Sequential Competitive Facility Location Problem in a Discrete Planar Space

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Abstract In this paper, there are two competitors in a planar market. The first competitor, called, the leader, opens \( p \) new facilities. After that, the second competitor, the follower, reacts to the leader’s action and opens \( r \) new facilities. The leader and the follower have got some facilities in advance in this market. The optimal locations for leader and follower are chosen among predefined candidate locations. In this paper, it is assumed that the demand is inelastic. Considering the huff model, customers share their demand among all facilities, probabilistically. The leader’s objective function is the maximization of its market share after the follower’s new facilities entry. The leader and follower problems are solved via an exact solution method.

Keywords Competitive Location, Leader-Follower, Huff-Type, Market Share.

1 Introduction

Research on competitive facility location problems started by Hotelling [1]. In his paper, it is assumed that all customers use the nearest facility. Huff [2] defined the utility function of facility for customers by considering not only the distance but also the quality of facility. In fact, Huff proposed a model for capturing market share assuming that the probability that a customer patronizes a facility is proportional to the quality of the facility and inversely proportional to a function of the distance to it.

One branch of competitive facility location models is competitive facility location with foresight. This kind of problem is known as the Stackelberg problem in game theory field. In this type of problems, two competitors locate their own new facilities sequentially. First, the leader locates its new facilities, then the leader’s competitor, the follower, reacts to the leader’s action and opens some new facilities too. This type of competitive facility location models was introduced by Hakimi [3] for the first time. He used the word "medianoid" for the follower problem when the follower wants to maximize his market share and the word "centroid" for the leader problem when maximization of her market share is equivalent to minmax of the follower’s market share. So far, many papers about this kind of problems have been published. Some of these papers have solved this problem for continuous and discrete spaces.

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An exact algorithm has been used for solving leader-follower problems by Drezner [4]. Bhadury et al. [5] solved the leader problem using alternating heuristic. Drezner and Drezner [6, 7] also suggested heuristics for Huff type problems. In [8, 9], the leader-follower problem has been proposed to determine the location and design of new facilities in a way that uses branch and bound method [9] and heuristics [8] for solving the problems. In this paper, huff type leader and follower problem in a discrete planar space is considered.

This paper is organized as follows: In section 2, the suggested model is proposed and its solution process is presented in section 3. In section 4, an example is described. In section 5, obtained results are presented.

2 Problem formulation

In this section, the proposed mathematical model to formulate the competitive facility location problem is presented. There are two competitors including the leader and follower that have some facilities in planar markets. The leader has plan to open $p$ new facilities in the predefined potential sites; however, she knows that the follower surely responds to its actions by opening $r$ new facilities. Generally, there are $m$ facilities in the space before opening new facilities by competitors and the $t$ facilities belong to the leader and the $m-t$ facilities correspond to the follower.

The demand is supposed as the demand point and there are $n$ demand points. The existing facilities are located in $m$ of $n$ demand points and the rest of $n-m$ points can be considered as potential locations for new facilities. We assumed gravity model for the demand points (Huff-type), thus the facilities probabilistically absorbed them. The sites and facilities exist in two dimension spaces.

- $n_{pot}^L$: The number of potential sites for the leader ($n_{pot}^L = n - m$)
- $n_{pot}^F$: The number of potential locations for the follower ($n_{pot}^F = n - m - p$)
- $z_i$: The location of $i^{th}$ existing facility
- $y_j$: The location of $j^{th}$ demand point
- $p_i^L$: The location of $s_i^L$ potential location for the leader
- $p_i^F$: The location of $s_i^F$ potential location for the follower
- $b_j$: Buying power of $j^{th}$ demand point (The population or total wealth represented by demand point $j$)
- $x_{kL}$: The location of $k^{th}$ leader’s new facility
- $x_{hF}$: The location of $h^{th}$ follower’s new facility
- $d_{ij}$: The distance between $i^{th}$ existing facility and $j^{th}$ demand point
- $d_{x_{kL}j}$: The distance between $k^{th}$ leader’s new facility and $j^{th}$ demand point
- $d_{x_{hF}j}$: The distance between $h^{th}$ follower’s new facility and $j^{th}$ demand point
- $q_{ij}$: Quality of $i^{th}$ existing facility for $j^{th}$ demand point
- $q_{kLj}$: Quality of leader’s new facilities for $j^{th}$ demand point
- $q_{hFj}$: Quality of follower’s new facility for $j^{th}$ demand point
- $XP_{s_{Lk}}$: A binary variable that is equal to 1 if leader opens $k^{th}$ new facility in $s_i^L$ potential location, 0 otherwise
- $XI_{s_{Fh}}$: A binary variable that is equal to 1 if the follower opens his $h^{th}$ new facility in $s_i^F$ potential location, 0 otherwise
As mentioned before, in accordance with Huff rule, the amounts of facility attraction for a customer has direct relationship with quality and reverse one with function of distance. The amount of \( i^{th} \) facility attraction for \( j^{th} \) customer equals:

\[
\frac{q_{ij}}{1 + d_{ij}^2}
\]

Market share is calculated by total of buying power of all customers multiplying the probability of customer attraction. The follower knows the sites that are located by the leader and he is beginning to locate his own facilities. The model for maximization of the follower’s market share is as follows:

\[
\begin{align*}
\text{Max}(F) = & \quad \frac{\sum_{j=1}^{n_F} b_j \sum_{i=1}^{n_F} \frac{q_{ij}}{1 + d_{ij}^2} \sum_{p=1}^{n_{pot}} q_{pj} X_{SP_{Fh}}}{\sum_{j=1}^{n_F} \sum_{i=1}^{n_F} \frac{q_{ij}}{1 + d_{ij}^2} \sum_{p=1}^{n_{pot}} q_{pj} X_{SP_{Fh}}} \\
\text{s.t.} & \quad \sum_{h=1}^{n_{pot}} X_{SP_{Fh}} = 1, \quad h = 1, 2, ..., r, \\
& \quad \sum_{r=1}^{n_{pot}} X_{SP_{Fh}} = 1, \quad s_F = 1, 2, ..., n_{pot}^F, \\
& \quad \sum_{h=1}^{r} \sum_{s_F=1}^{n_{pot}} X_{SP_{Fh}} = r, \\
& \quad X_{SP_{Fh}} \in \{0, 1\}.
\end{align*}
\]

Equation (1) is objective function that maximizes the follower’s market share. Constraint (2) ensures that each new facility of follower locates in only one of the potential sites. Constraint (3) shows that each potential location can be at most location of one of new facilities. The number of potential sites that are occupied by the new facilities will be exactly equal to the number of follower’s new facilities by constraint (4).

Let \( x^*_F(x_{kL}) \) is the optimal solution of the follower problem. The mathematical model for maximization of leader’s market share is as follows:

\[
\begin{align*}
\text{Max}(L) = & \quad \frac{\sum_{j=1}^{n_L} \frac{q_{ij}}{1 + d_{ij}^2} \sum_{p=1}^{n_{pot}} q_{pj} X_{SP_{Lk}}}{\sum_{j=1}^{n_L} \frac{q_{ij}}{1 + d_{ij}^2} \sum_{p=1}^{n_{pot}} q_{pj} X_{SP_{Lk}}} \\
\text{s.t.} & \quad \sum_{k=1}^{n_{pot}} X_{SP_{Lk}} = 1, \quad k = 1, 2, ..., p,
\end{align*}
\]

s.t.
The leader’s objective function (6) and the constraints (7-9) are described like the follower’s objective function and constraints, respectively.

3 Proposed solution method

In this paper, the follower is assumed a rational person. So, after locating the leader’s new facilities, the follower will surely open his own new facilities at the optimal locations. The follower problem can be solved by considering arbitrary locations for the leader facilities and then finding optimal locations for the follower facilities in order to maximize the follower’s market share. The follower problem is a mixed integer nonlinear programming problem.

In small scales, the follower problem is solved for all leaders’ potential locations (which equals \( n-m \) points) in order to obtain an exact solution for the leader problem. The exact solution of the follower problem which exists in the other optimal points (the follower can’t open his new facilities in the place the leader has located her facilities before) is obtained in each of these points. Since the leader problem in this paper is centroid, the maximum value of the leader’s market share is in the locations in those the maximum values of follower’s market share is minimum. Then, after solving the follower problem with all leader potential points, the best location for the leader new facilities is the solution which has the least optimal value for the follower’s objective function.

4 Numerical example

There are 16 demand points and 5 existing facilities in the 4 x 4 planar market. Three of these facilities belong to the leader and the two of them are considered as the follower’s facilities. The leader wants to open two new facilities and knows that the follower will open two new facilities after her action. Facilities are located in demand points, so potential points equal 11 for leader’s new facilities and 9 for follower’s new facilities. Each demand point has a different buying power from the others. The buying power is randomly generated for different demand points in a range of 1 to 10. Quality values are also determined randomly in a range of 1 to 5 for new and existing leader and follower facilities. The locations of demand points and the leader-follower existing facilities are stated in the following and are depicted in figure 1:

\[
\begin{align*}
    y_1 &= (0,0), (1,0), (2,0), (3,0), (0,1), (1,1), (2,1), (3,1), (0,2), (1,2), (2,2), (3,2), (0,3), (1,3), (2,3), (3,3) \\
    z_4 &= (0,1), (1,3), (3,3), (1,0), (3,2); \quad i=1, 2, 3 \text{ for leader and } i=4, 5 \text{ for follower}
\end{align*}
\]
Fig. 1 The location of leader’s and follower’s existing facilities (x for leader and ● for follower)

The buying power of demand points and the potential locations (Table 1) are stated respectively as follows:

\[ b_j = 6, 9, 8, 5, 10, 8, 6, 4, 9, 3, 8, 9, 6, 3, 3 \]

**Table 1** The potential locations for the given example

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For solving it, first, it is assumed that the leader opens her new facilities in \( x_{IL} = (0,0) \) and \( x_{IL}^* = (2,0) \). Then, the follower problem is solved. The maximum objective function for follower problem equals 50.43 at the points of \( x_{IF}^* = (1, 1) \) and \( x_{IF}^* = (1, 2) \). The follower problem is solved for all leader’s potential locations like the above one. The minimum value of maximized objective functions in each case is at the points of \( x_{IF}^* = (1, 1) \) and \( x_{IF}^* = (2, 1) \) which is the optimal solution for the leader problem because these points maximize the leader's objective function. In these points, the maximized follower’s objective function is 47.44 and the optimal locations for him are \( x_{IF}^* = (1, 2) \) and \( x_{IF}^* = (2, 2) \). Figure 2 illustrates the optimal locations both for the leader and the follower facilities.
Fig. 2 The location of leader’s and follower’s existing and new facilities

5 Conclusions

In this paper, we considered a sequential competitive location problem in a discrete planar market. A mixed-integer nonlinear programming model was proposed to formulate the leader and the follower problems, separately. The leader and follower opened the new facilities in two dimension spaces to maximize their market share where they had some opened facilities from the past. For the small-sized problem, the exact method was applied to solve the considered problem.

References