A NUMERICAL METHOD FOR A PLASMA-SHEATH MODELLING PROBLEM

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Abstract. A system of three coupled nonlinear differential equations which arise in the modelling of plasma sheaths is studied. Using a particular scaling of the variables, the original problem is reformulated so that in the transformed variables the problem explicitly involves a singular perturbation parameter. For one of the component equations theoretical results are given which show that a numerical method based on a simple linearization technique, a piecewise-uniform mesh of Shishkin type and a standard upwind finite difference operator generates parameter-uniform numerical approximations. Motivated by this, a numerical method for generating approximations to the solution of the full system of equations is constructed. Numerical results are presented which indicate that this method is parameter-uniform.

1. Introduction

Singularly perturbed differential equations arise frequently in many areas of science and engineering [9]. These differential equations are usually characterized by the presence of a small parameter multiplying some or all of the highest derivatives appearing in a non-dimensional form of the differential equation. Classical numerical methods (finite difference, finite volume or finite element methods) are known to be inappropriate for such problems [1, 11]. Their inadequacies include the appearance of non-physical spurious oscillations or the excessive numerical smearing of sharp profiles within the computed solutions [1]. Special numerical methods have been developed for a wide class of singularly perturbed differential equations which are parameter-uniform [1, 7, 11]. Parameter-uniform methods are numerical methods whose numerical approximations \( Z \) satisfy asymptotic error bounds of the form

\[
\|z - Z\|_{D,\infty} \leq CN^{-p}, \quad p > 0
\]  

(1.1)

where \( z \) is the solution of the associated continuous problem, \( N \) (independent of the singular perturbation parameter \( \varepsilon \)) is a measure of the order of the number of mesh elements used in each coordinate direction, the norm \( \|f\|_D = \max_{x \in D} |f(x)| \) is the maximum pointwise norm on the domain \( D \) and the error constant \( C \) is a constant which is independent of both \( N \) and \( \varepsilon \). In other words, the numerical approximations \( Z \) converge (as \( N \to \infty \)) to the analytical solution \( z \) for all values of the singular perturbation parameter \( \varepsilon \) to values of order one). For classical numerical methods, the error constant \( C \) typically depends adversely on the small parameter \( \varepsilon \) ( \( C \to \infty \) as \( \varepsilon \to 0 \)).

In this paper, we examine a system of three coupled nonlinear singularly perturbed differential equations which arise in the modelling of a plasma sheath. We use a simple linearization and standard upwinding in the discretization of the differential operators. The novel aspect in the numerical algorithm will be the use of a special distribution of the mesh points. In this paper, we employ an appropriate piecewise-uniform mesh of Shishkin type [1, 12], which is one of the simplest ways...
to construct a parameter uniform numerical method. For a large class of singularly perturbed differential equations [12], it has been established [1] that the numerical approximations generated on a Shishkin mesh have additional benefits such as parameter-uniform global accuracy of the linear interpolant and parameter-uniform approximations to the scaled derivatives of the solution. There is an expanding literature [1, 5, 11] establishing theoretical asymptotic error bounds of the form (1.1) for several types of linear singularly perturbed partial differential equations. Corresponding theoretical results for strongly nonlinear singularly perturbed partial differential equations or for systems of coupled partial differential equations are very few. In this paper, we do not present any theoretical study of the convergence properties of the numerical approximations to our system of differential equations modelling a plasma sheath. A numerical algorithm is constructed for this system based on appropriate parameter-uniform numerical methods for the component equations in the system. Extensive computational results are given which suggest that the use of the Shishkin mesh generates parameter-uniform numerical approximations to the dependent variables in the plasma system.

In §2, we describe the physical problem and in §3 we present the corresponding mathematical problem. In §4, we discuss properties of the piecewise-uniform Shishkin mesh in the context of a linear singularly perturbed ordinary differential equation and in §5 we discuss our approach to dealing with the key exponential nonlinearity in the system under investigation. In §6 we present the numerical algorithm and in the final two sections of the paper a selection of numerical results are given.

2. Problem formulation

Consider the interaction of a flowing plasma with a planar Langmuir probe [4]. Assume that the plasma flows in the plane of the probe surface and that the plasma consists solely of positive ions with density \( n_+ \) and electrons with density \( n_e \). Before the plasma encounters the probe, it is assumed to be quasi-neutral with \( n_+ = n_e = n_0 \), where \( n_0 \) is a constant. Downstream of the probe, the ions are moving with a velocity of \( \mathbf{u} = (u_i, u_v + u_F) \) and \( (\bar{u}_0, u_F) \) is the constant flow velocity of the ions upstream of the probe. Throughout this paper, it is assumed that \( u_F >> u_v \). We wish to consider the influence of the probe on the flow.

Let \( X \) be the horizontal distance to the right of the probe and \( Y \) the distance along the probe (from the tip of the probe). The leading edge of the probe is located at \( X = Y = 0 \). Assume also that the plasma upstream of the leading edge of the probe is not disturbed by the presence of the probe, so that the following conditions apply

\[
n_+(X, 0) = n_e(X, 0) = n_0, \quad \mathbf{u}(X, 0) = (\bar{u}_0, u_F).
\]

Assuming a collision-less plasma and that the ions are cold, the continuity equations for the ion density \( n_+ \) and momentum \( m_+ \mathbf{u} \) can be written [4] as

\[
\frac{\partial n_+}{\partial t} + \nabla \cdot (n_+ \mathbf{u}) = 0, \tag{2.2a}
\]

\[
m_+ n_+ \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) = e n_+ \mathbf{E}, \tag{2.2b}
\]

where \( \mathbf{E} = (E_X, E_Y) = -\nabla \phi \) is the electric field and \( m_+ \) is the mass of the ions. See Ha and Slemrod [3] for a discussion of global existence in the case of time-dependent sheaths. Our interest is in the steady state case and as \( u_F >> u_v \), we disregard terms involving \( u_v \). Hence the system given in (2.2) is approximated with
the system
\[
\frac{\partial}{\partial X}(n_+ u_i) + u_F \frac{\partial n_+}{\partial Y} = 0, 
\tag{2.3a}
\]
\[
u_i \frac{\partial u_i}{\partial X} + u_F \frac{\partial u_i}{\partial Y} = \frac{e}{m_+} E_X, 
\tag{2.3b}
\]
where (if \( E_X >> E_Y \)) then the electric field is determined from solving Poisson’s equation
\[
- \frac{\partial E_X}{\partial X} = \frac{\partial^2 \tilde{\phi}}{\partial X^2} = \frac{e(n_+ - n_e)}{\epsilon_0}, 
\tag{2.3c}
\]
Since \( m_e << m_+ \) and we assume that the electrostatic potential \( \tilde{\phi} \) tends to zero as one moves away from the probe, the electron density \( n_e \) is approximately related to the electrostatic potential \( \tilde{\phi} \) as \([4]\)
\[
n_e = n_0 \exp \left( - \frac{e\tilde{\phi}}{kT_e} \right), 
\tag{2.3d}
\]
where \( T_e \) is the electron temperature, \( k \) is Boltzmann’s constant, \( \epsilon_0 \) is the permittivity of free space and \( e \) is the electron charge.

3. Mathematical problem

In a similar fashion to the scaling of the variables used in \([6]\), we introduce the non-dimensional independent variables \( x, y \) and the non-dimensional dependent variables \( n, u \) and \( \phi \), which are defined as follows:
\[
n = \frac{n_+}{n_0}, \quad u = \frac{u_i}{c_s}, \quad \phi = \frac{-e\tilde{\phi}}{kT_e}, \quad x = \frac{X}{L}, \quad y = \frac{Y c_s}{u_F L},
\tag{3.1}
\]
The ion sound speed \( c_s \) and the electron Debyre length \( \lambda_D \) are defined respectively by \([4]\)
\[
c^2_s = \frac{kT_e}{m_+}, \quad \lambda^2_D = \frac{\epsilon_0 kT_e}{n_0 e^2}.
\]
The length \( L \) is a distance sufficiently far from the probe so that the effect of the probe on the plasma at this distance is negligible. Introduce the small parameter
\[
\varepsilon = \frac{\lambda_D}{L}, 
\tag{3.2}
\]
In practice, the following parameter ranges \( 10^{-4} \leq \varepsilon \leq 10^{-1}, \quad -100 \leq \phi(0) \leq -1 \) are typical. The mathematical problem we examine in this paper is as follows:
Find \( (\phi(x, y), u(x, y), n(x, y)) \), which satisfy the following system of differential equations, (applying the change of variables (3.1) to (2.3)) in the space domain \((x, y) \in (0, 1] \times [0, T]\):
\[
\frac{\partial n}{\partial y} + \frac{\partial (nu)}{\partial x} = 0, 
\tag{3.3a}
\]
\[
\frac{\partial u}{\partial y} + u \frac{\partial u}{\partial x} = - \frac{\partial \phi}{\partial x}, 
\tag{3.3b}
\]
\[
\varepsilon^2 \frac{\partial^2 \phi}{\partial x^2} = e^\phi - n, \quad \varepsilon = \lambda_D L^{-1}, 
\tag{3.3c}
\]
subject to the following set of boundary and initial conditions

\[
\begin{align*}
n_x(1, y) &= u_x(1, y) = 0, \quad y \geq 0, \\
\phi(0, y) &= -A, \quad \phi(1, y) = 0, \quad y \geq 0, \\
n(x, 0) &= 1, \quad u(x, 0) = u_0 < 0, \quad 0 < x \leq 1, \\
\phi(x, 0) &= \phi_0(x), \quad 0 < x < 1.
\end{align*}
\]

The parameters \( u_0 < 0 \) and \( A > 0 \) are assumed to be known constants. From this formulation, since (3.3c) is valid at \( y = 0 \) and \( n(x, 0) = 1 \), we deduce that the initial condition \( \phi_0(x) \) satisfies the boundary value problem

\[
\varepsilon^2 \phi''_0(x) = e^{\phi_0(x)} - 1, \quad \phi_0(0) = -A, \quad \phi_0(1) = 0.
\]

Note that we also have

\[
n_y(1, y) = -(nu_x)(1, y), \quad u_y(1, y) = -\phi_x(1, y), \quad y \geq 0.
\]

**Remark 1.** Note that the constant triple \( (\phi^{[0]}, u^{[0]}, n^{[0]}) = (0, u_0, 1) \) satisfies the system of differential equations (3.3a-c) and all the initial and boundary conditions except the initial condition \( \phi^{[0]}(x, 0) = \phi_0(x), 0 < x < 1 \) and the side boundary condition \( \phi(0, y) = -A \neq 0 \). Note also that the triple \( (\phi^{[1]}(x), u^{[1]}(x), n^{[1]}(x)) \) defined for \( x \in [0, 1] \) as the solution of

\[
\begin{align*}
\frac{d(n^{[1]}u^{[1]})}{dx} &= 0, \quad (3.4a) \\
u^{[1]} \frac{du^{[1]}}{dx} &= -\frac{d\phi^{[1]}}{dx}, \quad (3.4b) \\
\varepsilon^2 \frac{d^2\phi^{[1]}}{dx^2} &= e^{\phi^{[1]}} - n^{[1]}, \quad (3.4c) \\
\phi^{[1]}(0) &= -A, \quad u^{[1]}(1) = u_0, \quad n^{[1]}(1) = 1, \quad \phi^{[1]}(1) = 0, \quad (3.4d)
\end{align*}
\]

satisfies the system of differential equations (3.3a-c) and all the boundary conditions at \( x = 1 \) if \( (\phi^{[1]})'(1) = 0 \) and \( u_0 \neq 0 \). However, if \( A \neq 0 \), the initial conditions for \( n \) and \( u \) are not satisfied. Assuming \( n^{[1]}(x) > 0 \) (that the density is positive) and \( u_0 \neq 0 \), then we can determine this triple implicitly (see for example [10]). Solving (3.4) we have

\[
n^{[1]}(x) = \frac{|u_0|}{\sqrt{u_0^2 - 2\phi^{[1]}(x)}}, \quad u^{[1]}(x) = \frac{u_0}{|u_0|} \sqrt{u_0^2 - 2\phi^{[1]}(x)}, \\
(\varepsilon \frac{d\phi^{[1]}}{dx})^2 = 2 \left( e^{\phi^{[1]}} - 1 + |u_0| \left( \sqrt{u_0^2 - 2\phi^{[1]}(x)} - |u_0| \right) \right) + C_1, \quad 0 < x < 1,
\]

\[
\phi^{[1]}(0) = -A, \quad \phi^{[1]}(1) = 0, \quad C_1 = (\varepsilon \frac{d\phi^{[1]}}{dx}(1))^2.
\]

In equation (3.3c), the presence of the small parameter \( \varepsilon \) indicates that the system (3.3) is a singularly perturbed coupled system of nonlinear partial differential equations [1]. Due to the presence of the singular perturbation parameter, \( \varepsilon \), boundary layers can appear in the solutions. Boundary layers in the plasma are called sheaths [10] in the plasma physics literature [3]. In this paper, we will discretize the system (3.3) using an appropriate piecewise-uniform Shishkin mesh [1, 12].
4. Shishkin mesh

Consider the following linear singularly perturbed two point boundary value problem

\[ L_\varepsilon u_\varepsilon(x) \equiv -\varepsilon^2 u_\varepsilon'' + a(x) u_\varepsilon = f(x), \quad x \in \Omega = (0, 1), \tag{4.1a} \]
\[ a(x) \geq \beta^2 > 0, \tag{4.1b} \]
\[ u(0) = A, \quad u(1) = B. \tag{4.1c} \]

The solution of problem (4.1) can be written as the sum of a regular component \( v(x) \) and two singular components [1, 7, 8] \( w_l(x), w_r(x) \) of the form

\[ u_\varepsilon(x) = v(x) + (u_\varepsilon(0) - v(0)) w_l(x) + (u_\varepsilon(1) - v(1)) w_r(x), \tag{4.1d} \]

where

\[ L_\varepsilon v(x) = f(x), \quad v(0) = \left( \frac{f}{a} \right)(0), \quad v(1) = \left( \frac{f}{a} \right)(1); \]
\[ L_\varepsilon w_l(x) = 0, \quad w_l(0) = 1, \quad w_l(1) = 0; \]
\[ L_\varepsilon w_r(x) = 0, \quad w_r(0) = 0, \quad w_r(1) = 1. \]

The following parameter explicit bounds on these components and their derivatives can be established [7, 8] for \( 0 \leq k \leq 4 \)

\[ \left\| \frac{d^k v}{dx^k} \right\|_{\Omega, \infty} \leq C(1 + \varepsilon^{2-k}), \]
\[ \left| \frac{d^k w_l}{dx^k}(x) \right| \leq C\varepsilon^{-k}e^{-\beta x/\varepsilon}, \quad \left| \frac{d^k w_r}{dx^k}(x) \right| \leq C\varepsilon^{-k}e^{-\beta(1-x)/\varepsilon}, \]

where \( C \) is a constant that is independent of \( \varepsilon \). Throughout this paper, \( C \) is a generic constant which does not depend on \( N \) and \( \varepsilon \). Note that the singular component \( w_l \) is negligible away from the boundary point \( x = 0 \). That is

\[ |w_l(\tau)| \leq C\varepsilon^2, \quad \left| \frac{dw_l}{dx}(x) \right| \leq C\varepsilon \quad \text{and} \quad \left| \frac{d^2 w_l}{dx^2}(x) \right| \leq C \quad \text{for} \quad x \geq \tau = \frac{2 \varepsilon \ln 1/\varepsilon}{\beta}. \tag{4.1e} \]

However, within the region \( (0, C\varepsilon) \) the derivatives of the singular component become unbounded as \( \varepsilon \to 0 \). To obtain a reasonable numerical approximation to the solution \( u_\varepsilon \) using \( N \) mesh intervals (where in general \( \varepsilon << N^{-1} \)) it is necessary [1] to use a non-uniform mesh so that a significant proportion of the mesh elements are within the layer regions. One way to achieve this is to use a piecewise uniform Shishkin mesh [1, 7, 12]. Note that if \( a(1)u_\varepsilon(1) = f(1) \) then there are no steep gradients in the solution near the boundary \( x = 1 \) and then a coarse mesh will suffice in the vicinity of this boundary.

For the boundary value problem (4.1) with the additional assumption that

\[ u_\varepsilon(1) = \left( \frac{f}{a} \right)(1), \tag{4.1f} \]

an appropriate Shishkin mesh is defined as follows. The domain \( \Omega = [0, 1] \) is subdivided into the two subintervals

\[ [0, \sigma] \cup [\sigma, 1]. \tag{4.2} \]

On each subinterval a uniform mesh with \( \frac{N}{2} \) mesh-intervals is placed. The interior points of the mesh are denoted by \( \Omega_N^\varepsilon = \{ x_i : 1 \leq i < N \} \) and

\[ \sigma = \min \left\{ \frac{1}{2}, \frac{2\varepsilon}{\beta} \ln N \right\}. \tag{4.3} \]
The fine mesh size $h$ and the coarse mesh size $H$ are given by
\[ h = \frac{2\sigma}{N}, \quad H = \frac{2(1 - \sigma)}{N}. \] (4.4)

On this piecewise-uniform mesh $\Omega_{\varepsilon}^N$, a standard centred finite difference operator is used. Then the fitted mesh method for problem (4.1) is:

Find a mesh function $U_{\varepsilon}^N$ such that
\[ -\varepsilon^2 \delta^2 U_{\varepsilon}^N(x_i) + a(x_i)U_{\varepsilon}^N(x_i) = f(x_i) \quad \text{for all} \quad x_i \in \Omega_{\varepsilon}^N \] (4.5a)
\[ U_{\varepsilon}^N(0) = u_{\varepsilon}(0), \quad U_{\varepsilon}^N(1) = u_{\varepsilon}(1) \] (4.5b)

where $\delta^2$ is the standard centered finite difference operator defined for any mesh function $Z$ by
\[ \delta^2 Z(x_i) = \left( \frac{Z(x_{i+1}) - Z(x_i)}{x_{i+1} - x_i} - \frac{Z(x_i) - Z(x_{i-1})}{x_i - x_{i-1}} \right) \frac{1}{x_{i+1} - x_{i-1}}. \] (4.6)

Let $\bar{U}_{\varepsilon}^N$ denote the piecewise linear interpolant of the numerical solution $U_{\varepsilon}^N$ from the mesh $\Omega_{\varepsilon}^N$ to the domain $[0, 1]$. The following parameter-uniform pointwise error bound follows from the results in [7, 8]
\[ \|u_{\varepsilon} - \bar{U}_{\varepsilon}^N\|_{[0,1], \infty} \leq C(N^{-1} \ln N)^2, \] (4.7)

where $C$ is a constant independent of the singular perturbation parameter $\varepsilon$. Moreover, parameter uniform estimates on the discrete derivatives of the solution can be obtained on these piecewise-uniform Shishkin meshes. The forward difference of a mesh function $Z$ is denoted by $D^+ Z$ and is defined as
\[ D^+ Z(x_i) = \frac{Z(x_{i+1}) - Z(x_i)}{x_{i+1} - x_i}. \] (4.8)

Using the crude bound
\[ |D^+(U_{\varepsilon}^N - u_{\varepsilon})| \leq \frac{1}{(x_{i+1} - x_i)}(\|(U_{\varepsilon}^N - u_{\varepsilon})(x_{i+1})\| + ||(U_{\varepsilon}^N - u_{\varepsilon})(x_i)||) \] (4.9)

and noting that $H \geq CN^{-1}$, $\varepsilon \leq CNh$, we easily deduce that
\[ |D^+(U_{\varepsilon}^N - u_{\varepsilon})(x_i)| \leq \left\{ \begin{array}{ll} CN^{-1}(\ln N)^2, & \text{if } x_i \geq \sigma, \\ CN^{-1}(\ln N)^2/\varepsilon, & \text{if } x_i < \sigma. \end{array} \right. \] (4.10)

On the piecewise-uniform mesh we have the following result.

**Lemma 2.** Let $U_{\varepsilon}^N$ and $u_{\varepsilon}$ be the solutions of (4.5) and (4.1), then
\[ |(D^+ U_{\varepsilon}^N - u_{\varepsilon}')(x_i)| \leq \left\{ \begin{array}{ll} CN^{-1}(\ln N)^2, & \text{if } x_i \geq \tfrac{2\varepsilon \ln 1/\varepsilon}{\beta}, \\ CN^{-1}(\ln N)^2/\varepsilon, & \text{if } x_i < \tfrac{2\varepsilon \ln 1/\varepsilon}{\beta}. \end{array} \right. \] (4.11)

**Proof.** Note that if the mesh is uniform (when $\sigma = 0.5$) then a classical argument suffices to establish the bound
\[ |D^+ u_{\varepsilon} - u_{\varepsilon}'| \leq CN^{-1} \| u_{\varepsilon}' \| \leq CN^{-1} \varepsilon^{-2} \leq CN^{-1}(\ln N)^2. \] (4.12)

Now assume that $\sigma < 0.5$. Consider first the case of $\varepsilon \leq N^{-1}$, which implies that $\sigma \leq \tau$. Then, for $x_i \geq \tau$
\[ |D^+ u_{\varepsilon} - u_{\varepsilon}'| \leq |D^+ v - v'| + C|D^+ w_1 - w_1'| \leq CN^{-1} \|v''\| + CN^{-1} \|w_1''\|_{\tau, 1} \leq CN^{-1}, \]
since $|w_1''(x)| \leq C\varepsilon^{-2} e^{-\beta \tau/\varepsilon} \leq C$, when $x \geq \tau$. 

For \( \sigma \leq x_i < \tau \)
\[
\varepsilon |D^+ u_z - u'_z| \leq \varepsilon |D^+ v - v'| + C\varepsilon |D^+ w_l - w'_l|
\leq CN^{-1} \varepsilon \|v''\| + C\varepsilon \|u''_l\|_{\sigma,1}
\leq CN^{-2},
\]
since \( \varepsilon |w'_l(x)| \leq Ce^{-\beta \sigma/\varepsilon} \leq CN^{-2} \), when \( x \geq \sigma \).

For \( x_i < \sigma \)
\[
\varepsilon |D^+ u_z - u'_z| \leq \varepsilon |D^+ v - v'| + C\varepsilon |D^+ w_l - w'_l|
\leq C\varepsilon h \|v''\| + C\varepsilon h \|u''_l\|
\leq CN^{-1} \ln N,
\]
since \( h \leq C\varepsilon N^{-1} \ln N \).

In the other case \( \varepsilon > N^{-1} \) which implies \( \sigma > \tau \). For \( x_i \geq \tau \)
\[
|D^+ u_z - u'_z| \leq CN^{-1} \|v''\| + CN^{-1} \|u''_l\|_{\tau,1} \leq CN^{-1},
\]
and for \( x_i < \tau < \sigma \)
\[
|D^+ u_z - u'_z| \leq CN^{-1} \|v''\| + C\varepsilon h \|u''_l\| \leq CN^{-1} \ln N.
\]
Combine these bounds with (4.10) to complete the proof. \( \Box \)

Hence on the Shishkin mesh both the solution and the scaled derivative of the solution are uniformly approximated by \( U^N \) and \( D^+ U^N \). In (3.3b) note that the source term is the derivative of the potential. When discretizing the system of equations in (3.3), we wish to generate a uniform approximation to this source term. Hence, we will use a Shishkin mesh in the horizontal direction in our discretization of the domain.

Let us consider the higher order discrete approximation to the derivative defined by
\[
D^0 Z(x_i) = \frac{h_i D^+ Z(x_i) + h_{i+1} D^- Z(x_i)}{h_{i+1} + h_i}, \quad h_i = x_i - x_{i-1}
\]
which has the following truncation error
\[
D^0 u_z(x_i) - u'_z(x_i) = \frac{1}{h_{i+1} + h_i} \int_{x=x_{i-1}}^{x_{i+1}} \int_{t=t_{i-1}}^{t_i} \int_{s=s_{i-1}}^{s_i} u''_z(s) \, ds \, dt \, dx
+ \frac{h_{i+1} - h_i}{h_{i+1} + h_i} \frac{1}{h_i} \int_{x=x_i}^{x_{i+1}} \int_{t=t_{i-1}}^{t_i} \int_{s=s_{i-1}}^{s_i} \int_{\zeta=\zeta_{i-1}}^{\zeta_i} u'''_z(\zeta) \, d\zeta \, ds \, dt \, dx
+ \frac{1}{h_{i+1} + h_i} \int_{x=x_i}^{x_{i+1}} \int_{t=t_{i-1}}^{t_i} \int_{s=s_{i-1}}^{s_i} \int_{\zeta=\zeta_{i-1}}^{\zeta_i} u'''_z(\zeta) \, d\zeta \, ds \, dt \, dx
+ \frac{(h_{i+1} - h_i)(h_i - h_{i-1})}{6} u''_z(x_i).
\]
The solution of problem (4.1) can also be written as the sum of a modified regular component \( \tilde{v}(x) \) and the two singular components \( w_l(x) \), \( w_r(x) \) as
\[
u_z(x) = \tilde{v}(x) + (u_z(0) - \tilde{v}(0)) w_l(x) + (u_z(1) - \tilde{v}(1)) w_r(x)
\]
where
\[
L_z \tilde{v}(x) = f(x), \quad \tilde{v}(0) = \left( \frac{f}{a} \right)(0), \quad \tilde{v}(1) = \left( \frac{f}{a} \right)(1) - e^2 \left( \frac{f}{a} \right)(0)
\]
Then the following parameter explicit bounds can be established for \( 0 \leq k \leq 4 \)
\[
\left\| \frac{d^k \tilde{v}}{dx^k} \right\|_{\Omega,\infty} \leq C(1 + \varepsilon^{4-k}),
\]
(4.17)
where $C$ is a constant that is independent of $\varepsilon$. From these bounds it follows that at all internal mesh points (on an arbitrary mesh)
\[
|D^0\bar{v}(x_i) - \bar{v}'(x_i)| \leq CN^{-2}.
\]  
(4.19)

Consider a two-transition point mesh $\omega_2$ defined as follows:
\[
\omega_2 = [0, \sigma_2] \cup [\sigma_2, \tau_2] \cup [\tau_2, 1].
\]  
(4.20)

On each subdomain, a uniform mesh with $N, N, N, N$ mesh-intervals is used. The transition points are taken to be
\[
\sigma_2 = 2 \min \left\{ \frac{1}{8}, \frac{\varepsilon}{\beta} \ln N, \frac{\varepsilon}{\beta} \ln \frac{1}{\varepsilon} \right\}, \quad \tau_2 = 4 \min \left\{ \frac{1}{8}, \frac{\varepsilon}{\beta} \max \left\{ \ln N, \ln \frac{1}{\varepsilon} \right\} \right\}.
\]  
(4.21)

The three mesh widths involved in this piecewise-uniform mesh will be denoted by $h_1, h_2, H$ and are given by
\[
h_1 = 4 \frac{\sigma_2}{N}, \quad h_2 = 4 \frac{(\tau_2 - \sigma_2)}{N}, \quad H = 2 \frac{(1 - \tau_2)}{N}.
\]  
(4.22)

**Lemma 3.** For the mesh points $x_i \in \omega_2$ we have that
\[
|\langle D^0 w_i - \hat{w}_i \rangle(x_i) \rangle | \leq \begin{cases} 
CN^{-2}(\ln N)^2, & \text{if } x_i > \tau_2, \\
CN^{-2}(\ln N)^2/\varepsilon, & \text{if } x_i \leq \tau_2.
\end{cases}
\]  
(4.23)

**Proof.** The case $\tau_2 = 0.5$ is dealt with in a classical way. Assume that $\tau_2 < 0.5$.

Consider first the case $\varepsilon \leq N^{-1}$, which implies that
\[
\sigma_2 = 2 \frac{\varepsilon}{\beta} \ln N, \quad \tau_2 = 4 \frac{\varepsilon}{\beta} \ln (1/\varepsilon).
\]  
(4.24)

Then, for $x_i > \tau_2$, where the mesh is uniform,
\[
|D^0 w_i - \hat{w}_i | \leq CN^{-2} \|w_i'''\|_{[\tau_2, 1]} \leq CN^{-2} \varepsilon,
\]  
(4.25)

since $|w_i'''(x)| \leq C \varepsilon^{-3} e^{-\beta x_i/\varepsilon} \leq C \varepsilon$, when $x_i \geq \tau_2$.

For $\sigma_2 \leq x_i \leq \tau_2$,
\[
\varepsilon |D^0 w_i - \hat{w}_i | \leq C \varepsilon \|w_i'''\|_{[\sigma - h_1, 1]} \leq CN^{-2},
\]  
(4.26)

since $\varepsilon |w_i'(x)| \leq C \varepsilon^{-3} \varepsilon^{-1} \leq CN^{-2}$, when $x_i \geq \sigma$ and $e^{\beta h_1/\varepsilon} \leq C$.

For $x_i < \sigma_2$, where the mesh is fine and uniform,
\[
\varepsilon |D^0 w_i - \hat{w}_i | \leq C \varepsilon h_1^2 \|w_i'''\|_{[0, \sigma_2]} \leq C (N^{-1} \ln N)^2,
\]  
(4.27)

since $h_1 \leq C \varepsilon N^{-1} \ln N$.

In the second case $\varepsilon > N^{-1}$, which implies
\[
\sigma_2 = 2 \frac{\varepsilon}{\beta} \ln \frac{1}{\varepsilon}, \quad \tau_2 = 4 \frac{\varepsilon}{\beta} \ln N.
\]  
(4.28)

In the uniform mesh regions we have the following: For $x_i < \sigma_2$
\[
\varepsilon |D^0 w_i - \hat{w}_i | \leq C \varepsilon h_1^2 \|w_i'''\| \leq CN^{-2} (\ln \frac{1}{\varepsilon})^2 \leq C (N^{-1} \ln N)^2,
\]  
(4.29)

for $x_i > \tau_2$
\[
|D^0 w_i - \hat{w}_i | \leq CN^{-2} \|w_i'''\|_{[\tau_2, 1]} \leq CN^{-2} \varepsilon^{-3} \varepsilon^{-4} \leq CN^{-3},
\]  
(4.30)

and for $\sigma_2 < x_i < \tau_2$
\[
\varepsilon |D^0 w_i - \hat{w}_i | \leq C \varepsilon h_2^2 \|w_i'''\|_{[\sigma_2, 1]} \leq C (N^{-1} \ln N)^2.
\]  
(4.31)

At the second transition point $x_i = \tau_2$
\[
\varepsilon |D^0 w_i - \hat{w}_i | \leq C \varepsilon \|w_i\|_{[\tau_2 - h_2, 1]} \leq CN^{-2},
\]  
(4.32)

and, finally, for $x_i = \sigma_2$ we have that
\[
\varepsilon |D^0 w_i - \hat{w}_i | \leq C \varepsilon^3 (N^{-1} \ln N)^2 \|w_i'''\|_{[\sigma_2 - h_2, 1]} \leq C (N^{-1} \ln N)^2.
\]  
(4.33)
Thus the scaled discrete derivative incorporating $D^+$ has a first order truncation error, while the scaled discrete derivative incorporating $D^0$ has a second order truncation error. This motivates the choice of $D^0$ over $D^+$ in the discretization of the term $\partial\phi/\partial x$ in (3.3b).

5. Nonlinear ordinary differential equation

Let us now consider the nonlinear ordinary differential equation

\[-\varepsilon^2 y'' + e^y = n(x), \quad x \in \Omega = (0, 1),\]
\[y(0) = -A, \quad y(1) = 0,\]
\[e^{-A} < n(0) \leq n(x) \leq 1, \quad n(1) = 1.\]  

(5.1a)

Note that the Bernoulli function

\[b(y) = e^y - 1/y, \quad y \neq 0; \quad b(0) = 1\]  

(5.2)

satisfies

\[\frac{\partial b}{\partial y} > 0, \quad \forall y\]  

(5.3)

\[0 < \beta^2_1 = \frac{1 - e^{-A}}{A} \leq b(y) \leq b(0) = 1, \quad \frac{d}{dy}(b(y)y) \geq e^{-A} > 0, \quad y \in [-A, 0].\]  

(5.4)

Reformulate problem (5.1) into the form

\[-\varepsilon^2 y'' + e^y b(y) y = n(x) - 1 = f(x), \quad x \in \Omega = (0, 1),\]
\[f(1) = 0, \quad b(y) \geq \beta^2_1 > 0, \quad \forall y \in [y(0), y(1)]\]  

(5.5a)

\[y(0) = -A, \quad y(1) = 0 = f(1)\]  

(5.5b)

(5.5c)

Following the arguments in [2], we can establish the existence of a solution $y_\varepsilon$ to this nonlinear problem, which satisfies the inequality

\[y_\varepsilon(0) \leq y_\varepsilon(x) \leq 0;\]  

(5.6)

using (5.1c) and the lower and upper solutions of $y_\varepsilon(0)$ and 0, respectively. Define the reduced solution as $v_0(x) = \ln n(x)$, which is the solution of $e^{v_0} = n(x), \quad x \in [0, 1]$. The regular component $v_\varepsilon$ of any solution of (5.5) is defined to be the solution

\[-\varepsilon^2 v''_\varepsilon + b(v_\varepsilon) v_\varepsilon = n(x) - 1, \quad x \in (0, 1), \quad v_\varepsilon(0) = v_0(0) > y_\varepsilon(0), \quad v_\varepsilon(1) = 0.\]  

(5.7)

Note that, using (5.1c),

\[v_\varepsilon(0) \leq v_\varepsilon(x) \leq 0;\]  

(5.8)

and as in the case of [2], the derivatives of the regular component satisfy the bounds

\[\left\|v^{(k)}_\varepsilon\right\| \leq C(1 + \varepsilon^{2-k}), \quad k \leq 4.\]  

(5.9)

The singular component $w_\varepsilon = y_\varepsilon - v_\varepsilon$ is the solution of the problem

\[-\varepsilon^2 w''_\varepsilon + e^{v_\varepsilon} b(w_\varepsilon) w_\varepsilon = 0, \quad x \in (0, 1), \quad w_\varepsilon = y_\varepsilon(0) - v_\varepsilon(0) < 0, \quad w_\varepsilon(1) = 0.\]  

(5.10)

Note that

\[(y_\varepsilon(0) - v_\varepsilon(0))(1 - x) \leq w_\varepsilon(x) \leq 0.\]  

(5.11)

Consider the barrier function

\[\alpha(x) = (y_\varepsilon(0) - v_\varepsilon(0)) e^{-\gamma x/\varepsilon} < 0.\]  

(5.12)
Note that this is a lower solution if
\[ \gamma = \sqrt{\frac{e^{y_3(0)} - e^{y_3(0)}'}{y_e(0) - y_e(0)'} = \sqrt{\frac{n(0) - e^{y_2(0)}}{\ln n(0) - y_e(0)}}. \]  
(5.13)
This can be seen by observing that
\[ -\varepsilon^2 \alpha'' + e^{y_3} b(\alpha) = (y_e(0) - y_e(0)) e^{-\gamma x/\varepsilon} (e^{y_3} b(\alpha) - \gamma^2) \leq 0 \]  
(5.14)
as
\[ e^{y_3} b(\alpha) - \gamma^2 \geq e^{y_3(0)} b(y_e(0) - y_e(0)) - \gamma^2 = 0. \]  
(5.15)
Hence,
\[ |w_2(x)| \leq C e^{-\gamma x/\varepsilon} \]  
(5.16)
and using the differential equation (5.10) we can establish the derivative bounds
\[ |w_2^{(k)}(x)| \leq C \varepsilon^{-k} e^{-\gamma x/\varepsilon}, \quad k = 0, 1, 2 \]
\[ |w_2^{(k)}(x)| \leq C \varepsilon^{2-k} + C \varepsilon^{-k} e^{-\gamma x/\varepsilon}, \quad k = 3, 4. \]
We note that \( \gamma \leq \beta_1 \) and as \( n(0) \to 0, \gamma^2 \to e^{y_3(0)}. \)

Compare the nonlinear problem (5.5) to the linear problem (4.1). Motivated by the linear problem (4.1), we propose the following nonlinear numerical method for problem (5.5). The domain \([0, 1]\) is split into \([0, \sigma_3] \cup [\sigma_3, 1]\) and a uniform mesh is constructed on each of these subintervals. The numerical method is then: Find a mesh function \( Y_e^N \) such that
\[ -\varepsilon^2 \delta^2 Y_e^N(x_i) + b(Y_e^N) Y_e^N(x_i) = n(x_i) - 1 \quad \text{for all} \quad x_i \in \Omega_3^N \]  
(5.17a)
\[ \begin{align*} 
Y_e^N(0) &= y_e(0), & Y_e^N(1) &= y_e(1) 
\end{align*} \]  
(5.17b)
where the transition point in the piecewise uniform mesh \( \Omega_3^N \) is taken to be
\[ \sigma_3 = \min \left\{ \frac{1}{2}, \frac{2}{\gamma} \ln N \right\}. \]  
(5.18)
Using discrete lower and upper solutions and following the arguments in [2], we can establish the existence of a discrete solution \( Y_e^N \) to (5.17), which satisfies the inequality
\[ y_e(0) \leq Y_e^N(x_i) \leq 0. \]  
(5.19)

**Lemma 4.** Assume that \( N \) is sufficiently large (independent of \( \varepsilon \)). For all \( x_i \in \Omega_3^N \) we have that
\[ |(Y_e^N - y_e)(x_i)| \leq C \varepsilon^A N^{-1} (\varepsilon + N^{-1} (\ln N)^2) \]  
(5.20)
where \( Y_e^N \) is a solution of (5.17) and \( y_e \) is a solution of (5.1).

**Proof.** From the a priori bounds on the derivatives of the components, we can establish (in the case of \( \sigma_3 < 0.5 \)) the truncation error bounds
\[ |\varepsilon^2 (u''_e - \delta^2 u_e)(x_i)| \leq \begin{cases} 
C N^{-2}, & \text{if } x_i \neq \sigma_3, \\
C \varepsilon N^{-1}, & \text{if } x_i = \sigma_3.
\end{cases} \]
\[ |\varepsilon^2 (w''_e - \delta^2 w_e)(x_i)| \leq \begin{cases} 
C(N^{-1} \ln N)^2, & \text{if } x_i < \sigma_3, \\
C N^{-2}, & \text{if } x_i \geq \sigma_3.
\end{cases} \]
In the case of \( \sigma_3 = 0.5 \) the mesh is uniform and \( |\varepsilon^2 (y''_e - \delta^2 y_e)(x_i)| \leq C(N^{-1} \ln N)^2. \)

Hence,
\[ T = ||\varepsilon^2 (y''_e - \delta^2 y_e)|| \leq CN^{-1} (\varepsilon + N^{-1} (\ln N)^2). \]  
(5.21)
Now,
\[ -\varepsilon^2 \delta^2 Y_e^N + e^N = -\varepsilon^2 y''_e + e^N \]  
(5.22)
which implies that
\[-\varepsilon^2(\dot{Y}_N^N - y_e) + e^\nu b(Y_N^N - y_e)(Y_N^N - y_e) = -\varepsilon^2(\dot{y}_e'' - \delta^2 y_e)(x_i).\] (5.23)

Use the constant upper and lower discrete solutions of
\[A(x_i) = -2Te^A \quad \text{and} \quad B(x_i) = Te^A\] (5.24)
and the facts that \(e^{-A} \leq e^\nu \leq 1\), \(b(Te^A) \geq b(0)\) and for \(N\) sufficiently large \(b(-2Te^A) \geq 0.5\) to complete the proof. \(\square\)

**Remark 5.** Note the difference in the definitions of the parameters \(\sigma_3\) and \(\sigma\), defined respectively in (5.18) and (4.3). In the next section, for the sake of simplicity, we choose a transition parameter related to (4.3), which is independent of the variable \(n\), in the numerical algorithm for the coupled system (3.3).

The above argument motivates the use of a piecewise-uniform in the horizontal variable \(x\), when discretizing the potential equation (3.3c). Finally, we will motivate a piecewise-uniform mesh in the vertical direction \(y\), when discretizing the velocity equation (3.3b).

Consider the following system of two equations for a given function \(n(x)\)
\[
\frac{\partial u}{\partial y} + u \frac{\partial u}{\partial x} = -\frac{\partial \phi}{\partial x},
\]
\[
\varepsilon^2 \frac{\partial^2 \phi}{\partial x^2} = \phi - n(x).
\]

Note that
\[
\frac{\partial \phi}{\partial x} = O(\varepsilon^{-1}), \quad x < C\varepsilon.
\] (5.25)

Hence, the forcing term in the first order equation
\[
\frac{\partial u}{\partial y} + u \frac{\partial u}{\partial x} = -\frac{\partial \phi}{\partial x}, 0 \leq x < \varepsilon,
\] (5.26)
is unbounded with respect to \(\varepsilon\). Introduce the stretched variables
\[
\eta = \frac{x}{\varepsilon}, \quad \zeta = \frac{y}{\varepsilon}
\] (5.27)
and the transformed equation for \(\hat{u}(\eta, \zeta) = u(x, y)\) is
\[
\frac{\partial \hat{u}}{\partial \eta} + \hat{u} \frac{\partial \hat{u}}{\partial \zeta} = -\frac{\partial \hat{\phi}}{\partial \eta}, 0 < \eta < C, 0 < \zeta < C
\] (5.28)
where the forcing term is now of order one in the corner area \((\eta, \zeta) \in [0, 1] \times [0, 1]\).

Note also that if \(u(x, 0) = u_0\) then from (3.3b) we have that
\[
\frac{\partial u}{\partial y}(x, 0^+) = -\frac{\partial \phi}{\partial x}(x, 0^+).
\] (5.29)
The above observations motivate the use of a piecewise-uniform mesh in the vertical direction.

6. **Numerical Method in the case of \(u_0 < 0\).**

Note that the ions are drawn to the probe by the electric field (3.3b). So if \(u(x, 0) < 0\) then \(u(x, y) < 0\) for all \(y \geq 0\). Thus there will be no points where \(u(x, y) = 0\). This allows us to use a simple form of upwinding in our discretization.

Due to the presence of the singular perturbation parameter in (3.3c), we employ a piecewise-uniform mesh in the horizontal direction. Due to the unbounded (with respect to \(\varepsilon\)) source term in (3.3b) we will use a piecewise-uniform mesh in the
vertical direction. The domain $\Omega = \Omega_x \times \Omega_y$ is discretized by the tensor product mesh $\Omega^{N,M}$ where
\[
\Omega^{N,M} = \Omega_N^x \times \Omega_M^y.
\] (6.1)
The domains $\Omega_x, \Omega_y$ are composed of the subdomains
\[
\Omega_x = [0, \sigma_x] \cup [\sigma_x, \tau_x] \cup [\tau_x, 1]; \quad \Omega_y = [0, \sigma_y] \cup [\sigma_y, \tau_y] \cup [\tau_y, 1].
\] (6.2)
On each subdomain in the horizontal direction, a uniform mesh with $N_{x4}, N_{x4}, N_{x2}$ mesh-intervals is used in the respective subdomains. The transition points in the horizontal and vertical direction are taken to be
\[
\sigma_x = \min \left\{ \frac{1}{4}, \frac{4\varepsilon}{\beta_1} \min \left\{ \ln N, \ln \left( \frac{1}{\varepsilon} \right) \right\} \right\}, \quad \tau_x = \min \left\{ \frac{1}{2}, \frac{8\varepsilon}{\beta_1} \max \left\{ \ln N, \ln \left( \frac{1}{\varepsilon} \right) \right\} \right\},
\] (6.3a)
\[
\sigma_y = \min \left\{ \frac{1}{4}, \frac{4\varepsilon}{\beta_2} \min \left\{ \ln N, \ln \left( \frac{1}{\varepsilon} \right) \right\} \right\}, \quad \tau_y = \min \left\{ \frac{1}{2}, \frac{8\varepsilon}{\beta_2} \max \left\{ \ln N, \ln \left( \frac{1}{\varepsilon} \right) \right\} \right\},
\] (6.3b)
where
\[
\beta_2^2 = \frac{1 - e^{-A}}{A}.
\] (6.3e)
Initially the vertical mesh step is set at
\[
k_j = y_j - y_{j-1} = \begin{cases} 
\frac{4\sigma_y N^{-1}}{4(\tau_y - \sigma_y) N^{-1}}, & j \leq N/4, \\
2(1 - \tau_y) N^{-1}, & j > N/2.
\end{cases}
\] (6.4)
The system of differential equations (3.3a,b,c) is discretized using a standard upwind finite difference operator on this piecewise uniform mesh. When solving the nonlinear difference scheme, the vertical mesh step will sometimes be reduced in size. The resulting nonlinear finite difference method is linearized using the iterative algorithm given below.

Set the initial approximation for the density to be constant throughout the domain
\[
\mathcal{N}^0(x, y) = 1, \quad (x, y) \in [0, 1] \times [0, T].
\] (6.5a)
and determine an approximation $\Phi(x_i, 0)$ to the initial potential $\phi(x_i, 0)$ by smoothing out a linear initial condition using $N$ iterations of the sequence $\Phi_j$ generated from
\[
\Phi_0(x_i) = -A + Ax_i, \quad 0 \leq x_i \leq 1,
\] (6.5b)
\[
\varepsilon^2 \frac{\partial^2}{\partial x^2} \Phi_j(x_i) - \left( \frac{e^{\Phi_{j-1}(x_i) - 1}}{\Phi_{j-1}(x_i)} \right) \Phi_j(x_i) = 0, \quad 0 < x_i < 1,
\] (6.5c)
\[
\Phi_j(0) = -A, \quad \Phi_j(1) = 0.
\] (6.5d)
Hence, our initial conditions are
\[
\Phi(x_i, 0) = \Phi_N(x_i), \quad U(x_i, 0) = u_0 < 0, \quad \mathcal{N}(x_i, 0) = 1, \quad 0 \leq x_i \leq 1.
\] (6.5e)
At each subsequent vertical mesh level $y = y_j > 0$, an approximation
\[
(\Phi(x_i, y_j), U(x_i, y_j), \mathcal{N}(x_i, y_j))
\] is generated from a sequence of approximations
In the next section, we examine convergence properties of the related scaled ion current density

\[ J = -U\mathcal{N}. \]
Table 1. Computed orders for the ion current density $J$, when there is an applied potential corresponding to $\phi_0(0) = -10.0$ ($\phi_0(0) = -50.0$) and the initial supersonic velocity of $u(x, 0) = -2.0$ is towards the probe.

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>$N$</th>
<th>$2^{-4}$</th>
<th>$2^{-5}$</th>
<th>$2^{-6}$</th>
<th>$2^{-7}$</th>
<th>$2^{-8}$</th>
<th>$2^{-9}$</th>
<th>$2^{-10}$</th>
<th>$2^{-11}$</th>
<th>$2^{-12}$</th>
<th>$p^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>8</td>
<td>0.68(1.74)</td>
<td>0.69(1.02)</td>
<td>0.74(0.91)</td>
<td>-0.35(1.06)</td>
<td>-0.10(-0.18)</td>
<td>0.07(-0.10)</td>
<td>0.04(-0.04)</td>
<td>-0.06(-0.07)</td>
<td>0.54(0.40)</td>
<td>0.60(0.86)</td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>0.71(0.94)</td>
<td>0.72(0.56)</td>
<td>0.79(0.89)</td>
<td>0.69(0.96)</td>
<td>0.16(0.00)</td>
<td>0.19(0.14)</td>
<td>0.18(0.14)</td>
<td>0.54(0.40)</td>
<td>0.60(0.86)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>32</td>
<td>0.50(0.49)</td>
<td>0.84(0.33)</td>
<td>0.83(0.69)</td>
<td>0.82(0.89)</td>
<td>0.60(1.00)</td>
<td>0.60(1.04)</td>
<td>0.60(1.04)</td>
<td>0.60(1.04)</td>
<td>0.60(1.04)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>64</td>
<td>0.67(0.81)</td>
<td>0.80(0.72)</td>
<td>0.88(0.76)</td>
<td>0.92(0.83)</td>
<td>0.43(1.01)</td>
<td>0.58(0.71)</td>
<td>0.58(0.55)</td>
<td>0.58(0.55)</td>
<td>0.58(0.55)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>128</td>
<td>0.83(0.92)</td>
<td>0.82(0.90)</td>
<td>0.86(0.78)</td>
<td>0.94(0.80)</td>
<td>0.77(1.06)</td>
<td>0.77(0.86)</td>
<td>0.77(0.86)</td>
<td>0.77(0.86)</td>
<td>0.77(0.86)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>256</td>
<td>0.92(0.96)</td>
<td>0.88(0.95)</td>
<td>0.89(0.89)</td>
<td>0.89(0.79)</td>
<td>0.86(0.88)</td>
<td>0.86(0.90)</td>
<td>0.86(0.90)</td>
<td>0.86(0.90)</td>
<td>0.86(0.90)</td>
<td></td>
</tr>
</tbody>
</table>

7. Numerical results in the case of $u_0 < 0$.

We assume that, on the appropriate meshes $\Omega^N$, the piecewise linear interpolants $\bar{Z}^N$ of the numerical solutions $Z^N$ have been determined. We compute the parameter-uniform maximum pointwise two-mesh differences [1]

$$D^N = \max_{\varepsilon \in R^*} D_\varepsilon^N,$$

where $D_\varepsilon^N = ||Z^N - \bar{Z}^{2N}||_{\Omega^N, \infty}$. (7.1)

Approximations to the parameter-uniform order of convergence are computed using

$$p^* = \min_{N \in R_{\infty}} p_N^*, \text{ where } p_N^* = \log_2 \frac{D^N}{D^{2N}}.$$ (7.2)

In Figure 1 the currents along the probe for supersonic and subsonic flow are compared. In the Tables 1 and 2, computed orders of convergence for the computed

ion current density $J$ are given. These results suggest that the numerical algorithm presented in this paper is a parameter-uniform numerical method for the ion current density. In Tables 3 and 4 the number of time steps required for the algorithm to reach the final time level $T = 1$ are displayed. Note that in the case of supersonic flow ($u_0 < -1$), the number of time-steps appear to be independent of $\varepsilon$. However, in the case of subsonic flow ($-1 < u_0 < 0$), the number of time-steps increase as $\varepsilon \to 0$. 

![Figure 1](image-url)
A NUMERICAL METHOD FOR A PLASMA-SHEATH MODELLING PROBLEM

2. \( \epsilon \)
3. \( \eta \)
4. \( \alpha \)
5. \( \beta \)
6. \( \gamma \)
7. \( \delta \)
8. \( \zeta \)
9. \( \eta \)
10. \( \theta \)
11. \( \iota \)
12. \( \kappa \)
13. \( \lambda \)
14. \( \mu \)
15. \( \nu \)
16. \( \xi \)
17. \( \omicron \)
18. \( \pi \)
19. \( \rho \)
20. \( \sigma \)
21. \( \tau \)
22. \( \upsilon \)
23. \( \phi \)
24. \( \chi \)
25. \( \psi \)
26. \( \omega \)
27. \( \Gamma \)
28. \( \Delta \)
29. \( \Theta \)
30. \( \Lambda \)
31. \( \Xi \)
32. \( \Pi \)
33. \( \Sigma \)
34. \( \Upsilon \)
35. \( \Phi \)
36. \( \Chi \)
37. \( \Psi \)
38. \( \Omega \)
39. \( \alpha \)
40. \( \beta \)
41. \( \gamma \)
42. \( \delta \)
43. \( \epsilon \)
44. \( \zeta \)
45. \( \eta \)
46. \( \theta \)
47. \( \iota \)
48. \( \kappa \)
49. \( \lambda \)
50. \( \mu \)
51. \( \nu \)
52. \( \xi \)
53. \( \omicron \)
54. \( \pi \)
55. \( \rho \)
56. \( \sigma \)
57. \( \tau \)
58. \( \upsilon \)
59. \( \phi \)
60. \( \chi \)
61. \( \psi \)
62. \( \omega \)
63. \( \Gamma \)
64. \( \Delta \)
65. \( \Theta \)
66. \( \Lambda \)
67. \( \Xi \)
68. \( \Pi \)
69. \( \Sigma \)
70. \( \Upsilon \)
71. \( \Phi \)
72. \( \Chi \)
73. \( \Psi \)
74. \( \Omega \)
75. \( \alpha \)
76. \( \beta \)
77. \( \gamma \)
78. \( \delta \)
79. \( \epsilon \)
80. \( \zeta \)
81. \( \eta \)
82. \( \theta \)
83. \( \iota \)
84. \( \kappa \)
85. \( \lambda \)
86. \( \mu \)
87. \( \nu \)
88. \( \xi \)
89. \( \omicron \)
90. \( \pi \)
91. \( \rho \)
92. \( \sigma \)
93. \( \tau \)
94. \( \upsilon \)
95. \( \phi \)
96. \( \chi \)
97. \( \psi \)
98. \( \omega \)
99. \( \Gamma \)
100. \( \Delta \)
101. \( \Theta \)
102. \( \Lambda \)
103. \( \Xi \)
104. \( \Pi \)
105. \( \Sigma \)
106. \( \Upsilon \)
107. \( \Phi \)
108. \( \Chi \)
109. \( \Psi \)
110. \( \Omega \)
111. \( \alpha \)
112. \( \beta \)
113. \( \gamma \)
114. \( \delta \)
115. \( \epsilon \)
116. \( \zeta \)
117. \( \eta \)
118. \( \theta \)
119. \( \iota \)
120. \( \kappa \)
121. \( \lambda \)
122. \( \mu \)
123. \( \nu \)
124. \( \xi \)
125. \( \omicron \)
126. \( \pi \)
127. \( \rho \)
128. \( \sigma \)
129. \( \tau \)
130. \( \upsilon \)
131. \( \phi \)
132. \( \chi \)
133. \( \psi \)
134. \( \omega \)
135. \( \Gamma \)
136. \( \Delta \)
137. \( \Theta \)
138. \( \Lambda \)
139. \( \Xi \)
140. \( \Pi \)
141. \( \Sigma \)
142. \( \Upsilon \)
143. \( \Phi \)
144. \( \Chi \)
145. \( \Psi \)
146. \( \Omega \)
147. \( \alpha \)
148. \( \beta \)
149. \( \gamma \)
150. \( \delta \)
151. \( \epsilon \)
152. \( \zeta \)
153. \( \eta \)
154. \( \theta \)
155. \( \iota \)
156. \( \kappa \)
157. \( \lambda \)
158. \( \mu \)
159. \( \nu \)
160. \( \xi \)
161. \( \omicron \)
162. \( \pi \)
163. \( \rho \)
164. \( \sigma \)
165. \( \tau \)
166. \( \upsilon \)
167. \( \phi \)
168. \( \chi \)
169. \( \psi \)
170. \( \omega \)
171. \( \Gamma \)
172. \( \Delta \)
173. \( \Theta \)
174. \( \Lambda \)
175. \( \Xi \)
176. \( \Pi \)
177. \( \Sigma \)
178. \( \Upsilon \)
179. \( \Phi \)
180. \( \Chi \)
181. \( \Psi \)
182. \( \Omega \)

Table 2. Computed orders for the ion current density \( J \), when there is an applied potential corresponding to \( \phi_0(0) = -10.0 \) \( (\phi_0(0) = -50.0) \) and the initial subsonic velocity of \( u(x, 0) = -0.5 \) is towards the probe.

Table 3. Iteration counts when there is an applied potential corresponding to \( \phi_0(0) = -10.0 \) and an initial velocity of \( u(x, 0) = -2.0 \) \( (u(x, 0) = -0.5) \) towards the probe.

Table 4. Iteration counts when there is an applied potential corresponding to \( \phi_0(0) = -50.0 \) and an initial velocity of \( u(x, 0) = -2.0 \) \( (u(x, 0) = -0.5) \) towards the probe.

8. FURTHER NUMERICAL RESULTS

In the Tables 5–16, numerical orders of convergence are given for the individual variables \( \Phi, U, \mathcal{N} \) for \( \epsilon \in R_\epsilon = 2^{-4}, \ldots, 2^{-12} \) and \( N \in R_N = 2^3, \ldots, 2^6 \) in the four cases of \( \lambda = 10, w_0 = -2; \ A = 50, u_0 = -2; \ A = 10, u_0 = -0.5; \ A = 50, u_0 = -0.5. \) For sufficiently large \( N \), Tables 5–10 suggest that the method is parameter-uniform for the triple \( (\phi, u, n) \) in the case of supersonic flow. However, in the case
of subsonic flow, Tables 14–16 indicate a possible lack of uniform convergence in approximating the triple \( (\phi, u, n) \), when \( A \) is relatively large. In particular, observe the decreasing orders in one diagonal in Table 14. These results suggest that the algorithm presented here is robust (with respect to \( \varepsilon \)) in the case of supersonic flow, but that some improvement is required in the case of subsonic flow before a similar claim can be made.
In the Figures 2–5, we see that in the case of subsonic flow the sheath widens as one moves away from the tip of the probe. This is most noticeable in the case of the density.
Table 11. Computed orders for the computed scaled potential $\Phi$, when there is an applied potential corresponding to $\phi_0(0) = -10.0$ and $u(x, 0) = -0.5$ is the initial scaled velocity towards the probe.

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
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</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>8</td>
</tr>
<tr>
<td>$2^{-4}$</td>
<td>1.03</td>
</tr>
<tr>
<td>$2^{-5}$</td>
<td>1.34</td>
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<td>$2^{-6}$</td>
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</tr>
<tr>
<td>$2^{-7}$</td>
<td>-1.36</td>
</tr>
<tr>
<td>$2^{-8}$</td>
<td>0.60</td>
</tr>
<tr>
<td>$2^{-9}$</td>
<td>0.03</td>
</tr>
<tr>
<td>$2^{-10}$</td>
<td>0.09</td>
</tr>
<tr>
<td>$2^{-11}$</td>
<td>-0.26</td>
</tr>
<tr>
<td>$2^{-12}$</td>
<td>0.35</td>
</tr>
<tr>
<td>$p^N$</td>
<td>0.56</td>
</tr>
</tbody>
</table>

Table 12. Computed orders for the computed scaled velocity $U$, when there is an applied potential corresponding to $\phi_0(0) = -10.0$ and $u(x, 0) = -0.5$ is the initial scaled velocity towards the probe.

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>Number of Intervals $N$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>8</td>
</tr>
<tr>
<td>$2^{-4}$</td>
<td>1.25</td>
</tr>
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</tr>
<tr>
<td>$2^{-6}$</td>
<td>1.36</td>
</tr>
<tr>
<td>$2^{-7}$</td>
<td>-0.11</td>
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<tr>
<td>$2^{-8}$</td>
<td>0.76</td>
</tr>
<tr>
<td>$2^{-9}$</td>
<td>1.15</td>
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<td>1.47</td>
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<tr>
<td>$2^{-11}$</td>
<td>1.66</td>
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<tr>
<td>$2^{-12}$</td>
<td>1.73</td>
</tr>
<tr>
<td>$p^N$</td>
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</table>

Table 13. Computed orders for the computed scaled density $N$, when there is an applied potential corresponding to $\phi_0(0) = -10.0$ and $u(x, 0) = -0.5$ is the initial scaled velocity towards the probe.

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>Number of Intervals $N$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>8</td>
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<tr>
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<tr>
<td>$2^{-5}$</td>
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<tr>
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<td>-0.44</td>
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<tr>
<td>$2^{-8}$</td>
<td>-0.29</td>
</tr>
<tr>
<td>$2^{-9}$</td>
<td>-0.13</td>
</tr>
<tr>
<td>$2^{-10}$</td>
<td>0.06</td>
</tr>
<tr>
<td>$2^{-11}$</td>
<td>0.37</td>
</tr>
<tr>
<td>$2^{-12}$</td>
<td>0.41</td>
</tr>
<tr>
<td>$p^N$</td>
<td>-0.01</td>
</tr>
</tbody>
</table>
Table 14. Computed orders for the computed scaled potential $\Phi$, when there is an applied potential corresponding to $\Phi_0(0) = -50.0$ and $u(x,0) = -0.5$ is the initial scaled velocity towards the probe.

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>Number of Intervals N</th>
<th>8</th>
<th>16</th>
<th>32</th>
<th>64</th>
<th>128</th>
<th>256</th>
</tr>
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<tr>
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<td>0.88</td>
<td>1.04</td>
<td>0.97</td>
<td>0.96</td>
<td></td>
</tr>
<tr>
<td>$2^{-5}$</td>
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<td>0.81</td>
<td>0.93</td>
<td>0.94</td>
<td>1.07</td>
<td>0.71</td>
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<tr>
<td>$2^{-6}$</td>
<td>1.77</td>
<td>0.81</td>
<td>0.63</td>
<td>0.92</td>
<td>0.71</td>
<td>0.86</td>
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</tr>
<tr>
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<td>0.48</td>
<td>0.87</td>
<td>0.60</td>
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</tr>
<tr>
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<td>-0.17</td>
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<td>0.66</td>
<td>0.42</td>
<td>0.84</td>
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</tr>
<tr>
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<td>-0.30</td>
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<td>1.78</td>
<td>0.64</td>
<td>0.38</td>
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</tr>
<tr>
<td>$2^{-10}$</td>
<td>-3.30</td>
<td>-0.14</td>
<td>0.31</td>
<td>1.35</td>
<td>1.16</td>
<td>1.31</td>
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</tr>
<tr>
<td>$2^{-11}$</td>
<td>-3.36</td>
<td>-0.03</td>
<td>0.33</td>
<td>1.35</td>
<td>1.29</td>
<td>1.18</td>
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</tr>
<tr>
<td>$2^{-12}$</td>
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<td>1.35</td>
<td>1.39</td>
<td>1.08</td>
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</tr>
<tr>
<td>$p^N$</td>
<td>0.56</td>
<td>0.18</td>
<td>0.33</td>
<td>1.36</td>
<td>1.06</td>
<td>0.38</td>
<td></td>
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Table 15. Computed orders for the computed scaled velocity $U$, when there is an applied potential corresponding to $\Phi_0(0) = -50.0$ and $u(x,0) = -0.5$ is the initial scaled velocity towards the probe.

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>Number of Intervals N</th>
<th>8</th>
<th>16</th>
<th>32</th>
<th>64</th>
<th>128</th>
<th>256</th>
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<tbody>
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<td>0.96</td>
<td>0.87</td>
<td>0.93</td>
<td>0.97</td>
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</tr>
<tr>
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<td>1.12</td>
<td>0.61</td>
<td>0.92</td>
<td>1.05</td>
<td>1.04</td>
<td>0.86</td>
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</tr>
<tr>
<td>$2^{-6}$</td>
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<td>0.30</td>
<td>0.83</td>
<td>1.02</td>
<td>1.00</td>
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</tr>
<tr>
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<td>1.12</td>
<td>0.07</td>
<td>0.77</td>
<td>0.98</td>
<td></td>
</tr>
<tr>
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<td>1.77</td>
<td>1.04</td>
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<td>0.55</td>
<td>0.71</td>
<td></td>
</tr>
<tr>
<td>$2^{-9}$</td>
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<td>0.32</td>
<td>1.35</td>
<td>0.65</td>
<td>0.98</td>
<td>1.00</td>
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</tr>
<tr>
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<td>0.71</td>
<td>0.83</td>
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</tr>
<tr>
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<td>0.71</td>
<td>0.83</td>
<td></td>
</tr>
<tr>
<td>$2^{-12}$</td>
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<td>1.50</td>
<td>0.64</td>
<td>0.70</td>
<td>0.83</td>
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</tr>
<tr>
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<td>0.82</td>
<td>1.10</td>
<td>0.38</td>
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</table>

Table 16. Computed orders for the computed scaled density $N$, when there is an applied potential corresponding to $\Phi_0(0) = -50.0$ and $u(x,0) = -0.5$ is the initial scaled velocity towards the probe.

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>Number of Intervals N</th>
<th>8</th>
<th>16</th>
<th>32</th>
<th>64</th>
<th>128</th>
<th>256</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^{-4}$</td>
<td>1.23</td>
<td>0.73</td>
<td>0.35</td>
<td>0.74</td>
<td>0.88</td>
<td>0.94</td>
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<tr>
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<td>0.50</td>
<td>0.60</td>
<td>0.77</td>
<td>0.95</td>
<td>0.91</td>
<td>0.80</td>
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<tr>
<td>$2^{-6}$</td>
<td>0.70</td>
<td>0.48</td>
<td>0.02</td>
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<td>0.93</td>
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</tr>
<tr>
<td>$2^{-8}$</td>
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<td>1.36</td>
<td>0.70</td>
<td>1.00</td>
<td>0.59</td>
<td>0.57</td>
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</tr>
<tr>
<td>$2^{-9}$</td>
<td>-0.37</td>
<td>0.43</td>
<td>1.22</td>
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<td>0.88</td>
<td>0.94</td>
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<tr>
<td>$2^{-10}$</td>
<td>-0.31</td>
<td>0.43</td>
<td>1.25</td>
<td>0.55</td>
<td>0.58</td>
<td>0.71</td>
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</tr>
<tr>
<td>$2^{-11}$</td>
<td>-0.18</td>
<td>0.41</td>
<td>1.27</td>
<td>0.53</td>
<td>0.57</td>
<td>0.70</td>
<td></td>
</tr>
<tr>
<td>$2^{-12}$</td>
<td>-0.10</td>
<td>0.38</td>
<td>1.28</td>
<td>0.52</td>
<td>0.57</td>
<td>0.70</td>
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</tr>
<tr>
<td>$p^N$</td>
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<td>0.42</td>
<td>1.09</td>
<td>0.82</td>
<td>1.10</td>
<td>1.12</td>
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</table>
Figure 2. Contour plots of the computed scaled velocity $U$ and the computed scaled density $N$ for $\varepsilon = 2^{-6}$, 64 mesh points, $\phi_0(0) = -10.0$, initial scaled velocity $u(x, 0) = -0.5$.

Figure 3. Contour plots of the computed scaled velocity $U$ and the computed scaled density $N$ for $\varepsilon = 2^{-6}$, 64 mesh points, $\phi_0(0) = -10.0$, initial scaled velocity $u(x, 0) = -2.0$.

Figure 4. Surface plots for the computed scaled potential $\Phi$, computed scaled velocity $U$ and the computed scaled density $N$ for $\varepsilon = 2^{-6}$, 64 mesh points, $\phi_0(0) = -10.0$, initial scaled velocity $u(x, 0) = -2.0$.

References


Figure 5. Surface plots for the computed scaled potential $\Phi$, the computed scaled velocity $U$ and the computed scaled density $N$ for $\varepsilon = 2^{-6}$, 64 mesh points, $\phi_0(0) = -50.0$, initial scaled velocity $u(x, 0) = -0.5$.


