Around the Minimax Theorem

Mikio Nakayama [*]

1 von Neumann's Minimax Theorem [6, 1928]

A two-person zero-sum game is represented by $\Gamma = (N, X, Y, g)$ where

- $N = \{1, 2\}$: the set of players
- $X = \{x_1, \ldots, x_{\Sigma_1}\}$: a finite set of *pure* strategies x of player 1
- $Y = \{y_1, \ldots, y_{\Sigma_2}\}$: a finite set of *pure* strategies of player 2
- $g : X \times Y \to R$: the payoff function of player 1
- $-g : X \times Y \to R$: the payoff function of player 2

Example 1 The so-called *Morra,* or also called *gangster baccarat,* or *Paper, Stone, Scissors:*

$$\begin{aligned} \Sigma_1 = \Sigma_2 = 3, \quad & g(1,1) = 0, \; g(1,2) = 1, \; g(1,3) = -1, \\ & g(2,1) = -1, \; g(2,2) = 0, \; g(2,3) = 1, \\ & g(3,1) = 1, \; g(3,2) = -1, \; g(3,3) = 0. \end{aligned}$$

Definition 1 *A mixed strategy of player 1 is an element ξ of the set*

$$\{\xi \mid \xi_1 + \cdots + \xi_{\Sigma_1} = 1, \; \xi_1 \geq 0, \ldots, \xi_{\Sigma_1} \geq 0\}.$$

The mixed strategy of player 2 η is defined similarly.

Definition 2 *The payoff $h(\xi, \eta)$ to player 1 for the strategy pair (ξ, η) is given by*

$$h(\xi, \eta) = \sum_{x=1}^{\Sigma_1} \sum_{y=1}^{\Sigma_2} g(x, y) \xi_x \eta_y$$

Theorem 1 (Minimax Theorem) *In two-persin zero sum games,*

$$\text{Max}_\xi \, \text{Min}_\eta h(\xi, \eta) = \text{Min}_\eta \, \text{Max}_\xi h(\xi, \eta).$$

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Remark 1 The inequality

$$\text{Max}_{\xi}\ \text{Min}_{\eta} h(\xi, \eta) \leq \text{Min}_{\eta}\ \text{Max}_{\xi} h(\xi, \eta)$$

is obvious.

Remark 2 The minimax theorem has been proved in many ways by topological ones using fixed point theorems, convex analyses using separation theorems or completely algebraic ones. The original von Neumann's proof, however, does not involve fixed point theorems: it is a mixture of continuity and some topological considerations, but very hard to follow.

Remark 3 The concept of *mixed strategies* in its rigorous form was first defined by *Émile Borel [1]* as early as 1921. *Oskar Morgenstern [5]* also developed the idea of mixed strategies in 1928 in an informal form of the *Sherlock Holmes story*, see Suzuki [12].

2 Proof by Linear Programming

Here, we review the proof of the minimax theorem in the linear programming (LP) approach.

Let $A = (a_{ij})$ be the $m \times n$ matrix defining a zero-sum two-person game, and let

$$p = (p_1, \ldots, p_m), \quad q = (q_1, \ldots, q_n)$$

be mixed strategies of player 1 and 2, respectively. Without loss of generality, we may assume that $a_{ij} > 0$ for all i and j.

Now consider the LP problem:

$$\max v$$

subject to

$$\sum_{i=1}^{m} a_{ij} p_i \geq v, \ \ j = 1, \ldots, n,$$
$$\sum_{i=1}^{m} p_i = 1; \ \ p_i \geq 0, \ \ i = 1, \ldots, m.$$

This is the problem to find a *maxmin strategy* of player 1. Since this problem has a feasible solution yielding $v > 0$, by defining

$$p'_i = \frac{p_i}{v}, \quad i = 1, \ldots, m$$

the above problem can be transformed into (P):

$$\min \sum_{i=1}^{m} p'_i$$

subject to:

$$\begin{cases} \sum_{i=1}^{m} a_{ij} p'_i \geq 1, \ j = 1, \ldots, n, \\ p'_i \geq 0, \ i = 1, \ldots, m. \end{cases}$$

Similarly, consider the problem of finding a *minimax strategy* of player 2:

$$\min v$$

subject to:

$$\sum_{j=1}^{n} a_{ij} q_j \leq v, \quad i = 1, \ldots, m,$$

$$\sum_{j=1}^{n} q_j = 1; \ q_j \geq 0, \ j = 1, \ldots, n,$$

which, by defining

$$q'_j = \frac{q_j}{v}, \quad j = 1, \ldots, n,$$

is equivalent to the problem (D):

$$\max \sum_{j=1}^{n} q'_j$$

subject to:

$$\begin{cases} \sum_{j=1}^{n} a_{ij} q'_j \leq 1, \ i = 1, \ldots, n, \\ q'_j \geq 0, \ j = 1, \ldots, n. \end{cases}$$

Now, applying the *duality theorem* stating that *if the problem* (P) *and its dual,* (D) are both feasible, then both problems have optimal solutions giving rise to the same value of the objective functions, we conclude that there exist p' and q' such that

$$\sum_{i=1}^{m} p_i^{'*} = \sum_{j=1}^{n} q_j^{'*} = \frac{1}{v}$$

Thus, we have the value v; maximin strategy $p'v$, and minimax strategy $q'v$. This completes the proof of the minimax theorem.

3 Zermelo's Theorem [13, 1913]

We state here the so called Zermelo's theorem following the tradition of game theory [1].

[1] Recently, Schwalbe and Walker [8] have reported that this theorem is not exactly the same to what Zermelo proved. One of the theorems proved by Zermelo states, in particular, that the number of steps needed to win from a winning position is not more than the number of positions in the game.

Theorem 2 *In chess, either white can force a win, or black can force a win, or both can force at least a draw.*

This theorem can be expressed in the following two-person zero-sum game $\Gamma = (N, X, Y, g)$:

player's strategy Player i's strategy is a plan of moves against every conceivable contingency in the game. The number of strategies of a player in chess is finite, though very large.

The payoff function of Γ is defined by

player 1's payoff function

$$g(x, y) = \begin{cases} 1 & \text{if player 1 wins} \\ -1 & \text{if player 1 looses} \\ 0 & \text{if draw occurs} \end{cases}$$

Let G be the matrix with (x, y)-element being $g(x, y)$.

Player 1 can force a win $\iff$ G has at least one row x with $g(x, y) = 1$ for all $y = 1, \ldots, \Sigma_2$.

- This x is a maximin strategy of player 1; and any y is a minimax strategy of player 2.

Player 2 can force a win $\iff$ G has at least one column y with $g(x, y) = -1$ for all $x = 1, \ldots, \Sigma_1$

- This y is a minimax strategy of player 2; and any x is a maximin strategy of player 1.

Both can force a draw $\iff$ G has at least one pair (x, y) with $g(x, y) = 0$ for all x and y.

- This x is a maximin strategy of player 1, and this y is a minimax strategy of player 2.

4 Émile Borel [1, 1921]

In this short paper, Borel defined the concept of mixed strategies, and analyzed a two-person constant-sum game with payoffs being the probability of winning in the game. The payoff matrix is assumed to be skew symmetric with the following properties:

- m: the number of codes (pure strategies) of each player.

- a: the probability of player 1 winning; and b that of player 2 winning, so that $a + b = 1$.
- $\begin{cases} a = 1/2 + \alpha_{ik}, \\ b = 1/2 + \alpha_{ki}, \end{cases}$ where $-1/2 \leq \alpha_{ik}, \alpha_{ki} \leq 1/2$.
- $\alpha_{ik} + \alpha_{ki} = 0$.
- $\alpha_{ii} = 0$.

Borel then goes on to deal with the matrix by what is today called *the iterated elimination of weakly dominated strategies*.

Letting p and q be the mixed strategies of player 1 and 2 over the remaining $n(< m)$ strategies, respectively, the payoff to player 1, i.e., the probability of player 1 winning in the game is given by

$$\sum_{i=1}^{n} \sum_{k=1}^{n} (\frac{1}{2} + \alpha_{ik}) p_i q_k = \frac{1}{2} + \alpha,$$

where

$$\alpha = \sum_{i=1}^{n} \sum_{k=1}^{n} \alpha_{ik} p_i q_k.$$

The "best" strategy for player 1 is the one such that $\alpha = 0$ whatever the probabilities $q = (q_1, \ldots, q_n)$ may be. That this is the maximin strategy of player 1 should be clear. The solution is given for the case where $n = 3$.

The game of Morra can be expressed by the matrix $A = (a_{ik})$ such that

$$m = 3;\ \alpha_{12} = \alpha_{23} = \frac{1}{2},\ \alpha_{13} = -\frac{1}{2},$$

which will be equivalent to the one given in Example 1.1 by the utility transformation

$$f(a_{ik}) = 2a_{ik} - 1,\ i, k = 1, 2, 3.$$

Borel says without proof that when $n > 7$, the existence of the minimax solution will be restricted only for particular values of α_{ik}'s, which clearly contradicts the minimax theorem to be appeared seven years later.

Borel wrote several papers on two-person games since 1921, but none of these claimed the general existence of the "best" strategies.

In 1953, Fréchet [2] wrote a letter to *Econometrica* claiming that Émile Borel should be the one to initiate the modern theory of games. But, von Neumann [7] rejected this immediately on the ground that Borel did not prove the fundamental minimax theorem, even believed this to be false for a large number of strategies.

5 Hugo Steinhaus (and B.Knaster, S. Banach)

5.1 Games of Pursuit [9, Steinhaus 1925]

This paper was published in a journal editors of which were students of the university in Lwów, Poland. The original paper is not available now even in Polish, and the photostatic copy was provided by Polish mathematician Stan Ulam, who was a friend of von Neumann after immigrating to USA. The English translation is provided by Harold Kuhn.

The paper discusses three models of games: chess, naval pursuit and games of chance, among which we shall comment on the second one, the naval pursuit.
The model:

- A ship 1 is pursuing ship 2.
- $P_1 = (x_1, y_1)$: position of ship 1
- $P_2 = (x_2, y_2)$: position of ship 2
- $B(P_1, P_2)$: *mode* of pursuit, indicating the angle between the line of sight, connecting P_1 and P_2, and the direction of steering of pursuing ship.
- $C(P_1, P_2)$: *mode* of escape, representing the angle of escaping ship.
- the speed of each ship is given.

Let $t = F(B, C)$ be the duration of the chase from the beginning to the end of the manoeuvre. Then:

escaping ship 2's problem Given B, find a *mode* of escape $C_0 = F_1(B)$ that gives the maximum value of t,

$$t_{max} = F(B, C_0).$$

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pursuing ship 1's problem Find a *mode* of pursuit B_0 that attains the minimum value of t,

$$t_{min} = F(B_0, F_1(B_0)).$$

solution When the speed of the pursuing ship exceeds that of the escaping ship, a finite value of $t_{min} = t_0$ is obtained.

Remark 4

$$\begin{aligned} t_0 = F(B_0, F_1(B_0)) &= \min_B F(B, F_1(B)) \\ &= \min_B \max_C F(B, C) \geq \max_C \min_B \; F(B, C) \\ &= \min_B \; F(B, C_0). \end{aligned}$$

Steinhaus wrote in the letter attached to [9, Steinhaus 1925] that he did not know the above inequality holding in equality.

Remark 5 The problem is reminiscient of the *Stackelberg Oligopoly* in that the escaping ship is best replying to the pursuing ship, and knowing this the pursuing ship chooses it's best strategy. The only difference appears to be that the game is zero-sum.

5.2 Games of Fair Division [10]

Steinhaus was also interested in games of fair division, which seek a fair scheme or rule to divide a fixed size of, say, a cake ([10] and [11]). Games of fair division constitute an intuitive and interesting class of games that provide prototype considerations on fairness and equity.

Below is a summary of an n-person *divide and choose* method given by B.Knaster ans S.Banach reported by Steinhaus [10].

- There is a cake to be divided for n persons 1,2,...,n .
- 1 cuts from the cake an arbitrary part.
- 2 has then the right, but is not obliged, to diminish the slice cut.
- Whatever 2 does, 3 has the right, but is not obliged, to diminish still the already diminished or not diminished slice; and so on up to n.
- The last diminisher must take as his part the slice he was the last to touch.
- The remaining $n - 1$ persons then start the game with the remainder of the cake.
- After $n - 2$ persons are thus disposed of, the remaining two persons now apply the two-person divide-and-choose method.

Harold Kuhn [4] reformulated the game of fair division in an extensive form and show the method to obtain the fair division by a linear programming.

In [3], a physicist *George Gamow* who is famous as an initiator of the Big Bang Theory, and Marvin Stern also tell a story of dividing a fixed amount of brandy to three glasses, extending the divide-and-choose method. The three actors are Gamow, Stern and von Kalman who is known with the *Kalman filter*. How can do you think they attain a fair division?

Hugo Steinhaus completed his doctorate in 1911 under David Hilbert. He is remembered as a collaborator of S.Banach, but his interest extends to medicine, electricity, biology, geology and anthropology. At that time Lwów and Wroclaw have a number of excellent mathematicians such as Sierpinski, Banach, Ulam, Kratowski, etc. As far as game theory is concerned, however, Steinhaus worked in isolation being unaware of Borel and von Neumann's ongoing works.

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