Dynamics of the Helmholtz Oscillator with Fractional Order Damping

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Abstract: The dynamics of the nonlinear Helmholtz Oscillator with fractional order damping is studied in detail. The discretization of the differential equations according to the Grünwald-Letnikov fractional derivative definition in order to get numerical simulations is reported. Comparison between solutions obtained through a fourth-order Runge-Kutta method and the fractional damping system is commented when the fractional derivative of the damping term $\alpha$ is fixed at 1. That proves the good performance of the numerical scheme. The effect of taking the fractional derivative on the system dynamics is investigated using phase diagrams varying $\alpha$ from 0.5 to 1.75 with zero initial conditions. Periodic motions of the system are obtained at certain range of the damping term. On the other hand, escape of the trajectories from a potential well results at certain critical value of the fractional derivative. The history of the displacement as a function of time is shown also for every $\alpha$ selected.

Keywords: Helmholtz oscillator, Fractional damping, Grünwald-Letnikov fractional derivative.
INTRODUCTION

Fractional differential and integral operators are mathematical tools very useful in engineering and scientific applications. One of these is control systems. The Helmholtz equation, which is being used in many physical problems, is modified to study the dynamics with fractional order operator in the system. The rest of the parameters will keep constants during the work and the focus will be on the effect of just taking fractional order operator in the damping term. Because this parameter plays an important role in the dynamics characteristics, it is necessary to study the impact of its variation in order to get information about fractional dynamics. In this work the Grünewald-Letnikov fractional derivative [1,2,3] is considered for numerical simulations of the Helmholtz system.

FRACTIONAL DAMPED HELMHOLTZ SYSTEM

The Helmholtz oscillator is a nonlinear second order differential equation [4,5,6]. The motion of a particle with unit mass which undergoes a periodic forcing is given by

$$\ddot{x} + \mu \dot{x} + x - x^2 = F \cos(\omega t)$$

(1)

where $\mu, F$ and $\omega$ are positive constants. From the usual point of view, the damping term $\mu$ is proportional to the first order derivative of the displacement $x(t)$. In this work, the first order derivative $\dot{x}$ is replaced by a fractional derivative $D^{\alpha}x$, where $\alpha$ is the fractional damping exponent. The governing equation of the new system is

$$\ddot{x} + \mu D^{\alpha}x + x - x^2 = F \cos(\omega t)$$

(2)
The following property of fractional differential operators is very useful to get a new system with the purpose of numerical simulations.

\[ D^{\alpha_1}D^{\alpha_2}x(t) = D^{\alpha_1+\alpha_2}x(t) \]  

(3)

DISCRETIZATION SCHEME

Equation (3) can be transformed into a new system with three fractional differential equations, which are given by

\[ D^{\alpha}x = y \]  

(4.1)

\[ D^{1-\alpha}y = z \]  

(4.2)

\[ \frac{d}{dt}z = Dz = F\cos(wt) + x^2 - x - \mu y \]  

(4.3)

A numerical solution of the system above obtained by using Grünwald-Letnikov definition has the following form:

\[ x(t_k) = y(t_{k-1})h^{\alpha} - \sum_{j=0}^{k} c_j^{(\alpha)}x(t_{k-j}) \]  

(5.1)

\[ y(t_k) = z(t_{k-1})h^{1-\alpha} - \sum_{j=0}^{k} c_j^{(1-\alpha)}y(t_{k-j}) \]  

(5.2)

\[ z(t_k) = [F\cos(wt_k) + x(t_k)^2 - \mu y(t_k) - x(t_k)]h - \sum_{j=0}^{k} c_j^{(1)}z(t_{k-j}) \]  

(5.3)

where \( h \) is the discrete step. The coefficients \( c_j^{\alpha} \) are the binomial coefficients derived of the Grünwald-Letnikov fractional derivative. \( c_0^{\alpha} = 1 \) and

\[ c_j^{\alpha} = \left(1 - \frac{\alpha + 1}{j}\right)c_{j-1}^{\alpha} \]  

(6)
In the following zero-initial conditions are considered.

**NUMERICAL RESULTS**

In order to test the numerical solutions of system (5), a comparison with a fourth order Runge-Kutta method was implemented when $\alpha = 1$. The positive constants considered are $\mu = 0.8$, $F = 0.46$ and $\omega = 1$. To the discrete step and simulation time, $h = 0.01$ and $T = 200$ was considered for all of the simulations. The phase trajectory obtained with (5) and Runge-Kutta is shown in Fig. 1.

![Fig. 1 Phase trajectory comparison between fractional derivative (red) and Runge-Kutta method (blue).](image)

**Effects of the Fractional Order Damping**

In this work fractional order varies from 0.5 to 1.75 with the same parameters of the comparison with Runge-Kutta. In Fig. 2 are shown cases with various $\alpha$ values. Periodic motions are found through simulations at certain interval. The interval tested was $\alpha \in [0.5, 1.38]$. In Fig. 2(a)-(b), $\alpha = 0.5, 1$, respectively, there is no much qualitative difference in the behavior of the system. Cyclic orbits and the oscillations of $x$ versus $t$ shows that the main difference lies in the amplitude of such oscillations along the time of
the computational experiment. The behavior is very different when $\alpha = 1.38$, as shown in Fig. 2(c). Closed orbits remain, but the phase trajectory and $x(t)$ oscillations become irregular.

From $\alpha = 1.39$, with the same parameters and zero initial conditions, the orbits escape. The escape criteria chosen in this work is: *if* $x(t_k) > 3$ *the trajectory escapes.* When $\alpha = 1.39$ the orbit remains cyclic, but after $t = 51.23$ its escapes (see Fig 3(a)). For larger values of $\alpha$, for example, $\alpha = 1.75$, the escape time of the orbit is shorter. Such case is shown in Fig. 3(b).

**CONCLUSIONS**

The nonlinear dynamics of the Helmholtz oscillator with fractional damping is studied in this work. The *Grünwald-Letnikov* fractional derivative is implemented numerically in order to get numerical simulations of the fractional behavior of the damping in the system. When $\alpha = 1$ the results are the same that the ones obtained with a fourth-order Runge-Kutta method.

In the simulations, the fractional damping exponent $\alpha$ is in the interval from 0.5 to 1.75 and two important behaviors were found. According the results, the $\alpha$ exponent works as a control parameter of the system. When $\alpha \leq 1.38$ the orbits are periodic and bounded and remain between the walls of the potential well. On the other hand, one can argue that $\alpha = 1.39$ is a critical value of the control parameter, so that all the orbits escape. In particular, the bigger $\alpha$ the shorter the escape time from the well.
Fig. 2 Phase trajectory with various $\alpha$

Fig. 3 Escape of the trajectories at certain critical $\alpha$
REFERENCES


