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Gradient methods with selection technique for the multiple-sets split feasibility problem

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ABSTRACT
In this paper, we present two new iterative algorithms for approximating a solution of the multiple-sets split feasibility problem. The suggested algorithms are based on the gradient method with selection technique. Weak and strong convergence theorems are demonstrated.

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1. Introduction

Inverse problems in various disciplines can be expressed as split feasibility problems and their generalizations, such as the multiple-sets split feasibility problem (MSSFP) and the split common fixed point problem (see, e.g. [1–8]), and many iterative algorithms have been presented to solve these problems, see for example [9–28] and references therein.

In the present paper, we focus on the multiple-set split feasibility problem which is a general way to characterize various inverse problems arising in many real-world application problems, such as medical image reconstruction and intensity-modulated radiation therapy.

Let H_1 and H_2 be two real Hilbert spaces with their own inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let s and t be positive integers and let $\{C_i\}_{i=1}^{s}$ and $\{Q_j\}_{j=1}^{t}$ be two finite families of closed convex subsets of H_1 and H_2, respectively. Let $A\colon H_1 \to H_2$ be a bounded linear operator with its adjoint operator A^*. MSSFP

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is to find a point x^* such that

$$x^* \in \bigcap_{i=1}^{s} C_i \quad \text{and} \quad Ax^* \in \bigcap_{j=1}^{t} Q_j. \tag{1}$$

Assume MSSFP (1) is consistent, i.e. it is solvable, and $\emptyset \neq \Omega$ denotes its solution set.

The case where $s = t = 1$, called the split feasibility problem (SFP), was introduced by Censor and Elfving [29], modelling phase retrieval and other image restoration problems, and further studied and extended by many researchers; see, for instance, [30–41].

Note that x^* solves the MSSFP (1) implies that the distance from x^* to each C_i is zero and the distance from Ax^* to each Q_j is also zero. This motivates us to consider the proximity function

$$g(x) = \frac{1}{2}\sum_{i=1}^{s} \alpha_i \|x - P_{C_i}x\|^2 + \frac{1}{2}\sum_{j=1}^{t} \beta_j \|Ax - P_{Q_j}Ax\|^2 \tag{2}$$

where $\{\alpha_i\}$ and $\{\beta_j\}$ are positive real numbers, and P_{C_i} and P_{Q_j} are the metric projections onto C_i and Q_j, respectively.

It is known that x^* is a solution of MSSFP (1) iff $g(x^*) = 0$. Since $g(x) \geq 0$ for all $x \in H_1$, a solution of MSSFP (1) is a minimizer of g over any closed convex subset, with minimum value of zero. Note that this proximity function is convex and differentiable with gradient

$$\nabla g(x) = \sum_{i=1}^{s} \alpha_i (I - P_{C_i})x + \sum_{j=1}^{t} \beta_j A^*(I - P_{Q_j})Ax.$$

Since the gradient $\nabla g(x)$ is L-Lipschitz continuous [35] with constant

$$L = \sum_{i=1}^{s} \alpha_i + \sum_{j=1}^{t} \beta_j \|A\|^2,$$

one of the most popular methods for solving the minimization problem

$$\min_{x \in \Gamma} g(x) \tag{3}$$

is the gradient method that takes the following iterative manner

$$\begin{aligned} x_{n+1} &= x_n - \tau_n \nabla g(x_n) \\ &= x_n - \tau_n \left(\sum_{i=1}^{s} \alpha_i (I - P_{C_i})x_n + \sum_{j=1}^{t} \beta_j A^*(I - P_{Q_j})Ax_n \right), \quad n \geq 0. \end{aligned} \tag{4}$$

Here, the stepsize τ_n may be selected using various ways.

When $s=t=1$, we have the following gradient algorithm for solving two-sets split feasibility problem

$$x_{n+1} = x_n - \tau_n((I-P_C)x_n + A^*(I-P_Q)Ax_n), \quad n \geq 0. \tag{5}$$

On the other hand, we note that x^* is a solution of MSSFP (1) iff $f(x^*) = 0$, where

$$\begin{aligned} f(x) &= \frac{1}{2}\max_{1\leq i\leq s} \|x - P_{C_i}x\|^2 + \frac{1}{2}\max_{1\leq j\leq t} \|Ax - P_{Q_j}Ax\|^2 \\ &= \frac{1}{2}\|x - P_{C_{i(x)}}x\|^2 + \frac{1}{2}\|Ax - P_{Q_{j(x)}}Ax\|^2, \end{aligned} \tag{6}$$

in which

$$i(x) \in \left\{ i \mid \max_{1\leq i\leq s} \|x - P_{C_i}x\| \right\},$$

$$j(x) \in \left\{ j \mid \max_{1\leq j\leq t} \|Ax - P_{Q_j}Ax\| \right\}.$$

Hence, we can construct a version of (5) to solve MSSFP (1). In each iteration, our algorithm needs to compute only two projections, one from $\{P_{C_{i_n}}x_n\}_{i=1}^s$ and another one from $\{(I-P_{Q_{j_n}})Ax_n\}_{j=1}^t$.

It is our main purpose in this paper to present two new iterative algorithms for approximating a solution of MSSFP (1). The suggested algorithms are based on the gradient method with selection technique. Weak and strong convergence theorems are demonstrated.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. The (nearest point or metric) projection, denoted by P_C, from H onto C has the following characteristic [30] (for $v^\dagger \in H$)

$$\langle v^\dagger - P_C v^\dagger, v - P_C v^\dagger \rangle \leq 0, \tag{7}$$

for all $v \in C$.

It then follows that [42]

$$\langle v^\dagger - P_C v^\dagger, v^\dagger - v \rangle \geq \|v^\dagger - P_C v^\dagger\|^2, \tag{8}$$

for all $v \in C$ and $v^\dagger \in H$.

It is obvious that P_C is nonexpansive, i.e. $\|P_C u - P_C u^\dagger\| \leq \|u - u^\dagger\|$ for all $u, u^\dagger \in H$.

Lemma 2.1 ([43]): *Let C be a nonempty closed convex of a real Hilbert space H. Let $T\colon C \to C$ be a nonexpansive mapping. Then $I-T$ is demi-closed at 0, i.e. if $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$, then $x = Tx$.*

Given a sequence $\{x_n\}$ in H_1, $\omega_w(x_n)$ stands for the set of cluster points in the weak topology, that is,

$$\omega_w(x_n) = \{x : \exists x_{n_i} \to x \text{ weakly}\}.$$

Lemma 2.2 ([44]): *Let H be a real Hilbert space and $\{x_n\}$ a sequence in H such that there exists a nonempty closed set $\Omega \in H$ satisfying*

(i) *For every $z \in \Omega$, $\lim_{n\to\infty} \|x_n - z\|$ exists;*
(ii) $\omega_w(x_n) \subset \Omega$.

Then, there exists $\bar{z} \in \Omega$ such that $\{x_n\}$ weakly converges to $\bar{z}$.

Lemma 2.3 ([45]): *Assume that $\{\delta_n\}$ is a sequence of nonnegative real numbers such that*

$$\delta_{n+1} \leq (1 - \xi_n)\delta_n + \xi_n\sigma,$$

where $\{\xi_n\}$ is a sequence in $(0, 1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \xi_n = \infty$;
(ii) $\limsup_{n\to\infty} \sigma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\xi_n\sigma_n| < \infty$.

Then $\lim_{n\to\infty} \delta_n = 0$.

3. Main results

Let H_1 and H_2 be two real Hilbert spaces. Let s and t be positive integers and let $\{C_i\}_{i=1}^{s}$ and $\{Q_j\}_{j=1}^{t}$ be two finite families of closed convex subsets of H_1 and H_2, respectively. Let $A\colon H_1 \to H_2$ be a bounded linear operator with its adjoint operator A^*. Throughout, assume

$$\Omega = \left\{ \bar{z} : \bar{z} \in \bigcap_{i=1}^{s} C_i \text{ and } A\bar{z} \in \bigcap_{j=1}^{t} Q_j \right\} \neq \emptyset.$$

Next we present the following iterative algorithm to solve MSSFP (1).

Algorithm 3.1: Choose an arbitrary initial value $x_1 \in H_1$. Assume x_n has been constructed. Compute

$$\begin{aligned} z_n &= P_{C_{i_n}} x_n, \\ y_n &= A^*(I - P_{Q_{j_n}})Ax_n, \quad n \geq 1, \end{aligned} \tag{9}$$

where

$$i_n \in \left\{ i \mid \max_{i \in I_1} \|x_n - P_{C_i} x_n\|,\ I_1 = \{1, 2, \ldots, s\} \right\},$$
$$j_n \in \left\{ j \mid \max_{j \in I_2} \|Ax_n - P_{Q_j} Ax_n\|,\ I_2 = \{1, 2, \ldots, t\} \right\}. \tag{10}$$

If

$$\|x_n + y_n - z_n\| = 0, \tag{11}$$

then stop (in this case $x_n \in \Omega$ by Remark 3.1 below); otherwise, continue and construct x_{n+1} via the manner

$$x_{n+1} = x_n - \tau_n (x_n + y_n - z_n), \tag{12}$$

where

$$\tau_n = \lambda_n \frac{\|x_n - z_n\|^2 + \|y_n\|^2}{2\|x_n + y_n - z_n\|^2},$$

in which $\lambda_n > 0$.

Remark 3.1: The equality (11) holds if and only if x_n is a solution of MSSFP (1). First, assume the equality (11) holds. For any $z \in \Omega$, by (8), we have

$$\begin{aligned} 0 &= \langle x_n + y_n - z_n, x_n - z \rangle \\ &= \langle x_n - P_{C_{i_n}} x_n, x_n - z \rangle + \langle A^*(I - P_{Q_{j_n}})Ax_n, x_n - z \rangle \\ &= \langle x_n - P_{C_{i_n}} x_n, x_n - z \rangle + \langle (I - P_{Q_{j_n}})Ax_n, Ax_n - Az \rangle \\ &\geq \|x_n - P_{C_{i_n}} x_n\|^2 + \|(I - P_{Q_{j_n}})Ax_n\|^2 \end{aligned} \tag{13}$$

which implies that

$$\|x_n - P_{C_{i_n}} x_n\| = 0 \quad \text{and} \quad \|(I - P_{Q_{j_n}})Ax_n\| = 0. \tag{14}$$

According to the definitions of i_n and j_n, it follows from (14) that

$$\|x_n - P_{C_i} x_n\| = 0 \text{ for all } i \in I_1 \quad \text{and} \quad \|Ax_n - P_{Q_j} Ax_n\| = 0 \text{ for all } j \in I_2.$$

Hence, $x_n \in \bigcap_{i=1}^{s} C_i$ and $Ax_n \in \bigcap_{j=1}^{t} Q_j$. Therefore, $x_n \in \Omega$.

Assume that the sequence $\{x_n\}$ generated by Algorithm 3.1 is infinite. In other words, Algorithm 3.1 does not terminate in a finite number of iterations. Next, we demonstrate the convergence analysis of the sequence $\{x_n\}$ generated by Algorithm 3.1.

Theorem 3.1: *If* $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 4$, *then the sequence* $\{x_n\}$ *generated by Algorithm* 3.1 *converges weakly to a solution of* MSSFP *(1).*

Proof: Firstly, we show that the sequence $\{x_n\}$ is bounded. Picking up $z \in \Omega$, from (13), we have

$$\langle x_n + y_n - z_n, x_n - z \rangle \geq \|x_n - z_n\|^2 + \|y_n\|^2. \tag{15}$$

By (12) and (15), we deduce

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|x_n - z - \tau_n(x_n + y_n - z_n)\|^2 \\
&= \|x_n - z\|^2 - 2\tau_n \langle x_n + y_n - z_n, x_n - z \rangle \\
&\quad + \tau_n^2 \|x_n + y_n - z_n\|^2 \\
&\leq \|x_n - z\|^2 - \frac{\lambda_n(\|x_n - z_n\|^2 + \|y_n\|^2)^2}{\|x_n + y_n - z_n\|^2} \\
&\quad + \frac{\lambda_n^2(\|x_n - z_n\|^2 + \|y_n\|^2)^2}{4\|x_n + y_n - z_n\|^2} \\
&= \|x_n - z\|^2 - \lambda_n \left(1 - \frac{\lambda_n}{4}\right) \frac{(\|x_n - z_n\|^2 + \|y_n\|^2)^2}{\|x_n + y_n - z_n\|^2}.
\end{aligned} \tag{16}$$

This implies that $\lim_{n\to\infty} \|x_n - z\|$ exists. Thus, the sequence $\{x_n\}$ is bounded, and so are the sequences $\{Ax_n\}$, $\{P_{Q_j}x_n\}(j \in I_2)$ and $\{P_{C_i}x_n\}(i \in I_1)$.

We next show that every weak cluster point of the sequence $\{x_n\}$ belongs to the solution set, i.e. $\omega_w(x_n) \subset \Omega$.

In terms of (16), we get

$$\lambda_n(1 - \frac{\lambda_n}{4}) \frac{(\|x_n - z_n\|^2 + \|y_n\|^2)^2}{\|x_n + y_n - z_n\|^2} \leq \|x_n - z - \|x_{n+1} - z\|^2$$

It follows that

$$\lim_{n\to\infty} \frac{(\|x_n - z_n\|^2 + \|y_n\|^2)^2}{\|x_n + y_n - z_n\|^2} = 0.$$

This together with the boundedness of the sequence $\{x_n + y_n - z_n\}$ implies that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|y_n\| = 0.$$

Hence,

$$\lim_{n\to\infty} \|x_n - P_{C_i}x_n\| = 0 \quad \text{for all } i \in I_1,$$

and

$$\lim_{n\to\infty} \|Ax_n - P_{Q_j}Ax_n\| = 0 \quad \text{for all } j \in I_2.$$

By the demiclosedness (Lemma 2.1) of $I - P_{C_i}$ (for all $i \in I_1$) and $I - P_{Q_j}$ (for all $j \in I_2$), we deduce immediately $\omega_w(x_n) \subset \Omega$. To this end, the conditions of Lemma 2.2 are all satisfied. Consequently, the sequence $\{x_n\}$ converges weakly to a solution of MSSFP (1). This completes the proof. ■

Algorithm 3.1 has only weak convergence. Now, we present a new algorithm with strong convergence.

Algorithm 3.2: Let $u \in H_1$ and choose an arbitrary initial value $x_1 \in H_1$. Assume x_n has been constructed. Compute

$$\begin{aligned} z_n &= P_{C_{i_n}} x_n, \\ y_n &= A^*(I - P_{Q_{j_n}})Ax_n, \quad n \geq 1, \end{aligned} \tag{17}$$

where

$$\begin{aligned} i_n &\in \left\{ i \mid \max_{i \in I_1} \|x_n - P_{C_i} x_n\|, \ I_1 = \{1, 2, \ldots, s\} \right\}, \\ j_n &\in \left\{ j \mid \max_{j \in I_2} \|Ax_n - P_{Q_j} Ax_n\|, \ I_2 = \{1, 2, \ldots, t\} \right\}. \end{aligned} \tag{18}$$

If

$$\|x_n + y_n - z_n\| = 0, \tag{19}$$

then stop; otherwise, continue and construct x_{n+1} via the manner

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)[x_n - \tau_n(x_n + y_n - z_n)], \tag{20}$$

where $\alpha_n \in (0, 1)$ and

$$\tau_n = \lambda_n \frac{\|x_n - z_n\|^2 + \|y_n\|^2}{2\|x_n + y_n - z_n\|^2},$$

in which $\lambda_n \in (0, 4)$.

Assume that the sequence $\{x_n\}$ generated by Algorithm 3.2 is infinite. In other words, Algorithm 3.2 does not terminate in a finite number of iterations.

Theorem 3.2: *Suppose the sequences* $\{\alpha_n\}$ *and* $\{\lambda_n\}$ *satisfying the following conditions*:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 4$.

Then the sequence $\{x_n\}$ *generated by Algorithm* 3.2 *converges strongly to* $z = P_\Omega u$.

Proof: Set $u_n = x_n - \tau_n(x_n + y_n - z_n)$ for all $n \geq 0$. By (16), we have

$$\|u_n - z\|^2 \leq \|x_n - z\|^2 - \lambda_n \left(1 - \frac{\lambda_n}{4}\right) \frac{(\|x_n - z_n\|^2 + \|y_n\|^2)^2}{\|x_n + y_n - z_n\|^2}. \tag{21}$$

In particular, we have $\|u_n - z\| \leq \|x_n - z\|$. Thus, from (20), we obtain

$$\begin{aligned} \|x_{n+1} - z\| &= \|\alpha_n u + (1 - \alpha_n)u_n - z\| \\ &\leq \alpha_n \|u - z\| + (1 - \alpha_n)\|u_n - z\| \\ &\leq \alpha_n \|u - z\| + (1 - \alpha_n)\|x_n - z\| \\ &\leq \max\{\|x_n - z\|, \|u - z\|\}. \end{aligned}$$

By induction, we derive

$$\|x_{n+1} - z\| \leq \max\{\|x_0 - z\|, \|u - z\|\}.$$

Hence, $\{x_n\}$ is bounded and so are the sequences $\{Ax_n\}$, $\{P_{Q_j} x_n\}(j \in I_2)$ and $\{P_{C_i} x_n\}(i \in I_1)$.

From (20), we have

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|\alpha_n(u - z) + (1 - \alpha_n)(u_n - z)\|^2 \\ &\leq (1 - \alpha_n)\|u_n - z\|^2 + 2\alpha_n \langle u - z, x_{n+1} - z\rangle. \end{aligned} \tag{22}$$

By virtue of (21) and (22), we deduce

$$\begin{aligned} \|x_{n+1} - z\|^2 &\leq (1 - \alpha_n)\|x_n - z\|^2 + 2\alpha_n \langle u - z, x_{n+1} - z\rangle \\ &\quad - (1 - \alpha_n)\lambda_n (1 - \frac{\lambda_n}{4}) \frac{(\|x_n - z_n\|^2 + \|y_n\|^2)^2}{\|x_n + y_n - z_n\|^2} \\ &= (1 - \alpha_n)\|x_n - z\|^2 + \alpha_n \Bigg[2\langle u - z, x_{n+1} - z\rangle \\ &\quad - \frac{(1 - \alpha_n)}{\alpha_n} \lambda_n \left(1 - \frac{\lambda_n}{4}\right) \frac{(\|x_n - z_n\|^2 + \|y_n\|^2)^2}{\|x_n + y_n - z_n\|^2} \Bigg]. \end{aligned} \tag{23}$$

Set $\theta_n = \|x_n - z\|^2$ and

$$\delta_n = 2\langle u - z, x_{n+1} - z\rangle - \frac{(1 - \alpha_n)}{\alpha_n} \lambda_n \left(1 - \frac{\lambda_n}{4}\right) \frac{(\|x_n - z_n\|^2 + \|y_n\|^2)^2}{\|x_n + y_n - z_n\|^2}$$

for all $n \geq 1$. Then, from (23), we have

$$0 \leq \theta_{n+1} \leq (1 - \alpha_n)\theta_n + \alpha_n \delta_n, \quad n \geq 1. \tag{24}$$

It is obvious that

$$\delta_n \leq 2\langle u - z, x_{n+1} - z\rangle \leq 2\|u - z\|\|x_{n+1} - z\|.$$

So,

$$\limsup_{n\to\infty} \delta_n < \infty.$$

Next, we show that $\limsup_{n\to\infty} \delta_n \geq -1$ by contradiction. Assume that $\limsup_{n\to\infty} \delta_n < -1$. Then there exists m such that $\delta_n \leq -1$ for all $n \geq m$. It follows from (24) that

$$\begin{aligned}\theta_{n+1} &\leq (1-\alpha_n)\theta_n + \alpha_n\delta_n \\ &= \theta_n + \alpha_n(\delta_n - \theta_n) \\ &\leq \theta_n - \alpha_n,\end{aligned}$$

for all $n \geq m$.

Thus,

$$\theta_{n+1} \leq \theta_m - \sum_{i=m}^{n} \alpha_i.$$

Hence, by taking lim sup as $n \to \infty$ in the last inequality, we obtain

$$0 \leq \limsup_{n\to\infty} \theta_{n+1} \leq \theta_m - \limsup_{n\to\infty} \sum_{i=m}^{n} \alpha_i = -\infty,$$

which is a contradiction. Therefore, $\limsup_{n\to\infty} \delta_n \geq -1$ and it is finite. Consequently, we can take a subsequence $\{n_k\}$ such that

$$\begin{aligned}\limsup_{n\to\infty} \delta_n &= \lim_{k\to\infty} \delta_{n_k} \\ &= \lim_{k\to\infty} \Bigg[-\frac{(1-\alpha_{n_k})}{\alpha_{n_k}} \lambda_{n_k} \left(1 - \frac{\lambda_{n_k}}{4}\right) \frac{(\|x_{n_k} - z_{n_k}\|^2 + \|y_{n_k}\|^2)^2}{\|x_{n_k} + y_{n_k} - z_{n_k}\|^2} \\ &\quad + 2\langle u - z, x_{n_k+1} - z\rangle \Bigg]. \end{aligned} \tag{25}$$

Since $\{x_{n_k+1}\}$ is bounded, there exists a subsequence $\{x_{n_{k_i}+1}\}$ of $\{x_{n_k+1}\}$ such that the limit $\lim_{i\to\infty}\langle u - z, x_{n_{k_i}+1} - z\rangle$ exists. Consequently, from (25), the following limit also exists

$$\lim_{i\to\infty} -\frac{(1-\alpha_{n_{k_i}})}{\alpha_{n_{k_i}}} \lambda_{n_{k_i}} \left(1 - \frac{\lambda_{n_{k_i}}}{4}\right) \frac{(\|x_{n_{k_i}} - z_{n_{k_i}}\|^2 + \|y_{n_{k_i}}\|^2)^2}{\|x_{n_{k_i}} + y_{n_{k_i}} - z_{n_{k_i}}\|^2}.$$

This together with conditions (C1) and (C2) implies that

$$\lim_{i\to\infty} \frac{(\|x_{n_{k_i}} - z_{n_{k_i}}\|^2 + \|y_{n_{k_i}}\|^2)^2}{\|x_{n_{k_i}} + y_{n_{k_i}} - z_{n_{k_i}}\|^2} = 0,$$

which yields

$$\lim_{i\to\infty} \|x_{n_{k_i}} - z_{n_{k_i}}\| = 0 \quad \text{and} \quad \lim_{i\to\infty} \|y_{n_{k_i}}\| = 0.$$

By a similar proof as in Theorem 3.1, we conclude that any weak cluster point of $\{x_{n_{k_i}}\}$ belongs to Ω. Note that

$$\begin{aligned}\|x_{n_{k_i}+1} - x_{n_{k_i}}\| &\le \alpha_{n_{k_i}}\|u - x_{n_{k_i}}\| + \|u_{n_{k_i}} - x_{n_{k_i}}\| \\ &= \alpha_{n_{k_i}}\|u - x_{n_{k_i}}\| + \tau_{n_{k_i}}\|x_{n_{k_i}} + y_{n_{k_i}} - z_{n_{k_i}}\| \\ &= \alpha_{n_{k_i}}\|u - x_{n_{k_i}}\| + \frac{\|x_{n_{k_i}} - z_{n_{k_i}}\|^2 + \|y_{n_{k_i}}\|^2}{\|x_{n_{k_i}} + y_{n_{k_i}} - z_{n_{k_i}}\|} \\ &\to 0.\end{aligned}$$

This indicates that $\omega_w(x_{n_{k_i}+1}) \subset \Omega$. Without loss of generality, we assume that $x_{n_{k_i}+1}$ converges weakly to $x^\dagger \in \Omega$. Now by (25), we infer that

$$\begin{aligned}\limsup_{n\to\infty} \delta_n &= \lim_{i\to\infty} \delta_{n_{k_i}} \\ &= \lim_{i\to\infty} \Bigg[-\frac{(1-\alpha_{n_{k_i}})}{\alpha_{n_{k_i}}}\lambda_{n_{k_i}}(1 - \frac{\lambda_{n_{k_i}}}{4})\frac{(\|x_{n_{k_i}} - z_{n_{k_i}}\|^2 + \|y_{n_{k_i}}\|^2)^2}{\|x_{n_{k_i}} + y_{n_{k_i}} - z_{n_{k_i}}\|^2} \\ &\quad + 2\langle u - z, x_{n_{k_i}+1} - z\rangle \Bigg] \\ &\le \lim_{i\to\infty} 2\langle u - z, x_{n_{k_i}+1} - z\rangle \\ &= 2\langle u - z, x^\dagger - z\rangle \\ &\le 0\end{aligned}$$

due to the fact that $z = P_\Omega u$ and (8). Finally, applying Lemma 2.3 to (23), we conclude that $x_n \to z$. This completes the proof. ■

4. Concluding remarks

In this paper, we present two new iterative algorithms with selection technique for approximating a solution of the multiple-sets split feasibility problem. In each iteration, our algorithm needs to compute only two projections, one from $\{P_{C_{i_n}} x_n\}_{i=1}^s$ and another one from $\{(I - P_{Q_{j_n}})Ax_n\}_{j=1}^t$. The suggested algorithms are based on the gradient method. We prove that the presented two algorithms have weak and strong convergence, respectively.

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