Optimal Calderon Space for Bessel Potentials¹

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Presented by Academician V. A. Il’in March 17, 2014

Received April 11, 2014

DOI: 10.1134/S106456241406026X

The paper is devoted to Bessel potentials constructed by convolutions of Bessel–McDonald kernels with functions from the basic rearrangement invariant space, if the criterion is fulfilled for the embedding of potentials into the space of bounded continuous functions, we state the equivalent description for the cones of moduli of continuity of potentials in the uniform norm. This makes it possible to obtain the criterion for the embedding of potentials into the Calderon space. In the case of Bessel potentials constructed over the basic weighted Lorentz space, we explicitly describe the optimal Calderon space for such an embedding.

1. DESCRIPTION OF THE CONE OF CONTINUITY MODULI FOR POTENTIALS

Let \( E = E(R^n) \) be a rearrangement invariant space (RIS) with the norm \( ||f||_E \); let \( E^* = E'(R^n) \) be an associated space, and \( E = E(R_+) \) and \( E^* = E'(R_+) \) be their Luxemburg representations, i.e., an RIS on \( R_+ = (0, \infty) \) such that \( ||f||_E = ||f^*||_{E^*} \), \( ||g||_E = ||g^*||_{E^*} \), where \( f^* \), \( g^* \) are the decreasing rearrangements of the functions \( f, g \), and

\[
\|h\|_E = \sup \left\{ \int_0^\infty |f^*|^\alpha \, dt : f \in E, \|f\|_E \leq 1 \right\}
\]

(see definitions in [1; Ch. 1, 2]). Let us consider the space of Bessel potentials over the basic space \( E(R^n) \),

\[
H^\alpha_E(R^n) = \left\{ u = G_\alpha * f : f \in E(R^n) \right\},
\]

\[ 0 < \alpha < n, \]

equipped with the norm \( \|u\|_{H^\alpha_E} := ||f||_E \). Here,

\[
u(x) = (G_\alpha * f)(x) = \int G_\alpha(x - y)f(y)dy;
\]

\( G_\alpha \) is a Bessel–McDonald kernel:

\[
G_\alpha(y) = H_\nu(|y|), \quad y \in R^n; \quad 0 < \alpha < n,
\]

\[ \nu = n - \alpha; \]

\( H_\nu(z) = z^{-\nu}K_\nu(z), \quad z_0 > 0, \quad \text{where} \ K_\nu \ \text{is the modified Bessel function (see [2, 3]). Note that} \]

\[
H_\nu(z) \equiv z^{-2\nu}, \quad z \in (0, z_0); \quad H_\nu(z) \equiv z^{-\nu-1/2} e^{-z},
\]

\[ z \geq z_0 \]

with fixed \( z_0 > 0 \) (two-sided estimates with positive constants depending on \( \nu, z_0 \)), and the following criterion is valid concerning the embedding of potentials into the space of bounded uniform continuous functions with the norm

\[
\|u\|_{C} = \sup \{|u(x)| : x \in R^n\}.
\]

(4)

\( H^\alpha_E(R^n) \subset C(R^n) \iff \tau^{\alpha/n-1} \in \tilde{E}'(0, T), \quad T \in R_+ \)

(see, e.g., [4; Theorem 3.6]). Here, \( T = V_n z_0^n \), where \( V_n \) is the volume of the unit ball in \( R^n \), and \( \tilde{E}'(0, T) \) is the restriction of \( \tilde{E}'(R_+) \) onto \( (0, T) \). Under condition, (4) we consider a continuity modulus of order \( k \in N \) for the potential \( u \in H^\alpha_E(R^n) \). In the norm \( C(R^n) \),

\[
\omega_k(u; \tau) = \sup \left\{ \Delta^k_{\lambda}u \|_{C} : \tau \in R_+ \right\},
\]

and we introduce the following cone of continuity moduli of potentials with \( t \in (0, T) \):

\[
M = \left\{ h(t) = \omega_k(u; t^{1/n}) : u \in H^\alpha_E(R^n) \right\}.
\]

We supplement it with the functional

\[
\rho_M(h) = \inf \left\{ \|u\|_{H^\alpha_E} : u \in H^\alpha_E, \omega_k(u; t^{1/n}) = h(t) \right\},
\]

\[ h \in M. \]

¹ The article was translated by the authors.

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To describe this cone, we introduce a simpler cone of functions on \((0, T)\):
\[
K = \left\{ h(t) = \int_0^T \Omega(t, \tau) \sigma(\tau) d\tau : \sigma \in \tilde{E}_0(0, T) \right\},
\]
(7)
supplemented with the following functional:
\[
\rho_k(h) = \| \sigma \|_{\tilde{E}_0(0, T)}.
\]
(8)

Here,
\[
\tilde{E}_0(0, T) = \{ \sigma \in \tilde{E}(0, T) : 0 \leq \sigma \leq h \};
\]
\[
\Omega(t, \tau) = \frac{\sigma^{\alpha/n-1}}{(1 + \left( \frac{\tau}{t} \right)^{1/n})^{\alpha}}.
\]
(9)

Let us introduce the notions of covering and equivalence of the cones: \(M < K\) means that \(\exists c_0 \in R_+ ; \forall h_1 \in M \ \exists h_2 \in K:\n\rho_k(h_2) \leq c_0 \rho_M(h_1)\); \(h_2(t) \geq h_1(t), \ t \in (0, T)\).

Then,
\[
M \approx K \iff M < K < M.
\]
(11)

**Theorem 1.** Let \(0 < \alpha < n\), and the condition (4) be fulfilled. Then the equivalence of cones (5) and (7) takes place. The constants in the condition of mutual covering (11) depend on \(\alpha, n, T\) and on the norm of the embedding operator (4).

This result is based on the order-sharp estimates for the continuity moduli of potentials obtained in [5].

### 2. CRITERION OF EMBEDDING INTO THE CALDERON SPACE

The notion of the generalized Banach function space (GBFS) is needed here, which generalizes the well-known notion of the Banach function space (BFS) introduced by Bennett and Sharpley [1, Ch.1]. Let \(\mu\) be the Lebesgue measure on \((0, T), T \in (0, \infty)\).

**Definition 1.** A linear space \(X = X(0, T)\) of measurable functions on \((0, T)\) with the norm \(\| \cdot \|_X\) is called a GBFS if the following conditions are fulfilled:

(P1) \(\| f \|_X = 0 \iff f = 0 \mu\text{-almost everywhere on } (0, T)\);
(P2) \(| f | \leq g \in X \Rightarrow f, g, f \vee g \in X\);
(P3) \(0 \leq f_n \uparrow f, f_n \in X \Rightarrow \| f_n \|_X \uparrow \| f \|_X\);

and, for every measurable set \(B \subset (0, T)\) with \(\mu(B) > 0\) we have

(P4) \(\exists h_B > 0 \ \mu\text{-a.e. in } B : \int_B | f | d\mu \leq c_B \| f \|_X \forall f \in X\);

(P5) \(\exists \gamma_B \in X : f_B > 0 \ \mu\text{-a.e. in } B\).

For the BFS it was necessary in [1] that \(h_B = \gamma_B \) under conditions (P4) and (P5), where \(\gamma_B\) is the characteristic function of set \(B\). Note that if \(X = X(0, T)\) is a BFS, the function \(v\) is \(\mu\text{-measurable}, and } 0 < v < \infty \mu\text{-a.e. in } (0, T), then

\[
X_v = \{ f : f/v \in X; \| f/v \|_X := \| f \|_X/v \}
\]
is a GBFS. Moreover, a GBFS is really a Banach space, and its associated space is a GBFS, too. The duality principle holds for GBFSs (the twice-associated space coincides with the initial space, see [6]).

Let \(K = K(0, T)\) be some cone of nonnegative \(\mu\text{-measurable functions on } (0, T)\) equipped with positive homogeneous functional \(\rho_k\). For a GBFS \(X = X(0, T)\), embedding \(K \hookrightarrow X\) means that \(K \subset X\), and

\[
\exists c_k \in R_+ : \| h_k \| \leq c_k \rho_k(h) \forall h \in K.
\]
(12)

**Definition 2.** A GBFS \(X_0 = X_0(0, T)\) is called optimal for embedding \(K \hookrightarrow X\) if

(1) \(K \hookrightarrow X_0\);
(2) embedding \(K \hookrightarrow X\), where \(X\) is a GBFS, implies \(X \subset X_0\).

**Remark 1.** Let \(K\) and \(M\) be some cones of nonnegative \(\mu\text{-measurable functions on } (0, T)\) supplemented with functionals \(\rho_k\) and \(\rho_M\). If \(K \approx M\), then

(1) for every GBFS \(X = X(0, T)\) we have \(K \hookrightarrow X \implies M \hookrightarrow X\), and the ratio of the constants \(\frac{c_k}{c_m}\) in (12) depends on the constants of the mutual covering of the cones (see (10), (11));

(2) the same GBFS \(X_0 = X_0(0, T)\) is optimal for both embeddings \(K \hookrightarrow X\) and \(M \hookrightarrow X\).

Let \(X = X(0, T)\) be a GBFS, and \(k \in N\). We introduce the Calderon space (see, e.g., [7]; a more special variant was considered in [8]):

\[
\Lambda^k(C; X) = \{ u \in C(R^n) : \omega_k(u) ; t^{k/n} \in X(0, T) \};
\]
(13)
\[
\| u \|_{\Lambda^k(C; X)} = \| u \|_C + \| \omega_k(u) ; t^{k/n} \|_{X(0, T)}.
\]
(14)

The following nontrivial conditions hold:
\[
\Lambda^k(C; X) \neq \{ 0 \} \iff \| u \|_{ \Lambda^k(C; X) } < \infty;
\]
\[
\Lambda^k(C; X) \neq \Lambda^k(C; R^n) \iff \| | u \|_{X(0, T)} = \infty.
\]
(15)
(16)

Moreover, it is obvious that
\[
X_0(0, T) \subset X(0, T) \implies \Lambda^k(C; X_0) \subset \Lambda^k(C; X).
\]
(17)

Let us formulate the embedding criterion:
\[
H^s_x(R^n) \subset \Lambda^k(C; X).
\]
(18)

**Theorem 2.** Let \(0 < \alpha < n\), and the condition (4) be fulfilled. Then, the following equivalence holds:

\[
(18) \iff K \hookrightarrow X
\]
(19)

where \(K\) is the cone (7)–(9). The norm of the embedding operator in (18) depends on \(\alpha, n, T\), and on the norm of the embedding operator in (4).

**Corollary.** Let \(X_0 = X_0(0, T)\) be the optimal GBFS for embedding \(K \hookrightarrow X\), where \(K\) is the cone (7)–(9). Then, \(\Lambda^k(C; X_0)\) is the optimal Calderon space for embedding (18), i.e.,

\[
H^s_x(R^n) \subset \Lambda^k(C; X_0);
\]
(20)
\[
(18) \implies \Lambda^k(C; X_0) \subset \Lambda^k(C; X).
\]
3. DESCRIPTION OF THE OPTIMAL CALDERON SPACE

Let us concretize the results of Sections 1 and 2 in the case when the basic RIS \( E(R^n) \) coincides with the weighted Lorentz space:

\[
E(R^n) = \Lambda_q(v), \quad 1 \leq q < \infty;
\]

\[
\|f\|_{\Lambda_q(v)} := \left( \int_0^\infty \left( \int_0^\infty (\tau)^q v(\tau) d\tau \right)^{1/q} \right)^{1/q}.
\]  

(21)

Here, \( v > 0 \) is a \( \mu \)-measurable function. The general properties of Lorentz spaces are presented, e.g., in [9, 10]. Note that

\[
\Lambda_q(v) \neq \{0\} \iff V(t) := \int_0^t v(\tau) d\tau < \infty, \quad t \in R_+.
\]  

(22)

Quantity (21) is equivalent to the norm if \( q = 1 \), and \( t^{-1}V(t) \) almost decreases or \( 1 < q < \infty \), and \( \exists \epsilon \in R_+ \):

\[
\int_0^\infty t^{-3} v(\tau) d\tau \leq \epsilon V(t), \quad t \in R_+.
\]  

(23)

Below, we assume that these conditions are fulfilled.

Let \( \frac{1}{q} + \frac{1}{q'} = 1 \). For \( t \in (0, T) \), \( T \in R_+ \) determined by (4), we denote

\[
W(t) = V(t)^{-1} t^{\alpha/n};
\]  

(24)

\[
\Psi_1(t) = \sup_{\tau \in (0, t]} W(\tau), \quad q = 1;
\]  

(25)

\[
\Psi_q(t) = \left( \int_0^t W^{q'}(\tau) d\tau \right)^{1/q}, \quad 1 < q < \infty.
\]  

(26)

**Lemma 1.** With use of the above notation,

\( (4) \Leftrightarrow \Psi_q(T) < \infty \).

(27)

We denote

\[
\beta = \frac{\alpha - k}{n}; \quad \beta_+ = \max\{\beta; 0\};
\]  

(28)

\[
U_1(t) = \sup_{\tau \in [t, \infty]} \left( t^{\beta} V(\tau)^{-1} \right), \quad q = 1;
\]  

(29)

\[
U_q(t) = \left( \int_0^\infty t^{\beta} V(\tau)^{-1} \right)^{1/q} V(\tau) d\tau, \quad 1 < q < \infty.
\]  

(30)

Note that for \( \alpha \leq k \) we have

\[
U_1(t) = V(t)^{-1}; \quad U_q(t) = (q' - 1)^{-1} q V(\tau)^{-1/q}, \quad 1 < q < \infty.
\]

(31)

We use also the following notation:

\[
\|f\|_{X_0} = \left\{ \int_0^T \|f\|_{L_q(0, \tau)} \|d\Psi_q(t)\|_{\Psi_q(t)} \right\}^{1/q}.
\]  

(32)

Let \( T_1 \in (0, T) \) be such that \( \Psi_q(T_1) = 2^{-1}\Psi_q(T) \).

**Theorem 3.** For the above condition let \( \Psi_q(T) < \infty \), and the following condition be satisfied

\[
\exists \epsilon > 0: \quad U_q(t) \downarrow \epsilon, \quad t \in (0, T).
\]  

Then, the optimal GBFS \( X_0 = X_0(0, T) \) for embedding \( K \hookrightarrow X \) with cone \( K \) determined by (7)–(9) has the following norm:

\[
(1) \quad \text{if } q = 1, \text{ and } \Psi_1(+0) > 0, \text{ then } \|f\|_{X_0} = \|f\|_{L_q(0, T)};
\]  

(33)

\[
(2) \quad \text{if } q = 1, \text{ and } \Psi_q(0) = 0, \text{ or } 1 < q < \infty, \text{ then } \|f\|_{X_0} = \|f\|_{\nu_q(T, T_1)}.
\]  

(34)

**Remark 2.** In the case \( q = 1 \), and \( \Psi_q(0) = 0 \), the embedding (4) happens “on the smoothness limit,” and we obtain \( \Lambda_q(C; X_0) = C(R^n) \). According to the results in [5], in this case there exist functions \( u \in H^p_q(R^n) \) such that \( \omega_q(u; t^{1/n}) \rightarrow 0 (t \rightarrow +0) \) arbitrarily slowly.

**Theorem 4.** Let the assumptions of Theorem 3 be satisfied. Let \( q = 1 \), and \( \Psi_q(+0) > 0 \), or \( 1 < q < \infty \). Then, the optimal Calderon space for embedding (18) has the following norm:

\[
\|u\|_{\Lambda_q(C; X_0)} = \|u\| + \left\{ \int_0^T \omega_q(u; t^{1/n})^{q} \Psi_q(t) \right\}^{1/q}.
\]  

(35)

**4. SOME EXPLICIT DESCRIPTIONS OF THE OPTIMAL CALDERON SPACE**

We present two examples of the application of Theorem 4. Note that in Example 2, we develop some preceding results according to [8].

**Example 1.** Let \( 0 < \alpha < n, 1 \leq q < \infty, v = 1 \), so that \( E = L_q(R^n) \). For \( 0 < \alpha < n, q = 1 \) the space \( H^p_q(R^n) \) is not embedded into \( C(R^n) \). If \( 1 < q < \infty \), the criterion of embedding into \( C(R^n) \) has the form (4)

\[
(4) \Leftrightarrow \alpha > \frac{n}{q}.
\]

For \( \frac{n}{q} < \alpha < \min\left\{ n; k + \frac{n}{q} \right\} \), then the optimal Calderon space for embedding (18) has the norm

\[
\|u\|_{\Lambda_q(C; X_0)} = \|u\| + \left\{ \int_0^T \frac{\omega_q(u; t^{1/n})^{q}}{t^{\alpha/n - 1/q}} dt \right\}^{1/q}.
\]

This means it coincides with the classical Besov space \( B^{\alpha - n/q}_{q, q} (R^n) \) (see, e.g., [2] about the properties of Besov spaces).
Example 2. Let $0 < \alpha < n$, $1 \leq q < \infty$, $1 < p < \infty$, $E = \Lambda_q(\nu)$, where
\[
v(t) = t^{\alpha/p - 1} b(t), \quad t \in (0, T).
\]
Here, $b > 0$ is a slowly varying continuous function on $(0, T)$ (a function of logarithmic type), so that for every $\delta > 0$
\[
t^\delta b(t) \uparrow, \quad t^\delta b(t) \downarrow, \quad t \in (0, T)
\]
(i.e., $E(R^n)$ is a so-called Lorentz–Karamata space, see [8]).

The following results take place.

(1) For $0 < \alpha < n/p$, the space $H^s_{\nu}(R^n)$ is not embedded into $C(R^n)$.

(2) For $n/p < \alpha < \min\left\{n; k + n/p\right\}$, the optimal Calderon space for embedding (18) has norm (35), in which for $1 < q < \infty$
\[
\int_0^t b(t)^q \frac{d\tau}{\tau} \leq \frac{b(t)^q dt}{t}.
\]

(3) If $\alpha = n/p$ (the limiting case for embedding (4)), we have for $q = 1$
\[
(4) \Leftrightarrow \Psi(1) = \sup_{t \in (0, T)} b(t)^{-1} < \infty.
\]

(4) $\Leftrightarrow \Psi_q(t) = \left\{ \int_0^t b^{-q} d\tau \right\}^{1/q} < \infty, \quad t \in (0, T]$.

If these conditions are fulfilled (for $q = 1$, we also require $\Psi(t)(+0) = 0$), then the optimal Calderon space for embedding (18) has norm (35), in which for $1 < q < \infty$ we have
\[
\frac{d\Psi_q(t)}{\Psi_q(t)} \leq \frac{b(t)^q dt}{t}.
\]

ACKNOWLEDGMENTS

The research of the first author was partially supported by the Russian Science Foundation, project no. 14-11-00443.

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