Graphs on alphabets as models for large interconnection networks

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Abstract


Graphs on alphabets are constructed by labelling the vertices with words on a given alphabet, and specifying a rule that relates pairs of different words to define the edges. They have proved to be quite suitable to model large interconnection networks since their structure usually provides efficient routing algorithms. The aim of this paper is to present several infinite families of such graphs with a large number of vertices for given values of their diameter and maximum degree.

1. Introduction

The advent of very large scale integrated (VLSI) circuit technology has enabled the construction of very complex interconnection networks. By most accounts, the next generation of supercomputers will achieve its gains by increasing the number of processing elements (PE), rather than by using faster processors.

In these computers the time of interprocessor data communication may—and usually shall—overcome the actual computation time. Hence the interest is studying the design of interconnection networks which should consider several factors, as for instance layout, transmission delay and traffic density, reliability or fault tolerance, existence of efficient routing algorithms, cost, etc. For a survey about interconnection networks, see for instance the paper of Feng [15]. In this paper the concern is with the network topology. More precisely, we focus on two constraints inherent in such networks: the transmission delay should be small and each PE can be connected by links to just a few others.
It is well known that interconnection networks can be modeled by graphs. In our case, the vertices of the graph represent the PEs or nodes of the network and the edges represent the links between them. The distance between two vertices then represents the delay encountered in shortest path communication between the corresponding nodes, while the diameter of the graph measures the maximum possible delay. The degree of a vertex is the number of vertices it is connected to. For an account of results about graphs and interconnection networks, we refer the reader to the survey of Bermond, Bond, Paoli and Peyrat [3].

The graphs thus obtained can be directed or undirected, depending upon the links from each PE are used just for input or output, or for both. We are concerned here with undirected graphs only. Our aim is to propose several families of large graphs (i.e., graphs with a large number of vertices for given values of its degree and diameter) that have efficient routing algorithms. This last requirement puts aside several interesting constructions of large graphs described by Bermond, Dormal and Quisquater in [4]. In their classification the graphs presented here are graphs on alphabets, since each vertex of the graph may be thought of as a word on a given alphabet.

The organization of the paper is as follows. In the next section we state the basic concepts of graph theory related to the topology of networks. Then the best already known families of large graphs on alphabets are recalled. Sections 3, 4 and 5 contain our new proposals and Section 6 sums up the best families of such graphs known to date.

2. Basic concepts and known results

A graph $G = (V, E)$ consists of a set $V$ of vertices and a set $E$ of edges that join the vertices of $V$. The number of vertices $N = |V|$ is the order of the graph. If $(x, y)$ is an edge of $E$, it is said that $x$ and $y$ are adjacent, and it is usually written $x \sim y$. The degree of a vertex $\delta(x)$ is the number of vertices adjacent to $x$, and its maximum value over $V$ is the degree of $G$, $A = \Delta(G) = \max \{\delta(x) : x \in V\}$. If $\delta(x) = \Delta$ for all $x \in V$, it is said that the graph is regular of degree $\Delta$. The distance between two vertices $x$ and $y$, $d(x, y)$, is the length of a shortest path between $x$ and $y$, and its maximum value $D = \max \{d(x, y) : x, y \in V\}$ is the diameter of the graph. For the definitions not given here we refer the reader to [8].

The order of a graph with maximum degree $A \geq 3$ and diameter $D$ is easily seen to be bounded by

$$N \leq 1 + \Delta + \Delta(\Delta - 1) + \ldots + \Delta(\Delta - 1)^{D-1} = \frac{\Delta(\Delta - 1)^D - 2}{\Delta - 2}. \quad (2.1)$$

The right-hand side is called the Moore bound, and it is known that when $D \neq 1$ it can only be attained for $D = 2$ and $\Delta = 3, 7$ or possibly 57, see Bannai and Ito [2] or Damerell [9]. Hence the interest in finding graphs which have a large number of vertices, as close as possible to the Moore bound.
Besides de Bruijn and Kautz digraphs, one of the first historic examples of large graphs on alphabets must be Akers graphs, also known as odd graphs, see [11]. They are defined only for $\Delta = D + 1$, with each vertex represented by a $2\Delta - 1$ length sequence of $\Delta - 1$ 0's and $\Delta$ 1's, and where each vertex is adjacent to all vertices that have just a common 1 with it. Akers graphs are regular of degree $\Delta$, diameter $D = \Delta - 1$ and order $N = (2\Delta - 1)^{\Delta - 1}$. In most cases such graphs have less order than the graphs of the following sections. In fact Akers graphs belong to a more general family of graphs on alphabets known as Kneser graphs [25]. These graphs have as vertices the $r$-subsets of a $t$-set, $t \geq 2r + 1$, and two vertices are adjacent if the corresponding sets are disjoint.

Another example of large graphs on alphabets is the family of trivalent (i.e., $\Delta = 3$) graphs proposed by Leland and Solomon in [26]. They are defined on the vertex set $V = X^k$, where $X = \mathbb{Z}_2$ is the set of integers modulo 2, by the adjacency rules

$$
\begin{align*}
X_1X_2\ldots X_{k-1}X_k & \sim \left\{ \begin{array}{l}
X_2X_3\ldots X_kX_1, \\
\bar{X}_kX_1\ldots X_k-2X_{k-1}, \\
X_1\ldots X_{k-2}\bar{X}_{k-1}\bar{X}_k
\end{array} \right. \\
(\bar{X}_i &= X_i + 1 \mod 2).
\end{align*}
$$

It is shown that the diameter is upper bounded by $\lfloor 3k/2 \rfloor$ so that their order satisfies $N = 2^k \geq 2^{2D/3}$.

The first known infinite families of large $(\Delta, D)$ graphs, for even $\Delta$ and any $D$, were derived in a trivial way from the well-known families of de Bruijn digraphs [10] and Kautz digraphs [23,24]. It suffices to leave out the orientation of the arcs and remove the possible loops and parallel edges, that is, to take their underlying graphs.

Thus, de Bruijn graphs $UB(d, k)$ have vertex set $X^k$, $|X| = d$, adjacency conditions

$$
\begin{align*}
x_1x_2\ldots x_{k-1}x_k & \sim \left\{ \begin{array}{l}
x_2x_3\ldots x_{k-1}x_{k+1}, \quad x_{k+1} \in X, \\
x_0x_1\ldots x_{k-2}x_{k-1}, \quad x_0 \in X
\end{array} \right. \\
(2.2a)
\end{align*}
$$

degree $\Delta = 2d$ (if $k \geq 3$) and diameter $D = k$. Therefore, their number of vertices in terms of $\Delta$ and $D$ is

$$
N = \left( \frac{\Delta}{2} \right)^D.
(2.3)
$$

These graphs were generalized by Delorme and Farhi in [13]. They are also used by Jerrum and Skyum [22] to obtain the largest known graphs for fixed degree and very large diameter by substituting a graph with small average distance for each vertex of the de Bruijn graphs.

The Kautz graph $K(d, k)$ is the subgraph of the de Bruijn graph $UB(d + 1, k)$ obtained by considering only the vertices represented by words whose consecutive letters (elements of $X$) are different: $x_{i+1} \neq x_i$, $1 \leq i \leq k - 1$. For $k \geq 3$ and $d \geq 2$, Kautz
graphs have degree $A = 2d$, diameter $D = k$ and order

$$N = (d + 1)d^{k-1} = \left(\frac{A}{2}\right)^D - \left(\frac{A}{2}\right)^{D-1}.$$  \hspace{1cm} (2.4)

In [16,17,28] it is shown that the line digraph technique is very useful to obtain large digraphs. The same idea (i.e., line graph iteration) does not work for undirected graphs because it produces a steady increasing of their degree. However, using a similar approach, it is possible to construct some satisfactory families of large graphs.

The first of such families are by the sequence graphs, proposed in [18] as a generalization of line graphs. More precisely, given a connected graph $G = (V, E)$ with diameter $D$, its sequence graph of order $k \geq 1$, $S^k(G)$, is the graph whose vertices are the walks of length $k$ (sequences of $k + 1$ consecutive adjacent vertices), and two vertices of $S^k(G)$ are adjacent if their corresponding walks are adjacent in $G$.

To be more precise, the adjacency rules are:

$$x_0x_1\ldots x_{k-1}x_k \sim \begin{cases} yx_0\ldots x_{k-1}, & (y, x_0) \in E, \\ x_1x_2\ldots x_k y, & (x_k, y) \in E. \end{cases}$$

Note that, for $k = 1$, $S^1(G)$ is actually the line graph of $G$.

By studying how a given sequence can be shifted to another one, it is easily proved that the diameter $D_k$ of $S^k(G)$ satisfies $D_k \leq D + k$. Moreover, if $G$ is bipartite and $k$ is odd, $D_k \leq D + k - 1$. For both results, see [18]. In the latter case, the largest graphs are obtained when $G$ is the complete bipartite graph $K_{d_1, d_2}$ on $d_1 + d_2$ vertices. In this case the sequence graphs can be defined as graphs on alphabets as follows.

The graph $S^{2k-1}(K_{d_1, d_2}) = S(d_1, d_2, k)$ has vertex set $V = X^k \times Y^k$, $|X| = d_1$, $|Y| = d_2$, and its elements are represented by the sequences $x_1y_1x_2y_2\ldots x_ky_k$. The adjacency rules are:

$$x_1y_1x_2y_2\ldots x_ky_k \sim \begin{cases} x_{k+1}y_kx_k\ldots y_2x_2y, & x_{k+1} \in X, \\ y_kx_{k-1}x_k\ldots y_1x_1y_0, & y_0 \in Y. \end{cases}$$  \hspace{1cm} (2.5a)

Hence, (assuming $k, d_1, d_2 \geq 2$) the graph $S(d_1, d_2, k)$ has degree $A = d_1 + d_2$ and diameter $2k$, since the above result gives $D_{2k-1} \leq 2 + (2k - 1) - 1 = 2k$ and obviously there exist vertices at distance $2k$. If $d_1 = d_2 - d$, these graphs have the same number of vertices as de Bruijn graphs:

$$N = (d_1d_2)^k = \left(\frac{A}{2}\right)^D$$  \hspace{1cm} (2.6)

with even degree $A$ and even diameter $D$. On the other hand, if $d_1 = d_2 - 1 = d$, we obtain a family of graphs with $A$ odd, $D$ even, and order

$$N = \left(\frac{A - 1}{4}\right)^D = \left(\frac{A - 1}{2}\right)^D + \frac{D}{2} \left(\frac{A - 1}{2}\right)^{D-1} + \ldots.$$  \hspace{1cm} (2.7)
The graphs $B(d_1, d_2, k)$, proposed by Bond in [6] (see also [7]), are defined in the same way as sequence graphs $S(d_1, d_2, k)$ except that, in the sequences representing the vertices, successive elements of $X$ must be different, i.e., $x_{i+1} \neq x_i$, $1 \leq i \leq k - 1$. They have $N = d_1(d_1 - 1)^{k-1}d_2^k$ vertices, degree $\Delta = d_1 + d_2 - 1$ and diameter $D = 2k$. If $d_1 = d_2 = d$, the graph $B(d, d, k)$ has odd degree $\Delta$, even diameter $D$ and number of vertices

$$N = \left(\frac{A + 1}{2}\right)^{\frac{D}{2} + 1} \left(\frac{A - 1}{2}\right)^{\frac{D}{2} - 1} = \left(\frac{A - 1}{2}\right)^D + \left(\frac{D}{2} + 1\right)\left(\frac{A - 1}{2}\right)^{D - 1} + \cdots. \tag{2.8}$$

If $d_1 - 1 = d_2 = d$, the graph $B(d + 1, d, k)$ has even degree $\Delta$, even $D$, and the same order as Kautz graphs

$$N = \left(\frac{A}{2}\right)^D + \left(\frac{A}{2}\right)^{D - 1}. \tag{2.9}$$

In [11] Delorme groups together the ideas used in some of his own constructions, see [12,13], and the one involved in sequence graphs, to find a good family of large graphs with even degree and odd diameter. Let $k$ be an odd integer. The Delorme’s graph $D(d, k)$ has vertex set $\mathbb{Z}_2 \times X^k$, $|X| = d \geq 2$, and adjacency conditions

$$a; x_1x_2\ldots x_{k-1}x_k \sim \begin{cases} a; x_{k+1}x_kx_{k-1}\ldots x_3x_2, & x_{k+1} \in X, \\ a + 1; x_{k+1}x_kx_{k-2}\ldots x_1x_0, & x_0 \in X. \end{cases} \tag{2.10a}$$

$$a; x_1x_2\ldots x_{k-1}x_k \sim \begin{cases} a; x_{k+1}x_kx_{k-1}\ldots x_3x_2, & x_{k+1} \in X. \\ a + 1; x_{k+1}x_kx_{k-2}\ldots x_1x_0, & x_0 \in X. \end{cases} \tag{2.10b}$$

It has degree $\Delta = 2d$, order $N = 2(\Delta/2)^k$ and diameter $D = k$. In terms of $\Delta$ (even) and $D$ (odd), the number of vertices of Delorme’s graphs is twice the order of de Bruijin graphs:

$$N = 2\left(\frac{\Delta}{2}\right)^D. \tag{2.11}$$

3. Bisequence graphs

We present in this section a new construction which generalizes Delorme’s proposal.

Let now $k$ be an integer such that $k = 4n + 1$ ($n \geq 1$). The bisequence graph $BS(d_1, d_2, k)$, has vertex set $V = \{X^{2n+1} \times Y^{2n}\} \cup \{X^{2n} \times Y^{2n+1}\}$, $|X| = d_1$, $|Y| = d_2$. Their elements are represented by either of the sequences

$$u = x_1x_2y_3y_4x_5x_6\ldots y_{k-2}y_{k-1}x_k$$

or

$$v = y_1y_2x_3x_4y_5y_6\ldots x_{k-2}x_{k-1}y_k.$$
with $x_i \in X$ and $y_j \in Y$. Thus, $BS(d_1, d_2, k)$ has $N = (d_1 + d_2)(d_1d_2)^{2n}$ vertices. The adjacency rules are

$$x_{k+1}x_k y_{k-1}y_k \cdots y_k x_{k-2}y_{k-1}x_k \sim \begin{cases} x_{k+1}x_k y_{k-1}y_k \cdots y_4 y_3 x_2, & x_{k+1} \in X, \quad \text{(3.1a)} \\ y_{k-1}y_{k-2} \cdots y_2 y_3 x_2 y_1, & y_0 \in Y, \quad \text{(3.1b)} \end{cases}$$

and

$$y_1 y_2 x_3 y_4 y_5 y_6 \cdots x_{k-2} x_{k-1} x_k \sim \begin{cases} y_{k+1} y_k x_{k-1} x_{k-2} \cdots x_4 x_3 y_2, & y_{k+1} \in Y, \quad \text{(3.1c)} \\ x_{k-1} x_{k-2} \cdots x_4 x_3 y_2 y_1 x_0, & x_0 \in X. \quad \text{(3.1d)} \end{cases}$$

The degree of $BS(d_1, d_2, k)$ is therefore $\Delta = d_1 + d_2$. With regard to its diameter we have the following result.

**Theorem 3.1.** *The graph $BS(d_1, d_2, k)$, $k = 4n + 1$, has diameter $D = k$.***

**Proof.** Let us consider the vertices $u$ and $v$ as above. We then have the following path of length $k$ that joins them:

$$u = x_1 x_2 y_3 y_4 x_5 y_6 \cdots y_{k-2} y_{k-1} x_k$$

$$y_{k-1} y_{k-2} x_{k-3} \cdots y_4 y_3 x_2 y_1$$

$$y_2 y_1 x_3 y_2 x_3 y_4 \cdots x_{k-3} y_{k-2}$$

$$x_{k-3} x_{k-4} \cdots y_4 y_3 x_2 y_1 y_2 x_3$$

$$\vdots$$

$$x_{k-1} x_k x_{k-2} x_{k-3} x_{k-4} \cdots x_5 x_4 y_2 y_1 x_0.$$

Analogously, from vertex $u$ to a vertex $w = s_1 s_2 t_3 t_4 s_5 s_6 \cdots t_{k-2} t_{k-1} s_k$, $s_i \in X$, $t_j \in Y$, we have the following path of the same length

$$u = x_1 x_2 y_3 y_4 x_5 y_6 \cdots y_{k-2} y_{k-1} x_k$$

$$s_k x_k y_{k-1} y_{k-2} \cdots x_6 x_5 y_4 y_3 x_2$$

$$y_3 y_4 x_5 \cdots y_{k-2} y_{k-1} x_k s_k t_k t_{k-1}$$

$$t_{k-2} t_{k-1} s_k x_k y_{k-1} y_{k-2} \cdots y_4$$

$$\vdots$$

$$x_k s_k t_k t_{k-2} t_{k-3} t_{k-4} \cdots s_6 s_5 t_4 t_3 s_2$$

$$s_1 s_2 t_3 t_4 s_5 s_6 \cdots t_{k-2} t_{k-1} s_k = w.$$

As before we conclude that $D = k$. \( \square \)

When $d_1 = d_2 = d$, the graphs $BS(d, d, k)$, $k = 4n + 1$, have even degree, odd diameter $D = 1 \mod 4$ and the same number of vertices as Delorme's graphs

$$N = 2 \left( \frac{\Delta}{2} \right)^D. \quad (3.2)$$
In fact, one can easily check that both families are isomorphic. (In this case, the above construction can also be obtained when \( k \equiv 3 \mod 4 \).)

When \( d_1 = d_2 - 1 \), the graphs \( BS(d, d + 1, k), \ k = 4n + 1 \), have odd \( A \) and \( D \), \( D \equiv 1 \mod 4 \), and order

\[
N = \Delta \left( \frac{A^2 - 1}{4} \right)^{(D-1)/2} = 2 \left( \frac{A-1}{2} \right)^D + D \left( \frac{A-1}{2} \right)^D - 1 + \ldots
\]  

improving, for these values of \( D \), those of (2.7) and (2.8) corresponding to sequence and Bond’s graphs respectively.

4. Graphs of type \( Z \)

The new constructions proposed in this section are inspired in those of [13,14,27]. Let us begin describing the graphs that we denote by \( Z(m, d, k) \). These graphs have vertex set \( V = \mathbb{Z}_m \times X^k \), \( |X| = d \), where \( m \) is an odd integer, \( 3 \leq m \leq k + 1 \), and adjacency conditions

\[
\begin{align*}
\alpha; x_1 x_2 \ldots x_{k-1} x_k & \sim \begin{cases} 
  a+1; x_2 x_3 \ldots x_k x_{k+1}, & x_{k+1} \in X, \\
  a-1; x_0 x_1 \ldots x_{k-2} x_{k-1}, & x_0 \in X, \\
  -a; x_1 x_2 \ldots x_{k-1} x_k 
\end{cases} \tag{4.1a}
\end{align*}
\]

so that they have \( N = m d^k \) vertices and degree \( \Delta = 2d + 1 \).

**Theorem 4.1.** The graph \( Z(m, d, k) \) has diameter \( D = k + 1 \).

**Proof.** Let us consider the vertices \( x = a; x_1 x_2 \ldots x_{k-1} x_k \) and \( y = b; y_1 y_2 \ldots y_{k-1} y_k \). We will prove that there always exists a path of length \( k + 1 \) between them. The necessity of paths of this length between some vertices will be clear when considering the case \( b - a \not\equiv k \mod m \). From \( x \), and after \( i \) steps (4.1a), \( 0 \leq i \leq k \), we can reach the vertex

\[
a + i; x_{i+1} \ldots x_k y_1 \ldots y_i
\]

which is adjacent, by (4.1c), to the vertex

\[
-(a + i); x_{i+1} \ldots x_k y_1 \ldots y_i.
\]

From it, and after \( (k + 1) - (i + 1) = k - i \) additional steps (4.1a), we arrive to the vertex

\[
y' = k - 2i - a; y_1 y_2 \ldots y_{k-1} y_k. \tag{4.2}
\]

Then \( y' = y \) iff \( k - 2i - a \equiv b \mod m \), that is

\[
2i = k - a - b \mod m, \tag{4.3}
\]

but this equation has always a solution \( i, 0 \leq i \leq m - 1 \leq k \), for any \( a \) and \( b \) since \( \gcd(2, m) = 1 \).
Note that the "clockwise" path considered above is not unique since it is easily checked that, using \( j \) steps (4.1b), one step (4.1c) and \( k-j \) steps (4.1b), \( 0 \leq j \leq k \) ("counterclockwise" path), we reach the vertex
\[
y'' = -k + 2j - a; y_1 y_2 \ldots y_{k-1} y_k
\]
and the condition \( y'' = y \) now leads to the equation
\[
2j = k + a + b \mod m
\]
which, as before, has always a solution. □

The best results are obtained with the choices \( m = k + 1 \) and \( m = k \). When \( m = k + 1 \) the resulting graphs have odd diameter and odd degree, and their number of vertices is
\[
N = D \left( \frac{A-1}{2} \right)^{D-1}. \tag{4.6}
\]
When \( m = k \) we obtain graphs with even diameter, odd degree and order
\[
N = (D-1) \left( \frac{A-1}{2} \right)^{D-1}. \tag{4.7}
\]

Let us now see how to improve the values (4.6) and (4.7) by a simple technique already used in preceding constructions.

The graphs \( Z'(m, d, k) \) are defined like the graphs \( Z(m, d, k) \), but consecutive symbols (elements of \( X \)) must now be different, \( x_{i+1} \neq x_i, \ 1 \leq i \leq k-1 \). Hence, they have degree \( \Delta = 2d-1 \) and order \( N = md(d-1)^k \). Moreover, the next result shows that they have the same diameter as the graphs \( Z(m, d, k) \).

**Theorem 4.2.** The graph \( Z'(m, d, k) \) has diameter \( D = k + 1 \).

**Proof.** Let \( x = a; x_1 x_2 \ldots x_{k-1} x_k \) and \( y = b; y_1 y_2 \ldots y_{k-1} y_k \) be two generic vertices. We will exhibit a path of length \( k-1 \) or \( k \) between them. First, if \( y_1 \neq x_k \), this path is the same as in the proof of Theorem 4.1. Otherwise, it is easily proved that from \( x \), and after \( k \) steps \( i \) of type (4.1a), one of type (4.1c) and \( k-(i+1) \) of type (4.1a), it is possible to reach the vertices
\[
-a-2i-2; y_1 y_2 \ldots y_{k-1} y_k
\]
where \( 0 \leq i \leq k-1 \). For these \( k \) values of \( i \), the first digit in (4.8) takes all values in \( \mathbb{Z}_m \) except, when \( m = k + 1 \), the value \( b = -a - 2(m-1) - 2 = -a \). Thus, it only remains to consider the case \( x = a; x_1 x_2 \ldots x_{m-2} y_1 \) and \( y = -a; y_1 y_2 \ldots y_{m-2} y_{m-1} \).

1. If \( a \neq 0 \), we first go to the vertex, adjacent to \( x, -a; x_1 x_2 \ldots x_{m-2} y_1, -a \neq a \), which can be seen as the new initial vertex with \( -(a) \neq b \) and we are in the case above.
2. If \( a = 0 \), then \( b = 0 \), and a path of length \( m \) from \( x \) to \( y \), which only uses
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conditions of type (4.1a), is the following:

\[
x = 0; x_1 x_2 \ldots x_{m-2} y_1 \\
1; x_2 x_3 \ldots y_1 y_0, y_0 \neq y_1 \\
2; x_3 x_4 \ldots y_1 y_0 y_1 \\
3; x_4 x_5 \ldots y_0 y_1 y_2 \\
\vdots \\
0; y_1 y_2 \ldots y_{m-2} y_{m-1} = y.
\]

When \( m = k + 1 \) we obtain the graphs \( Z'(m, d, m-1) \) with odd degree and odd diameter, and whose order is

\[
N = D \left( \frac{A-1}{2} \right)^{D-1} + D \left( \frac{A-1}{2} \right)^{D-2}.
\] (4.9)

When \( m = k \), the resulting graphs \( Z'(m, d, m) \) have even diameter, odd degree and number of vertices

\[
N = (D-1) \left( \frac{A-1}{2} \right)^{D-1} + (D-1) \left( \frac{A-1}{2} \right)^{D-2}.
\] (4.10)

(Values to be compared with those of (4.6) and (4.7) respectively.)

From the consideration of the two paths showed before for the graphs \( Z(m, d, k) \), we shall next see how the values of (4.9) and (4.10) can be further improved. We concentrate our discussion upon the main case \( k = \pm 1 \mod m \). Let us first show that, in these graphs, roughly half of the edges of the form \( (a; x_1 x_2 \ldots x_{m-1}, -a; x_1 x_2 \ldots x_{m-1}) \) are redundant in the sense that they can be removed without increasing the diameter. This fact allows us to double the order by joining two copies of the resulting graphs through the deficient vertices. Starting from (4.2) and (4.4) with \( k = m - 1 \), we see that the clockwise and counterclockwise paths lead to the same vertex if and only if

\[-2i - a - 1 \equiv 2j - a + 1 \mod m.
\]

Since \( m \) is odd, this implies

\[-j - i \equiv 1 \mod m \]

or

\[(a - j) - (a + i) \equiv 1 \mod m\] (4.11a)

That is illustrated in Fig. 1 which also shows, using a self-explanatory graph, the changes of the first digit \( a \) induced by the adjacency conditions (4.1).

Let \( E_0 \) and \( E_1 \) denote the sets of edges of \( Z(m, d, k) \) defined by condition (4.1c) when \( a = 0, \pm 2, \pm 4, \ldots \) and \( a = \pm 1, \pm 3, \ldots \) respectively. The following result shows that (4.11) implies the redundancy of either \( E_0 \) or \( E_1 \).
Lemma 4.3. The graphs obtained from \( Z(m, d, k) \), \( k \equiv -1 \mod m \), by leaving out the edges of either \( E_0 \) or \( E_1 \) have diameter \( D = k + 1 \).

Proof. Obviously, it suffices to consider the case \( k = m - 1 \). Then, considering the same type of paths between two generic vertices \( x = a; x_1 x_2 \ldots x_{m-2} x_{m-1} \) and \( y = b; y_1 y_2 \ldots y_{m-2} y_{m-1} \), equations (4.3) and (4.5) can be restated in the form

\[
b \equiv -1 - a - 2i \mod m \quad (4.12)
\]

and

\[
b \equiv +1 - a + 2j \mod m \quad (4.13)
\]

respectively.

We consider the following cases:

(a) \( m = 4n - 1 \).

(a.0) Remove \( E_0 \): condition (4.1c) only applies when \( a = \pm 1, \pm 3, \ldots, \pm (2n - 1) \).

(a.1) Remove \( E_1 \): condition (4.1c) only applies when \( a = 0, \pm 2, \ldots, \pm (2n - 2) \).

(b) \( m = 4n + 1 \).

(b.0) Remove \( E_1 \): condition (4.1c) only applies when \( a = 0, \pm 2, \ldots, \pm 2n \).

(b.1) Remove \( E_0 \): condition (4.1c) only applies when \( a = \pm 1, \pm 3, \ldots, \pm (2n - 1) \).

Figure 2(a) shows the cases (a.0) and (a.1) when \( m = 7 \). In case (a.0) we see that, in (4.12) and (4.13), each of \( i \) and \( j \) can take \( 2n \) values (those satisfying \( a + i \) or \( a - j \) equal to \( \pm 1, \pm 3, \ldots, \pm (2n - 1) \)) which, according to (4.11b), give all possible values for \( b \) since there only exists one repetition when \( a - j = -(2n - 1) \equiv 2n \mod m \) and \( a + i = 2n - 1 \).
On the other hand, in case (a.1) each of $i$ and $j$ can take only $2n - 1$ values (those that satisfy $a + i$ or $a - j$ equal to $0, \pm 2, \ldots, \pm (2n - 2)$) which, according to (4.11b), give in (4.12) and (4.13) $4n - 2$ different values of $b$. From the above, we obviously conclude that the missing value is that corresponding to $a + i = 2n - 1$ or $a - j = -(2n - 1) \equiv 2n \mod m$, that is $b = a$. Then, to reach the corresponding vertex $a; y_1y_2\ldots y_{m-1}$ it suffices to consider $m$ steps of type (4.1a) or (4.1b) as in case (2) of the proof of Theorem 4.2.

Cases (b.0) and (b.1) are analogous to (a.0) and (a.1) respectively. □

When $m = 4n - 1$ we can now obtain a new graph by joining together two copies of the graph that results in case (a.1) according to the adjacency conditions (4.1c) that apply in (a.0), and similarly when $m = 4n + 1$. More precisely, an... going back to the general case, the graphs $Z^n(m, d, k)$ have vertex set $\mathbb{Z}_2 \times \mathbb{Z}_m \times X^k$, where $m$ is an odd integer and $k = -1 \mod m$. Hence, each vertex is defined by a sequence $a, a; x_1x_2\ldots x_{k-1}x_k$ with $a \in \mathbb{Z}_2$, $a \in \mathbb{Z}_m$ and $x_i \in X$. To define the adjacency conditions it is necessary to distinguish two cases:

(a) If $m = 4n - 1$,

\[
\begin{align*}
\alpha, a; x_1x_2\ldots x_k &\sim \begin{cases}
\alpha, a + 1; x_2\ldots x_kx_{k+1}, & x_{k+1} \in X, \\
\alpha, a - 1; x_0x_1\ldots x_{k-1}, & x_0 \in X,
\end{cases} & \text{(4.14a)} \\
\alpha, -a; x_1x_2\ldots x_k, & \text{if } a = 0, \pm 2, \ldots, \pm (2n - 2), & \text{(4.14c)} \\
\delta, a; x_1x_2\ldots x_k, & \text{if } a = \pm 1, \pm 3, \ldots, \pm (2n - 1). & \text{(4.14d)}
\end{align*}
\]
(b) If \( m = 4n + 1 \),

\[
\begin{align*}
\alpha, a; x_1 x_2 \ldots x_k & \rightarrow \\
\alpha, a + 1; x_1 x_2 \ldots x_{k+1}, & \quad x_{k+1} \in X, \\
\alpha, a - 1; x_0 x_1 \ldots x_{k-1}, & \quad x_0 \in X, \\
\alpha, -a; x_1 x_2 \ldots x_k, & \quad \text{if } a = \pm 1, \pm 3, \ldots, \pm (2n - 1), \\
\alpha, -a; x_1 x_2 \ldots x_k, & \quad \text{if } a = 0, \pm 2, \ldots, \pm 2n.
\end{align*}
\]

(4.15a) \( (4.15b) \) \( (4.15c) \) \( (4.15d) \)

Figure 2(b) shows the \( \ldots \) ge of the digits \( \alpha, a \) induced by the adjacency conditions (4.14) when \( m = 7 \).

From the above, the graphs \( Z^\alpha(m, d, k) \) have order \( N = 2md^k \) and degree \( \Delta = 2d + 1 \).

**Theorem 4.4.** The graph \( Z^\alpha(m, d, k) \), \( k \equiv -1 \mod m \), has diameter \( D = k + 1 \).

**Proof.** The proof of this result is a straightforward consequence of Lemma 4.3 and the way \( Z^\alpha(m, d, k) \) has been constructed from two copies of the graphs considered there. Indeed, the existence of a \( k + 1 \) length path between any pair of vertices \( x \) and \( y \) stems from cases (a.1) and (b.1) [(a.0) and (b.0)] considered in the proof of the lemma, if both vertices belong to the same [different] copy \( \Box \)

The two most interesting particular cases follow.

When \( k = m - 1 \), the graphs \( Z^\alpha(m, d, m - 1) \) have odd diameter and degree, and their order is

\[ N = 2D \left( \frac{\Delta - 1}{2} \right)^{D-1}. \]  

(4.16)

When \( k = 2m - 1 \), we obtain the graphs \( Z^\alpha(m, d, 2m - 1) \) with even diameter, odd degree and number of vertices

\[ N = D \left( \frac{\Delta - 1}{2} \right)^{D-1}. \]  

(4.17)

5. Graphs \( T(d, k) \)

In this section we use several ideas involved in the preceding constructions to obtain a new family of graphs for (any) odd values of \( \Delta \) and \( D \).

Let \( |X| = d + 1 \) and consider the graph \( T(d, k) \), \( k \) odd, whose vertices are represented by the sequences \( a; x_1 x_2 \ldots x_{k-1} x_k, \ a \in \mathbb{Z}_2, \ x_i \in X, \ x_i \neq x_{i+1}, \ 1 \leq i \leq k - 1. \) Hence, \( T(d, k) \) has order \( N = 2(d + 1)d^{k-1} \).

To define the adjacency relations consider the bijections

\[ \pi(a; x_1 x_2 \ldots x_{k-1} x_k) = \bar{a}; x_k x_{k-1} \ldots x_2 x_1, \]  

(5.1)

\[ r(a; x_1 x_2 \ldots x_{k-1} x_k) = \bar{a}; x_1 x_2 \ldots x_{k-1} x_k \]  

(5.2)
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(remember that \( a = a + 1 \mod 2 \)), and let \( \phi \) denote any one of the \( d \) mappings

\[
\phi(a; x_1 x_2 \ldots x_{k-1} x_k) = \overline{a}; x_0 x_1 \ldots x_{k-2} x_{k-1}
\]

for the different choices of \( x_0 \in X \), \( x_0 \neq x_1 \).

Note that, with these definitions,

\[
\phi^{-1}(a; x_1 x_2 \ldots x_{k-1} x_k) = \overline{a}; x_2 x_3 \ldots x_k x_{k+1}
\]

where \( x_{k+1} \neq x_k \), \( n^2 = 1 \) (identity), and \( r \) commutes with \( \pi \), \( \phi \) and \( \phi^{-1} \). Moreover, we have \( \pi \phi \pi = \phi^{-1} \) since

\[
\pi \phi (a; x_1 x_2 \ldots x_{k-1} x_k) = \pi \phi (\overline{a}; x_k x_{k-1} \ldots x_2 x_1)
\]

\[
= \pi(a; x_{k+1} x_k \ldots x_3 x_2)
\]

\[
= \overline{a}; x_2 x_3 \ldots x_k x_{k+1}
\]

\[
= \phi^{-1}(a; x_1 x_2 \ldots x_{k-1} x_k).
\]

Now any vertex \( x = a; x_1 x_2 \ldots x_{k-1} x_k \) of the graph \( T(d, k) \) is defined to be adjacent to the \( A = 2d + 1 \) vertices \( n \pi(x) \), \( n \gamma(x) \) and \( r(x) \). That is, if \( a = 1 \),

\[
\pi \phi \pi (x) = \phi^{-1}(x) = 0; x_2 x_3 \ldots x_k x_{k+1},
\]

\[
\pi \phi (x) = 1; x_{k-1} x_{k-2} \ldots x_1 x_0,
\]

\[
r(x) = 0; x_1 x_2 \ldots x_{k-1} x_k,
\]

and, if \( a = 0 \) (\( \pi^0 = 1 \)),

\[
\phi \pi (x) = 0; x_{k+1} x_k \ldots x_2 x_3,
\]

\[
\phi (x) = 1; x_0 x_1 \ldots x_{k-2} x_{k-1},
\]

\[
r(x) = 1; x_1 x_2 \ldots x_{k-1} x_k.
\]

Notice that, as a consequence of the condition \( x_i \neq x_{i+1} \), all these vertices are different. Besides, \( x \) is not adjacent to itself since otherwise and assuming \( a = 0 \) (the case \( a = 1 \) being similar) we would have \( x = 0; x_1 x_2 \ldots x_{k-1} x_k = 0; x_{k+1} x_k \ldots x_2 x_1 = \phi \pi (x) \) and, in particular, \( x_{(k+1)/2} = x_{(k+3)/2} \) contradicting the afore-mentioned condition. Thus, the graph \( T(d, k) \), on \( N = 2((A - 1)/2)^k + 2((A - 1)/2)^{k-1} \) vertices, is \( \Delta \)-regular. For instance, the graph \( T(2, 3) \) is shown in Fig. 3.

In order to find the diameter of \( T(d, k) \), it is useful to consider the following equalities

\[
(\phi \pi)(\pi \phi \pi)(\phi)(\phi) = \phi^4,
\]

\[
(\pi \phi \pi)(\phi \pi)(\phi \pi \pi)(\phi \pi) = \pi \phi^4 \pi = \phi^{-4},
\]

since they provide a pattern to form a given sequence from any other one by adding successive digits.
Theorem 5.1. The graph \( T(d, k) \) has diameter \( D = k \).

**Proof.** We shall prove that, between any pair of vertices \( x \) and \( y \), there exists a path of length \( k \) or \( k - 1 \). Using the graph automorphism \( \pi \) we can fix \( u = 0 \) in vertex \( x \). Then, with the notation \( x = 0; x_1 x_2 \ldots x_{k-1} x_k \), \( y^0 = y_1 y_2 \ldots y_{k-1} y_k \), \( y^1 = 1; y_1 y_2 \ldots y_{k-1} y_k \), it is necessary to consider the following six cases:

(a) \( k = 4n + 1 \),

(a.1) \( x_1 = y_k \);

(a.2.1) \( x_1 \neq y_k, x_k = y_k \);

(a.2.2) \( x_1 \neq y_k, x_k \neq y_k \).
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Let us see how, in every case, the application to $x$ of $k$ or $k - 1$ suitable adjacency mappings leads to both $y^0$ and $y^1$.

(a.1) $k = 4n + 1$, $x = 0; y_k x_{k-1} ... x_1$. In $k - 1$ steps we have, by (5.7), $\phi^{4n}(x) = y^0$; and, in $k$ steps, $r \phi^{4n}(x) = y^1$.

(a.2.1) $k = 4n + 1$, $x = 0; x_1 x_2 ... x_{k-1} y_k$. Similarly, in $k$ steps, $\phi \phi^{4n}(x) = y^1$.

To reach $y^0$ is a little more involved. Using the different properties of the adjacency rules we obtain, in $k$ steps, $(\phi \pi)r(\pi \phi)\phi (\phi \pi)\phi^{-4(n-1)}(x) = r \pi \phi^{4n}(x) = y^0$.

(a.2.2) $k = 4n + 1$, $x = 0; x_1 x_2 ... x_{k-1} y_k, x_1 \neq y_k, x_k \neq y_k$. As before, $y^1$ is easily obtained: $\phi \phi^{4n}(x) = y^1$. Moreover, $(\phi \pi)\phi^{-4n}(x) = (\phi \pi)\phi^{4n}(x) = \phi^{4n+1}(x) = y^0$.

The other three cases are similar and their corresponding equalities self-explanatory:

(b.1) $k = 4n + 3$, $x = 0; y_k x_{k-1} ... x_1$. $(\pi \phi)\phi \phi^{4n}(x) = \pi \phi^{4n+2}(x) = y^1$; $r \pi \phi^{4n+2}(x) = y^0$.

(b.2.1) $k = 4n + 3$, $x = 0; x_1 x_2 ... x_{k-1} y_k, x_1 \neq y_k, x_k \neq y_k$. Vertex $y^0$ is obtained as in the preceding case; $(\pi \phi)\phi \phi^{-4n}(x) = \phi^{-(4n+3)}(x) = y^1$. □

According to the above result, we have constructed regular graphs with odd degree $\Delta$ and diameter $D$ on

$$N = 2\left(\frac{\Delta - 1}{2}\right)^D + 2\left(\frac{\Delta - 1}{2}\right)^{D-1}$$  (5.8)

vertices.

6. Conclusions

Several infinite families of dense graphs for any given values of their degree and diameter have been presented. It is noteworthy that until recently only two such infinite families were known, namely de Bruijn and Kautz graphs, both with even degree. The following list summarizes the best families described in the paper.

- When $\Delta$ is even and $D$ is odd Delorme's graphs achieve

$$N = 2\left(\frac{\Delta}{2}\right)^D.$$  

- When $\Delta$ and $D$ are even both Kautz and Bond's graphs achieve

$$N = \left(\frac{\Delta}{2}\right)^D + \left(\frac{\Delta}{2}\right)^{D-1}.$$
When \( \Delta \) is odd and \( D \) is even the best families are Bond’s graphs \( B(d,d,k) \),

\[
N = \left( \frac{\Delta + D + 1}{2} \right) \left( \frac{\Delta - 1}{2} \right)^{D-1} + \cdots
\]

or \( Z(m,d,2m-1) \) graphs,

\[
N = D \left( \frac{\Delta - 1}{2} \right)^{D-1},
\]
depending upon the particular values of \( \Delta \) and \( D \).

Finally, when both \( \Delta \) and \( D \) are odd the best families are \( T(d,k) \) graphs,

\[
N = (\Delta + 1) \left( \frac{\Delta - 1}{2} \right)^{D-1},
\]

\( Z''(m,d,m-1) \) graphs,

\[
N = 2D \left( \frac{\Delta - 1}{2} \right)^{D-1},
\]
cr, when \( D \equiv 1 \mod 4 \), \( BS(d,d+1,k) \) graphs,

\[
N = (\Delta + D - 1) \left( \frac{\Delta - 1}{2} \right)^{D-1} + \cdots,
\]
depending upon the particular values of \( \Delta \) and \( D \).

Of course these are all quite general families and for individualized values of \( \Delta \) and/or \( D \) (mostly for small values) there are particular constructions that improve the above results. See, for instance, [4,20,21,26].

The search for dense graphs seems to be a hard problem. While de Bruijn digraphs and mainly Kautz digraphs are very good dense digraphs with orders close to the Moore bound, the best known families of dense graphs are still far away of the analogous Moore bound. Even the asymptotic behaviour of \( NA^{-D} \) as \( D \) grows is, for most values of \( \Delta \), far away from the value 1 predicted by Bollobás and de la Vega’s work [5].

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References

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