suppose without loss of generality that $T^2_{q+1} = \prod_{i=1}^5 (c_i)$ where $c_1 = c_5 = b_1$ and $c_3 = b_5$. With $T_q := (b_5, c_2) \prod_{i=1}^5 (b_i)$ and $T_{q+1} := (b_9, c_4) \prod_{i=5}^9 (b_i)$ then $(T_q, T_{q+1})T^1_1 T^2_1 T^2_2$ is a $K'_{a,a}$-realisation of the sequence $\tau = (6)^p$.

(3132) If $p = q + 1$, then $t_p = 8a - 18, t_q \geq 8a - 22$ and $I^1 := [1, q - 1], j := q, k := p \rightarrow (C9)$.

(32) $t_q = 4$.

(321) If $t_l \geq 10$, then $I^1 := [1, q - 1], j := l \rightarrow (C8)$.

(322) If $t_l = 6$, then $I^1 := [1, q - 1] \cup \{l\} \rightarrow (C7)$.

(4) If $\sum_{i=1}^q t_i \leq a^2 - 9a + 16$, then $I^1 := [1, q], j := q + 1 \rightarrow (C8)$.
References


Authors’ addresses: Sylwia Cichacz, AGH University of Science and Technology, Al.Mickiewicza 30, 30-059 Kraków, Poland, e-mail: cichacz@agh.edu.pl; Mirko Horňák, P. J. Šafárik University, Jesenná 5, 040 01 Košice, Slovakia, e-mail: mirko.hornak@upjs.sk.