Some experiments with complete and elementary symmetric functions

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Abstract

The complete and elementary symmetric functions are special cases of Schur functions. It is well-known that the Schur functions can be expressed in terms of complete or elementary symmetric functions using two determinant formulas: Jacobi-Trudi identity and Nägelsbach-Kostka identity. In this paper, we study new connections between complete and elementary symmetric functions.

Keywords: symmetric functions, Stirling numbers, determinants

MSC 2010: 05E05, 15A15

1 Introduction

For any positive integers n and k, the unsigned Stirling number of the first kind, denoted by

$$\begin{bmatrix} n \\ k \end{bmatrix},$$

is defined as the number of permutations of an n-set with k disjoint cycles, and the Stirling number of the second kind, denoted by

$$\begin{Bmatrix} n \\ k \end{Bmatrix},$$

is defined as the number of partitions of an n-set with k non-empty subsets.

We start with three experiments involving Stirling numbers of both kinds.

Experiment 1. Let

$$A_n^{(k)} = \left(\begin{bmatrix} i+k \\ j \end{bmatrix} \right)_{1 \leqslant i,j \leqslant n} \qquad \text{and} \qquad B_n^{(k)} = \left(\begin{Bmatrix} i+k \\ j \end{Bmatrix} \right)_{1 \leqslant i,j \leqslant n}$$

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be two square matrices where k is a non-negative integer. Using Maple to compute the values of $\det A_n^{(k)}$ and $\det B_n^{(k)}$ for various values of n and k, we note the following two identities

$$\det\left(\begin{bmatrix} i+k \\ j \end{bmatrix}\right)_{1\leqslant i,j\leqslant n} = k!^n \tag{1.1}$$

and

$$\det\left(\begin{Bmatrix} i+k \\ j \end{Bmatrix}\right)_{1\leqslant i,j\leqslant n} = n!^k. \tag{1.2}$$

Experiment 2. Let

$$A_n^{(k)} = \left(\begin{bmatrix} i+k \\ j+k-1 \end{bmatrix}\right)_{1\leqslant i,j\leqslant n} \quad \text{and} \quad B_n^{(k)} = \left(\begin{Bmatrix} i+k \\ j+k-1 \end{Bmatrix}\right)_{1\leqslant i,j\leqslant n}$$

be two square matrices where k is a positive integer. Using Maple to compute the values of $\det A_n^{(k)}$ and $\det B_n^{(k)}$ for various values of n and k, we note the following two identities

$$\det\left(\begin{bmatrix} i+k \\ j+k-1 \end{bmatrix}\right)_{1\leqslant i,j\leqslant n} = \begin{Bmatrix} n+k \\ k \end{Bmatrix} \tag{1.3}$$

and

$$\det\left(\begin{Bmatrix} i+k \\ j+k-1 \end{Bmatrix}\right)_{1\leqslant i,j\leqslant n} = \begin{bmatrix} n+k \\ k \end{bmatrix}. \tag{1.4}$$

Experiment 3. We consider the square matrices

$$A_n^{(k)} = \left(\begin{bmatrix} i+k \\ j+1 \end{bmatrix}\right)_{1\leqslant i,j\leqslant n} \quad \text{and} \quad B_n^{(k)} = \left(\begin{Bmatrix} i+k \\ j+1 \end{Bmatrix}\right)_{1\leqslant i,j\leqslant n}$$

where k is a positive integer. Using Maple to compute $\det A_n^{(k)}$ for various values of n and k, we note that the sequences $\left\{\det A_n^{(2)}\right\}_{n>0}$, $\left\{\det A_n^{(3)}\right\}_{n>0}$, $\ldots$, $\left\{\det A_n^{(7)}\right\}_{n>0}$, respectively $\left\{\det A_n^{(8)}\right\}_{n>0}$ are related to the sequences $A000225$, $A001240$, $A001241$, $A001242$, $A111886$, $A111887$, respectively $A111888$ from OEIS [6]. In this way, we assume that the generating function for the sequence $\left\{\det A_n^{(k)}\right\}_{n>0}$ is given by

$$1 + \sum_{n>0} \det A_n^{(k)} x^n = \prod_{j=1}^{k} \left(1 - \frac{k!}{j} x\right)^{-1}$$

and

$$\det\left(\begin{bmatrix} i+k \\ j+1 \end{bmatrix}\right)_{1\leqslant i,j\leqslant n} = k!^n \sum_{j=1}^{k} (-1)^{j-1} \frac{1}{j^n} \binom{k}{j}. \tag{1.5}$$

On the other hand, we note that the sequences $\left\{\det B_n^{(k)}\right\}_{n>0}$, $k \in \{2,3,4,5,6\}$ are related to the sequences $A000254$, $A000424$, $A001236$, $A001237$ and $A001238$ from OEIS [6]. Thus we assume that

$$\det\left(\left\{ {i+k \atop j+1} \right\}\right)_{1\leqslant i,j\leqslant n} = (n+1)!^{k-1}\sum_{j=1}^{n+1}(-1)^{j-1}\frac{1}{j^{k-1}}\binom{n+1}{j}. \tag{1.6}$$

In this paper, we show that these experimental results are very special cases of more general identities. In fact, it is well-known [4, 5] that the unsigned Stirling numbers of the first kind are the elementary symmetric functions of the numbers $1, 2, \ldots, n$, i.e.,

$$\left[{n+1 \atop n+1-k} \right] = e_k(1,2,\ldots,n) \tag{1.7}$$

and the Stirling numbers of the second kind are the complete homogeneous symmetric functions of the numbers $1, 2, \ldots, n$, i.e.,

$$\left\{ {n+k \atop n} \right\} = h_k(1,2,\ldots,n). \tag{1.8}$$

Therefore, new connections between complete and elementary symmetric functions will be studied in the paper.

2 Main results

We recall [3, Ch. I.3] that the Schur function $s_{\lambda,n} = s_\lambda(x_1, x_2, \ldots, x_n)$ can be defined as the ratio of two determinants as follows:

$$s_\lambda(x_1,x_2,\ldots,x_n) = \frac{\det\left(x_i^{\lambda_j+n-j}\right)_{1\leqslant i,j\leqslant n}}{\det\left(x_i^{n-j}\right)_{1\leqslant i,j\leqslant n}},$$

where $\lambda = [\lambda_1, \ldots, \lambda_n]$ is an integer partition of length $\leqslant n$ and $x_1, \ldots, x_n$ are independent indeterminates.

If $\lambda = [1^k]$, then $s_{\lambda,n}$ is the kth elementary symmetric function $e_{k,n}$,

$$e_{k,n} = e_k(x_1,x_2,\ldots,x_n) = \sum_{1\leqslant i_1<i_2<\ldots<i_k\leqslant n} x_{i_1}x_{i_2}\ldots x_{i_k}.$$

When $\lambda = [k]$, $s_{\lambda,n}$ is the kth complete homogeneous symmetric function $h_{k,n}$,

$$h_{k,n} = h_k(x_1,x_2,\ldots,x_n) = \sum_{1\leqslant i_1\leqslant i_2\leqslant\ldots\leqslant i_k\leqslant n} x_{i_1}x_{i_2}\ldots x_{i_k}.$$

In particular, we have $e_{0,n} = 1$ and $h_{0,n} = 1$. For $k < 0$ it is convenient to define $e_{k,n} = 0$ and $h_{k,n} = 0$. When $k > n$ we have $e_{k,n} = 0$ and

$$h_k(x_1,\ldots,x_n) = h_k(x_1,\ldots,x_n,\underbrace{0,\ldots,0}_{k-n}).$$

The generating function for the elementary symmetric function $e_{k,n}$ is

$$\sum_{k\geqslant 0} e_{k,n}t^k = \prod_{i=1}^{n}(1+x_it)$$

and the generating function for the $h_{k,n}$ is

$$\sum_{k\geqslant 0} h_{k,n}t^k = \prod_{i=1}^{n}(1-x_it)^{-1}. \tag{2.1}$$

The following relations

$$e_{k,n} = \det\left(h_{i-j+1,n}\right)_{1\leqslant i,j\leqslant k}$$

and

$$h_{k,n} = \det\left(e_{i-j+1,n}\right)_{1\leqslant i,j\leqslant k}$$

are well-known. The first is a special case of Jacobi-Trudi identity [2, eq. 0.2] and the second is a special case of Nägelsbach-Kostka identity [2, eq. 0.3].

In this paper, we shall prove:

Theorem 2.1. *Let a and k be two nonnegative integers such that $a \leqslant k$. Then*

$$\det(e_{i-j+a,i-1+k})_{1\leqslant i,j\leqslant n} = \det(e_{i-j+a,k})_{1\leqslant i,j\leqslant n}.$$

Theorem 2.2. *Let a and k be two non-negative integers. Then*

$$\det(h_{i-j+a,j+k})_{1\leqslant i,j\leqslant n} = \det(h_{i-j+a,n+k})_{1\leqslant i,j\leqslant n}.$$

Due to Giambelli's determinant formulas [1], on the right hand side of equations from Theorems 2.1 and 2.2, we have the Schur functions $s_{[n^a],k}$ and $s_{[a^n],n+k}$ respectively.

The case $a = k$ of Theorem 2.1 can be written as

Corollary 2.1. *For $k, n > 0$,*

$$\det\left(e_{i-j+k}(x_1, x_2, \ldots, x_{i-1+k})\right)_{1\leqslant i,j\leqslant n} = (x_1x_2\cdots x_k)^n.$$

By Theorem 2.2, with k replaced by 0, we get

Corollary 2.2. *For $a, n > 0$,*

$$\det\left(h_{i-j+a}(x_1, x_2, \ldots, x_j)\right)_{1\leqslant i,j\leqslant n} = (x_1x_2\cdots x_n)^a.$$

Example. The case $k = 2$ of Corollary 2.1 is given by

$$\begin{vmatrix} e_{2,2} & e_{1,2} & 1 & & \\ e_{3,3} & e_{2,3} & e_{1,3} & 1 & \\ e_{4,4} & e_{3,4} & e_{2,4} & e_{1,4} & 1 \\ e_{5,5} & e_{4,5} & e_{3,5} & e_{2,5} & e_{1,5} \\ e_{6,6} & e_{5,6} & e_{4,6} & e_{3,6} & e_{1,6} \\ \cdots & & & & \end{vmatrix}_{n\times n} = e_{2,2}^n$$

and the case $a = 2$ of Corollary 2.2 can be written as

$$\begin{vmatrix} h_{2,1} & h_{1,2} & 1 & & \\ h_{3,1} & h_{2,2} & h_{1,3} & 1 & \\ h_{4,1} & h_{3,3} & h_{2,3} & h_{1,4} & 1 \\ h_{5,1} & h_{4,3} & h_{3,3} & h_{2,4} & h_{1,5} \\ h_{6,1} & h_{5,3} & h_{4,3} & h_{3,4} & h_{2,5} \\ \cdots & & & & \end{vmatrix}_{n\times n} = e_{n,n}^2.$$

Replacing x_i by i in these corollaries, we obtain the identities (1.1) and (1.2). A generalization of the symmetry between the relations (1.3) and (1.4) can be obtain replacing a by 1 in our theorems.

Corollary 2.3. *For* $k, n > 0$*,*

$$\det\left(e_{i-j+1}(x_1, x_2, \ldots, x_{i-1+k})\right)_{1\leqslant i,j\leqslant n} = h_n(x_1, x_2, \ldots, x_k), \quad \textit{and}$$
$$\det\left(h_{i-j+1}(x_1, x_2, \ldots, x_{j-1+k})\right)_{1\leqslant i,j\leqslant n} = e_n(x_1, x_2, \ldots, x_{n+k-1}).$$

Example. By Corollary 2.3 for $k = 2$, we have

$$\begin{vmatrix} e_{1,2} & 1 & & \\ e_{2,3} & e_{1,3} & 1 & \\ e_{3,4} & e_{2,4} & e_{1,4} & 1 \\ e_{4,5} & e_{3,5} & e_{2,5} & e_{1,5} \\ \cdots & & & \end{vmatrix}_{n\times n} = h_{n,2}.$$

and

$$\begin{vmatrix} h_{1,2} & 1 & & \\ h_{2,2} & h_{1,3} & 1 & \\ h_{3,2} & h_{2,3} & h_{1,4} & 1 \\ h_{4,2} & h_{3,3} & h_{2,4} & h_{1,5} \\ \cdots & & & \end{vmatrix}_{n\times n} = e_{n,n+1}.$$

The cases $a = k-1$ of Theorem 2.1 and $k = 1$ of Theorem 2.2 can be written as

Corollary 2.4. *For* $k, n > 0$*,*

$$h_n\left(\frac{1}{x_1}, \ldots, \frac{1}{x_{k+1}}\right) = \frac{\det\left(e_{i-j+k}(x_1, \ldots, x_{i+k})\right)_{1\leqslant i,j\leqslant n}}{(x_1 \cdots x_{k+1})^n}$$

and

$$h_k\left(\frac{1}{x_1},\ldots,\frac{1}{x_{n+1}}\right)=\frac{\det\left(h_{i-j+k}(x_1,\ldots,x_{j+1})\right)_{1\leqslant i,j\leqslant n}}{(x_1\cdots x_{n+1})^k}.$$

We see that the identities (1.5) and (1.6) follow directly from this corollary, taking into account (1.7), (1.8) and (2.1).

3 Proofs of Theorems

To prove the first theorem, we take into account that

$$e_{k,n}=x_ne_{k-1,n-1}+e_{k,n-1}.$$

We have

$$\begin{vmatrix} e_{a,k} & e_{a-1,k} & \cdots & e_{a-n+1,k} \\ e_{a+1,k+1} & e_{a,k+1} & \cdots & e_{a-n+2,k+1} \\ \vdots & \vdots & & \vdots \\ e_{a+n-2,k+n-2} & e_{a+n-3,k+n-2} & \cdots & e_{a-1,k+n-2} \\ e_{a+n-1,k+n-1} & e_{a+n-2,k+n-1} & \cdots & e_{a,k+n-1} \end{vmatrix}$$

$$=\begin{vmatrix} e_{a,k} & e_{a-1,k} & \cdots & e_{a-n+1,k} \\ e_{a+1,k+1} & e_{a,k+1} & \cdots & e_{a-n+2,k+1} \\ \vdots & \vdots & & \vdots \\ e_{a+n-2,k+n-2} & e_{a+n-3,k+n-2} & \cdots & e_{a-1,k+n-2} \\ e_{a+n-1,k+n-2} & e_{a+n-2,k+n-2} & \cdots & e_{a,k+n-2} \end{vmatrix}$$

$$\vdots$$

$$=\begin{vmatrix} e_{a,k} & e_{a-1,k} & \cdots & e_{a-n+1,k} \\ e_{a+1,k} & e_{a,k} & \cdots & e_{a-n+2,k} \\ \vdots & \vdots & & \vdots \\ e_{a+n-2,k+n-3} & e_{a+n-3,k+n-3} & \cdots & e_{a-1,k+n-3} \\ e_{a+n-1,k+n-2} & e_{a+n-2,k+n-2} & \cdots & e_{a,k+n-2} \end{vmatrix}$$

$$\vdots$$

$$=\begin{vmatrix} e_{a,k} & e_{a-1,k} & \cdots & e_{a-n+1,k} \\ e_{a+1,k} & e_{a,k} & \cdots & e_{a-n+2,k} \\ \vdots & \vdots & & \vdots \\ e_{a+n-2,k} & e_{a+n-3,k} & \cdots & e_{a-1,k} \\ e_{a+n-1,k} & e_{a+n-2,k} & \cdots & e_{a,k} \end{vmatrix}.$$

Theorem 2.1 is proved.

Considering the relation

$$h_{k,n}=x_nh_{k-1,n}+h_{k,n-1},$$

we can write

$$\begin{vmatrix} h_{a,n+k} & h_{a-1,n+k} & \cdots & h_{a-n+2,n+k} & h_{a-n+1,n+k} \\ h_{a+1,n+k} & h_{a,n+k} & \cdots & h_{a-n+3,n+k} & h_{a-n+2,n+k} \\ \vdots & \vdots & & \vdots & \vdots \\ h_{a+n-1,n+k} & h_{a+n-2,n+k} & \cdots & h_{a+1,n+k} & h_{a,n+k} \end{vmatrix}$$

$$= \begin{vmatrix} h_{a,n-1+k} & h_{a-1,n+k} & \cdots & h_{a-n+2,n+k} & h_{a-n+1,n+k} \\ h_{a+1,n-1+k} & h_{a,n+k} & \cdots & h_{a-n+3,n+k} & h_{a-n+2,n+k} \\ \vdots & \vdots & & \vdots & \vdots \\ h_{a+n-1,n-1+k} & h_{a+n-2,n+k} & \cdots & h_{a+1,n+k} & h_{a,n+k} \end{vmatrix}$$

$$\vdots$$

$$= \begin{vmatrix} h_{a,n-1+k} & h_{a-1,n-1+k} & \cdots & h_{a-n+2,n-1+k} & h_{a-n+1,n+k} \\ h_{a+1,n-1+k} & h_{a,n-1+k} & \cdots & h_{a-n+3,n-1+k} & h_{a-n+2,n+k} \\ \vdots & \vdots & & \vdots & \vdots \\ h_{a+n-1,n-1+k} & h_{a+n-2,n-1+k} & \cdots & h_{a+1,n-1+k} & h_{a,n+k} \end{vmatrix}$$

$$\vdots$$

$$= \begin{vmatrix} h_{a,1+k} & h_{a-1,2+k} & \cdots & h_{a-n+2,n-1+k} & h_{a-n+1,n+k} \\ h_{a+1,1+k} & h_{a,2+k} & \cdots & h_{a-n+3,n-1+k} & h_{a-n+2,n+k} \\ \vdots & \vdots & & \vdots & \vdots \\ h_{a+n-1,1+k} & h_{a+n-2,2+k} & \cdots & h_{a+1,n-1+k} & h_{a,n+k} \end{vmatrix}.$$

The proof of Theorem 2.2 is finished.

4 Proof of Corollary 2.4

We can write

$$\begin{aligned} h_n\left(\frac{1}{x_1},\dots,\frac{1}{x_k}\right) &= s_{[n]}\left(\frac{1}{x_1},\dots,\frac{1}{x_k}\right) \\ &= \frac{(x_1\cdots x_k)^{2n}\det\left(x_i^{-(\lambda_j+k-j)}\right)_{1\leqslant i,j\leqslant k}}{(x_1\cdots x_k)^{2n}\det\left(x_i^{-(k-j)}\right)_{1\leqslant i,j\leqslant k}} \\ &= \frac{\det\left(x_i^{2n-(\lambda_j+k-j)}\right)_{1\leqslant i,j\leqslant k}}{(x_1\cdots x_k)^{n}\det\left(x_i^{n-(k-j)}\right)_{1\leqslant i,j\leqslant k}} \\ &= \frac{s_{[n^{k-1}]}(x_1,\dots,x_k)}{(x_1\cdots x_k)^n}, \end{aligned}$$

where

$$\lambda_j = \begin{cases} n, & j=1, \\ 0, & j>1. \end{cases}$$

According to Giambelli's determinant formulas [1], we have

$$s_{[n^{k-1}]}(x_1,\ldots,x_k) = \det(e_{i-j+k-1}(x_1,\ldots,x_k))_{1\leqslant i,j\leqslant n}.$$

Taking into account Theorem 2.1, the proof of the first relation is finished.

The proof of the second relation is similar to the proof of the first relation, invoking another special case of Giambelli's determinant formulas [1],

$$s_{[k^n]}(x_1,\ldots,x_{n+1}) = \det(h_{i-j+k}(x_1,\ldots,x_{n+1}))_{1\leqslant i,j\leqslant n},$$

and Theorem 2.2.

5 Concluding remarks

The paper calculates some determinants involving the complete and elementary symmetric functions. Some specialization of these results are given in the paper. These specializations are determinant formulas for Stirling numbers of both kinds. The similar determinant formulas involving r-Stirling numbers, r-Whitney numbers, Legendre-Stirling numbers, Jacobi-Stirling numbers, and central factorial numbers can be obtained as well. This fact is possible because these numbers are specializations of complete and elementary symmetric functions.

Acknowledgments. The author expresses his gratitude to Dr. Oana Merca for the careful reading of the manuscript and helpful remarks. Finally, special thanks go to the anonymous referee for many suggestions to and comments on the original version of this paper.

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