On some power sums of sine or cosine

Mircea Merca

Department of Mathematics
University of Craiova
Craiova, 200585 Romania

Abstract

In this note, using the multisection series method, we establish the formulas for various power sums of sine or cosine functions.

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1. Introduction

We start with the multisection formula first published by T. Simpson (see [2, Ch. 16], [6, Ch. 4, S. 4.3] and [8]) as early as 1759: if

\[ f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \]

is a finite or infinite series, then for \(0 \leq r < n\) the sum

\[ a_r x^r + a_{r+n} x^{r+n} + a_{r+2n} x^{r+2n} + \cdots \]

is given by

\[ \sum_{k \geq 0} a_{r+kn} x^{r+kn} = \frac{1}{n} \sum_{k=0}^{n-1} z^{-kr} f(z^k x), \quad (1) \]

where \(z = e^{2\pi i/n}\) is the \(n\)th root of 1.

Applying the multisection formula (1) to \(f(x) = (1 + x)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^k\),

one can find a general result proved by C. Ramus (see [1, p. 70, Problem 38] and [5]) in 1834:

\[ \sum_{k \geq 0} \binom{p}{r+kn} = \frac{2^p}{n} \sum_{k=0}^{n-1} \cos^p \left( \frac{k\pi}{n} \right) \cos \frac{k(p-2r)\pi}{n}. \quad (2) \]
By (2), with $p$ replaced by $2p$ and $r$ replaced by $p \mod n$, we obtain a formula for the cosine power sums
\[
\sum_{k=0}^{n-1} \cos^{2p} \left( \frac{k\pi}{n} \right) = \frac{n}{2^{2p}} \sum_{k=-\left\lceil \frac{p}{n} \right\rceil}^{\left\lfloor \frac{p}{n} \right\rfloor} \left( \frac{2p}{p+kn} \right),
\] where $\lfloor x \rfloor$ denotes the largest integer not greater than $x$. We note that the formula proved in [7] is the case $p<n$ of this formula.

When $n \equiv p \pmod{2}$, by Ramus’s formula (2), with $r$ replaced by $p + n \mod n$, we obtain
\[
\sum_{k=0}^{n-1} (-1)^k \cos \left( \frac{k\pi}{n} \right) = \frac{n}{2^{2p}} \sum_{k=-\left\lceil \frac{p}{n} \right\rceil}^{\left\lfloor \frac{p}{n} \right\rfloor} \left( \frac{p}{p+kn} \right),
\] where $\lceil x \rceil$ denotes the nearest integer to $x$, i.e., $\lceil x \rceil = \lceil x + \frac{1}{2} \rceil$. Note that the conjecture published in [4] is solved by this formula.

Taking into account that
\[
\sum_{k=1}^{n-1} f(2k-1) = \frac{1}{2} \left( \sum_{k=1}^{2n-1} f(k) - \sum_{k=1}^{2n-1} (-1)^k f(k) \right),
\] for $f(k) = \cos^{2p} \left( \frac{k\pi}{n} \right)$, by (3) (with $n$ replaced with $2n$) and (4) (with $n$ and $p$ replaced by $2n$ and $2p$, respectively), we deduce the following relation
\[
\sum_{k=1}^{n} \cos^{2p} \left( \frac{2k-1}{n} \cdot \frac{\pi}{2} \right) = \frac{n}{2^{2p}} \sum_{k=-\left\lceil \frac{p}{n} \right\rceil}^{\left\lfloor \frac{p}{n} \right\rfloor} (-1)^k \left( \frac{2p}{p+kn} \right).
\] In this paper, we shall prove:

**Theorem 1.** Let $n$, $p$ and $r$ be three non-negative integers, such that $0 \leq r < n$. Then
\[
\sum_{k \geq 0} (-1)^{|p/2|+r+kn} \left( \frac{p}{r+kn} \right) = \frac{2p}{n} \sum_{k=0}^{n-1} \sin^{p} \left( \frac{k\pi}{n} \right) f_{p} \left( \frac{k(p-2r)\pi}{n} \right),
\] where
\[
f_{p}(x) = \begin{cases} \cos(x), & \text{for } p \text{ even}, \\ \sin(x), & \text{for } p \text{ odd}. \end{cases}
\]

**Corollary 1.** Let $n$ and $p$ be two positive integers. Then
\[
\sum_{k=0}^{n-1} \sin^{2p} \left( \frac{k\pi}{n} \right) = \frac{n}{2^{2p}} \sum_{k=-\left\lceil \frac{p}{n} \right\rceil}^{\left\lfloor \frac{p}{n} \right\rfloor} (-1)^{kn} \left( \frac{2p}{p+kn} \right).
\]
Corollary 2. Let \( n \) and \( p \) be two positive integers. Then
\[
\sum_{k=0}^{2n-1} (-1)^k \sin^{2p} \left( \frac{k\pi}{2n} \right) = (-1)^n \frac{n}{2^{2p-1}} \sum_{k=-\left[\frac{n}{2}\right]}^{\left[\frac{n}{2}\right]} \left( \frac{2p}{p + n + 2kn} \right).
\]

Corollary 3. Let \( n \) and \( p \) be two positive integers. Then
\[
\sum_{k=1}^{n} \sin^{2p} \left( \frac{2k-1}{n} \cdot \frac{\pi}{2} \right) = \frac{n}{2^{2p}} \sum_{k=-\left[\frac{n}{2}\right]}^{\left[\frac{n}{2}\right]} (-1)^{k(n+1)} \left( \frac{2p}{p + kn} \right).
\]

Corollary 4. Let \( n \) and \( p \) be two positive integers, both of them odd. Then
\[
\sum_{k=1}^{n} (-1)^k \sin^{2p} \left( \frac{2k-1}{n} \cdot \frac{\pi}{2} \right) = (-1)^{\left[\frac{n}{2}\right]} \frac{n}{2^{2p-1}} \sum_{k=-\left[\frac{n}{2}\right]}^{\left[\frac{n}{2}\right]} \left( \frac{p+n}{2p} + 2kn \right).
\]

Note that, Corollaries 1, 2 and 4 are obtained by replacing \( n \) with \( 2n \) in Theorem 1, the differences being given by \( r \) which is replaced with \( p \mod n \) for Corollary 1, \( (p+n) \mod 2n \) for Corollary 2 respectively \( p + n \mod 2n \) for Corollary 4. We get Corollary 3 using the relation (5) for \( f(k) = \sin^{2p} \left( \frac{k\pi}{2n} \right) \) and Corollaries 1 and 2.

2. Proof of Theorem 1

We apply the multisection formula (1) for
\[
f(x) = (1 - x)^p = \sum_{k=0}^{p} (-1)^k \binom{p}{k} x^k.
\]

When \( x = 1 \), we get
\[
\sum_{k\geq0}(-1)^{r+nk} \binom{p}{r+kn} = \frac{1}{n} \sum_{k=0}^{n-1} z^{-kr} (1 - z^k)^p,
\]
where \( z = e^{\frac{2\pi i}{n}} \). Having
\[
e^{-ir\phi} (1 - e^{i\phi})^p = e^{-ir\phi} e^{\frac{i\phi}{2}} \left( e^{\frac{i\phi}{2}} - e^{-\frac{i\phi}{2}} \right)^p = e^{-ir\phi} e^{\frac{i\phi}{2}} \left( e^{-\frac{i\phi}{2}} - e^{\frac{i\phi}{2}} \right)^p = (-1)^p (2i)^p \sin^p \frac{\phi^2}{2} e^{i(p-2r)\frac{\phi}{2}},
\]
we can write
\[
z^{-kr} (1 - z^k)^{2p} = (-1)^p 2^p 2^{2p} \sin^{2p} \left( \frac{k\pi}{n} \right) e^{i(p-r)\frac{2k\phi}{n}}.
\]

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and
\[ z^{-kr} (1 - z^k)^{2p+1} = (-1)^{p+1} 2^{2p+1} \sin^{2p+1} \left( \frac{k\pi}{n} \right) \cdot i e^{i(2p+1-2r)\frac{k\pi}{n}}. \]  \hspace{1cm} (10)

As we know that the left side of (8) is real, we deduce that the real parts of the right sides of (9) and (10) are

\[ (-1)^p 2^{2p} \sin^{2p} \left( \frac{k\pi}{n} \right) \cos \frac{2k(p-r)\pi}{n} \]

and

\[ (-1)^p 2^{2p+1} \sin^{2p+1} \left( \frac{k\pi}{n} \right) \sin \frac{k(2p+1-2r)\pi}{n}, \]

respectively. Theorem 1 is proved.

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References


