Phase models represent the ideal framework to investigate the synchronization of a nonlinear oscillator with an external forcing. While many researches focused the attention to their analysis, little work has been done about the reduction of a physical system to the corresponding phase model. In this paper we show how, resorting to averaging techniques, it is possible to obtain the phase model corresponding to a given set of state equations. As examples, we derive the phase equations and investigate the synchronization properties of two popular nonlinear oscillators.

Keywords: phase model; synchronization; averaging technique; nonlinear oscillators.

1. Introduction

The behavior of a periodically driven nonlinear oscillator, and the interaction between two coupled oscillatory systems, are classical problems in nonlinear dynamics. In particular the phenomenon of synchronization, that is the adjustment of the rhythms due to the driving signal or the interaction, has received an increasing amount of attention finding numberless applications in engineering,\textsuperscript{1,2} physics,\textsuperscript{3} chemistry,\textsuperscript{4,5} and biology.\textsuperscript{6-8}

The investigation of collective rhythmicity usually focuses on ascertaining the mechanisms responsible for its onset. To achieve a collective coherent behavior, a coupling among the oscillators or an external driving mechanism are required. In contrast, the unavoidable small differences between the oscillators, and thus between their natural frequencies, are destructive to mutual entrainment and to the formation of collective rhythmicity. The conflict between these opposite tendencies, not only induces a threshold between synchronous and asynchronous behaviors, but also determines a plethora of possible entrained states, depending on the relative
phase difference between the system and the forcing.

The research of a model that was, on one hand capable to reproduce the wealth of admissible dynamical behaviors, and on the other simple enough to be mathematically tractable, lead to the introduction of the Kuramoto model. If the unperturbed system has a stable limit cycle in its phase space, we can describe this state by its amplitude and phase. Any perturbation transverse to the cycle will be adsorbed, being associated to negative Lyapunov exponents. Conversely, any perturbation along the cycle will be neither adsorbed nor amplified, since this variable corresponds to the null Lyapunov exponent and thus to the sole neutrally stable direction. This shows why the phase is a privileged variable, and we are justified in describing the state of the system by looking at its phase only.

The equation governing the evolution of the phase, i.e. the phase equation, has been recently studied with great wealth of details, investigating the existence and stability of periodic solutions in chains of oscillators, the onset of synchronization, the effect of the coupling range, the influence of delays, and the formation of patterns. These researches have revealed a great amount of details about the dynamics of coupled or perturbed oscillators from the qualitative point of view, but are less informative from the quantitative point of view. It is interesting to observe that much less attention has been paid to the derivation of phase models. Starting from the physics of a given system, one usually derives a set of state equations, and the problem of reducing these equations to the phase equation arise. The ambition of this paper, is to give a contribution to fill this gap.

Historically, three approaches have been devised to reduce the state equation to the corresponding phase model. Malkin’s theorem provides an analytical formula for the phase deviation, i.e. the deviation from the natural phase of the oscillator due to the coupling or the driving signal, but requires the frequencies of the oscillator and the forcing to be resonant. Winfree and Kuramoto approaches, which differ in details but are equivalent in nature, are based on the concept of isochrones, which have to be computed numerically in Winfree approach, while are determined by a coordinates dependent phase function following Kuramoto.

In this paper we focus the attention on Kuramoto approach, because of its conceptual simplicity and mathematical elegance. Despite these positive aspects, the method does not provide a direct way to find the proper phase function. We show how, resorting to an averaging technique, it is possible to find an analytical approximation of the phase function, and reduce a weakly forced nonlinear oscillator to its phase model. This represents the ideal framework to investigate the conditions under which the nonlinear oscillation synchronizes with the driving signal.

The paper is organized as follows. Section 2 introduces the ideas of phase model and phase dynamics. Section 3 recalls the basic concepts of the method of averaging. Section 4 shows how the phase equation can be derived through the method of averaging and how it can be used to investigate synchronization with an external forcing. In section 5 we apply the method to two popular nonlinear systems, the Stuart–Landau and the Van der Pol oscillators. Section 6 is devoted to conclusions.
2. Phase Model

Synchronization of nonlinear oscillations under the action of an external forcing signal is a classical problem in nonlinear dynamics and engineering that has recently returned topical. For instance, this problem is encountered in the analysis of some micro-circuits for radio frequency applications.\textsuperscript{21,22} We consider a system of nonlinear ordinary differential equations (ODEs)

\[ \dot{x}(t) = f(x(t)) \]  

under the effect of a weak, periodic perturbation, i.e.

\[ \dot{x}(t) = f(x(t)) + \epsilon g(t), \]  

where \( x \in \mathbb{R}^n \) and \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \). We assume that (1) has a stable attractive limit cycle with period \( T_0 \), \( \gamma_0(t) = \gamma_0(t + T_0) \), \( \epsilon \ll 1 \) represents the smallness of the perturbation and \( g(t) \) is assumed to have period \( T \), in general different from \( T_0 \). The weak interaction hypothesis, which at first glance may look restrictive, is justified by the observation that strong interactions impose too strong limitation on the motion of the subsystems, making natural to consider the whole system non decomposable. In other words, strong couplings make the system unified.\textsuperscript{3}

Since \( \gamma_0 \) is homeomorphic to the unit cycle, there exists an homeomorphism, i.e. a continuous mapping with continuous inverse, mapping the points of the cycle to the points of the unit circle (see Fig. 1). We can introduce a parametrization on the limit cycle such that each point \( x \in \gamma_0 \) is associated to a certain phase \( \phi(x) \), which grows monotonically with time, i.e using the chain rule,

\[ \dot{\phi} = \nabla \phi(x) \cdot \dot{x} = 1, \]  

where, without lost of generality, we have assumed the angular frequency in the new parametrization equal to one. This, together with (1) gives

\[ \dot{\phi} = \nabla \phi(\gamma_0(t)) \cdot f(\gamma_0(t)) = 1. \]  

\[ \Gamma \]

\[ \gamma \]

\[ S^1 \]

Fig. 1. Parametrization \( \Gamma \) of the limit cycle \( \gamma \) using a phase variable \( \phi \in S^1 \).

Therefore, the phase function we are looking for is the solution of the partial differential equation (PDE) (4). Since the limit cycle is assumed to be attractive,
every solution with initial conditions lying in a small enough neighborhood of \( \gamma_0 \), asymptotically converges to the limit cycle. We can extend the definition of phase to these trajectories by introducing the concept of isochrones.23,24 Observing the dynamical system (1) stroboscopically, with time interval equal to the period \( T_0 \), we obtain a mapping

\[ \mathbf{x}(t) \mapsto \mathbf{x}(t + T_0) = \Phi(\mathbf{x}). \] (5)

All the points on the limit cycle are fixed points of this mapping. Conversely, for any initial condition in the vicinity of \( \gamma_0 \), we obtain a sequence of points converging to the cycle. Taking \( \mathbf{x}^* \in \gamma_0 \), all the points that are attracted to \( \mathbf{x}^* \) under the action of \( \Phi(\mathbf{x}) \) form a \((n-1)\)-dimensional surface, called isochrone, transverse to the limit cycle at \( \mathbf{x}^* \). The concept of phase is extended to the neighborhood of the limit cycle demanding the phase to be constant on each isochrone. These surfaces are invariant sets of the mapping (5)(see fig. 2).

![Fig. 2. An isochrone \( I(\phi) \) in the vicinity of a stable limit cycle. The solution with initial condition \( x_0 \), observed stroboscopically with time interval \( T_0 \), generates the sequence \( \{x_0, x(T_0), x(2T_0), \ldots\} \) which converges to \( x^* \). The set of all the points that are attracted to \( x^* \) under the action of (5) forms the isochrone, which is an invariant set for this mapping.](image)

Weak perturbations drive the trajectory of (2) away from the limit cycle. Having defined the phase in the vicinity of \( \gamma_0(t) \), it makes sense to talk about the phase of these perturbed orbits. Let us denote by \( \gamma(t) \) the solution (not necessarily periodic) of (2), the phase along this orbit evolves according to

\[ \dot{\phi} = \nabla \phi(\gamma(t)) \cdot (f(\gamma(t)) + \epsilon g(t)). \] (6)

Since \( \gamma_0(t) \) is assumed to be stable, a weak perturbation will not significantly change the trajectory. Up to the first perturbative order, we can approximate \( \gamma(t) \) with \( \gamma_0(t) \), and using (4) eq. (6) yields

\[ \dot{\phi} = 1 + \epsilon \nabla \phi(\gamma_0(t)) \cdot g(t). \] (7)

In order to solve this equation we need to know both the unperturbed limit cycle \( \gamma_0(t) \), solution of the nonlinear ODEs (1), and to compute the gradient of the phase function \( \phi(x) \), which is solution of the PDE (4). In the following sections, we shall show how to obtain analytical approximation of the cycle and the phase function.
3. The Method of Averaging

The simplest way to attain these information is through numerical simulations, but this approach becomes very time consuming when the whole set of initial conditions and parameters values must be explored.

Alternatively, one can resort to approximate analytical methods to obtain the desired information. Spectral techniques like the Harmonic Balance, or perturbative expansions like the Lindstedt–Poincaré method, provide a very accurate analytical approximation of the limit cycle, and have been successfully applied to investigate nonlinear oscillators (both a single oscillator and networks) and their bifurcations. Unfortunately, these methods are unable to capture the transient behavior and thus are not suitable to describe trajectories approaching \( \gamma_0 \). Conversely, the method of averaging (also called method of Krylov–Bogoliubov–Mitropolski), describes weakly nonlinear oscillations in terms of slowly varying amplitude and phase, representing the solution in the ideal form for phase model reduction.

For our purposes we specify (1) as a second order nonlinear system

\[
\ddot{x}(t) + x(t) + \alpha f(x, \dot{x}) = 0. \tag{8}
\]

This equation represents a harmonic oscillator with angular frequency equal to one, plus a nonlinear term, the strength of the nonlinearity measured by \( \alpha \). We remark that any nonlinear oscillator can be rewritten in this form by introducing a properly scaled time variable. We search for a solution of (8) in the form

\[
x(t) = a(t) \cos(t + \Omega(t)) \tag{9}
\]

which implies

\[
\dot{x}(t) = \dot{a}(t) \cos(t + \Omega(t)) - a(t) \left(1 + \dot{\Omega}(t)\right) \sin(t + \Omega(t)). \tag{10}
\]

Additionally, we require the solution to be such that

\[
\dot{a}(t) \cos(t + \Omega(t)) - a(t) \dot{\Omega}(t) \sin(t + \Omega(t)) = 0. \tag{11}
\]

This requirement guarantees that the solution of the form (9) is unique, and is equivalent to require that \( x(t) \) and \( \dot{x}(t) \) are in quadrature. This condition is fulfilled by linear systems, by some strongly nonlinear ones as the Stuart–Landau oscillator, and it is satisfied with a good approximation by weakly nonlinear systems. Using (9) and (11), eq. (8) gives

\[
\dot{a}(t) \sin(t + \Omega(t)) + a(t) \dot{\Omega}(t) \cos(t + \Omega(t)) = \alpha f(a(t), \Omega(t)) \tag{12}
\]

where, to simplify the notation, we have written \( f(a(t), \Omega(t)) \) instead of \( f(a(t) \cos(t + \Omega(t)), -a(t) \sin(t + \Omega(t))) \). Eq. (11) and (12) lead to

\[
\begin{align*}
\dot{a}(t) &= \alpha f(a(t), \Omega(t)) \sin(t + \Omega(t)) \\
\dot{\Omega}(t) &= \frac{\alpha}{a(t)} f(a(t), \Omega(t)) \cos(t + \Omega(t)).
\end{align*} \tag{13}
\]
Eqs. (13) are still exact, since no approximations have been introduced so far, but are not easier to solve than (8). However, if $\alpha$ is a small parameter, $a(t)$ and $\Omega(t)$ are slowly varying (also called nearly constant) variables. Thus we can substitute to the right hand sides of eqs. (13) their mean values over one period, without introducing a great error. This procedure is known as averaging, and yields

$$
\begin{cases}
\dot{a}(t) = \alpha F(a(t)) \\
\dot{\Omega}(t) = \alpha G(a(t)) \frac{a(t)}{a(t)}
\end{cases}, 
$$

(14)

where

$$
F(a(t)) = \frac{1}{2\pi} \int_0^{2\pi} f(a(t), \Omega(t)) \sin(t + \Omega(t)) \, d\Omega
$$

(15)

and

$$
G(a(t)) = \frac{1}{2\pi} \int_0^{2\pi} f(a(t), \Omega(t)) \cos(t + \Omega(t)) \, d\Omega.
$$

(16)

The first of (14) is a nonlinear, first order differential equation, that in some significant case can be easily solved. In particular, at steady state the nonlinear system is expected to oscillate with constant amplitude, and thus equilibrium points are of particular importance. The second of (14), together with (9) inform us that the angular frequency is equal to a constant plus an amplitude dependent function. Thus, in general, a point rotates along solutions associate to different initial conditions with different speeds, and the isochrones are not simple straight lines. In the next section we shall introduce a new phase such that the angular frequency is the same for all the trajectories in the vicinity of the limit cycle.

4. Phase Dynamics, Isochrones and Synchronization

In the phase space, the solution of the nonlinear differential equation (8) is represented by the curve

$$
\dot{\gamma}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = a(t) \begin{pmatrix} \cos(t + \Omega(t)) \\ -\sin(t + \Omega(t)) \end{pmatrix}.
$$

(17)

Inverting these formulas we have the amplitude and phase

$$
a(t) = \sqrt{x^2(t) + y^2(t)}
$$

(18)

$$
\theta(t) = t + \Omega(t) = -\arctan \frac{y(t)}{x(t)}.
$$

(19)

As stated in section 2, we want to introduce a new phase $\phi(x)$, which grows monotonically in time. Let us consider

$$
\phi(a, \theta) = \theta(t) + H(a(t))
$$

(20)

where $H(a(t))$ is an unknown function. By differentiating with respect to time

$$
\dot{\phi} = \dot{\theta}(t) + H'(a(t)) \dot{a}(t),
$$

(21)
using (14) and (19) and integrating we have
\[ H(a(t)) = -\int \frac{G(a)}{a F(a)} \, da \]  \hspace{1cm} (22)
where, without loss of generality, we have taken the arbitrary integration constant to be zero. This definition only makes sense out of the limit cycle (where \( a(t) = \text{const} \) and \( F(a(t)) = 0 \)). On the cycle we simply have \( \phi = \theta \).

Eqs. (18), (19) and (20) allow us to rewrite the phase of the trajectory as a function of the state variables
\[ \phi(x, y) = -\arctan \frac{y}{x} + H(\sqrt{x^2 + y^2}). \]  \hspace{1cm} (23)
Since we expect that all trajectories approach the limit cycle, using (20) we can write
\[ \gamma_\theta(t) = a_\infty \left( \begin{array}{c} \cos(\phi - H(a_\infty)) \\ \sin(H(a_\infty) - \phi) \end{array} \right) \]  \hspace{1cm} (24)
where

\[ a_\infty = \lim_{t \to +\infty} a(t) \]  

(25)

provided the limit exists. Eq. (24) is of the form \( \gamma_0(t) = \Gamma(\phi, a_\infty) \), we can rewrite (7) in the final form

\[ \dot{\phi} = 1 + \epsilon \nabla \phi(\Gamma(\phi, a_\infty)) \cdot \mathbf{g}(t). \]  

(26)

We can now tackle the problem of synchronization, i.e. the situation in which the nonlinear oscillator locks its frequency to that of the external forcing, and the system (2) admits a periodic solution. We consider the most general case of \( m:n \) frequency locking, that is the case \( T_0/T = m/n \), where \( m \) and \( n \) are relative prime nonnegative integers (they do not have a common divisor greater than one). Introducing the phase difference between the free oscillation and the forcing term \( \psi(t) = \phi(t) - \frac{n}{m} \omega t \), with \( \omega = 2\pi/T \), equation (26) yields

\[ \dot{\psi}(t) = \nu + \epsilon q(\psi + \frac{n}{m} \omega t, t) \]  

(27)

where \( \nu = 1 - \frac{n}{m} \omega \) is the frequency mismatch (we have fixed the free running frequency equal to one), and \( q(\psi + \frac{n}{m} \omega t, t) = \nabla \phi(\Gamma(\phi, a_\infty)) \cdot \mathbf{g}(t) \). Close to resonance, \( \nu \to 0 \) and \( \psi \) is a slowly varying function. Thus we can remove the time dependency in (27) by substituting \( q(\psi + \frac{n}{m} \omega t, t) \) with its mean value over one period, obtaining

\[ \dot{\psi}(t) = \nu + \epsilon Q(\psi) \]  

(28)

where

\[ Q(\psi) = \frac{1}{T} \int_0^T q(\psi + \frac{n}{m} \omega t, t) \, dt. \]  

(29)

Equation (28) is a nonlinear, first order differential equation, entrainment with the external forcing occurs if it has a stable equilibrium point. The admissible values of the frequency mismatch lie in the interval

\[ \epsilon \min(-Q(\psi)) \leq \nu \leq \epsilon \max(-Q(\psi)), \]  

(30)

that can be rewritten as

\[ \frac{m}{n} [1 - \epsilon \max(-Q(\psi))] \leq \omega \leq \frac{m}{n} [1 - \epsilon \min(-Q(\psi))]. \]  

(31)

These inequalities define regions in the \((\nu, \epsilon)\) plane, known as Arnold’s tongues, where synchronization can be attained. The phase difference and the stability of the synchronous states are found by looking at zeroes and the sign of the r.h.s. of (28).
5. Applications

5.1. Stuart–Landau oscillator

As a first example we consider the Stuart–Landau oscillator, described by the following ordinary differential equation

\[ \dot{z}(t) = (1 + i \alpha)z(t) - (1 + i \beta)|z(t)|^2, \]  

under the influence of the periodic forcing \( g(t) = (\cos(n \omega t), 0)^T \). Equation (32) admits an exact solution without need to resort to the method of averaging. We consider this example because, introducing polar coordinates, we derive coupled equations for the amplitude and the phase of the oscillations, analogous to equations (14). The instructive part of the example will be the derivation of the phase function and the phase equation. Introducing polar coordinates

\[ \Re\{z\} = a(t) \cos \theta(t), \quad \Im\{z\} = a(t) \sin \theta(t) \]  

equation (32) is transformed into

\[
\begin{cases}
\dot{a}(t) = a(t) \left(1 - a^2(t)\right) \\
\dot{\theta}(t) = \alpha - \beta a^2(t)
\end{cases}
\]  

which can be easily integrated giving

\[ a(t) = \left(1 + \frac{1 - a_0^2}{a_0^2} e^{-2t}\right)^{\frac{1}{2}} \]  

\[ \theta(t) = \theta_0 + (\alpha - \beta)t - \frac{\beta}{2} \ln \left(a_0^2 + (1 - a_0^2) e^{-2t}\right). \]  

where \((a_0, \theta_0)\) are the initial conditions. For any initial condition the trajectory converges toward oscillations of constant amplitudes \(a_\infty = 1\) and constant angular velocity \(\dot{\theta} = \alpha - \beta\). Equation (22) gives

\[ H(a(t)) = -\beta \ln a(t) \]  

and (20)

\[ \phi = \theta(t) - \beta \ln a(t). \]  

We can now give the expression for the phase obtaining

\[ \phi(x, y) = \arctan \frac{y}{x} - \frac{\beta}{2} \ln \left(x^2 + y^2\right). \]  

Computing the partial derivatives we finally obtain the phase equation

\[ \dot{\phi} = \alpha - \beta - \epsilon \left(\beta \cos \phi + \sin \phi, \beta \sin \phi - \cos \phi\right) \cdot g(t). \]  

For the sake of simplicity we consider \(1:n\) resonances only, the time averaged equation for the phase difference can be easily derived

\[ \dot{\psi} = \nu - \frac{\epsilon}{2} \left(\beta \cos \psi + \sin \psi\right) \]
with $\nu = \alpha - \beta - n\omega$. By inspecting the right hand side of (41), we observe that synchronization can be achieved if

$$-\frac{\epsilon}{2} \sqrt{\beta^2 + 1} \leq \nu \leq \frac{\epsilon}{2} \sqrt{\beta^2 + 1}. \quad (42)$$

The corresponding Arnold tongues for different values of $n$ are shown in fig. 4. For $-\frac{\epsilon}{2} \sqrt{\beta^2 + 1} < \nu < \frac{\epsilon}{2} \beta$ we have a stable synchronous state if

$$\psi = -\arccos \left( \frac{2\nu/\epsilon + \sqrt{\beta^2 + 1 - (2\nu/\epsilon)^2}}{\beta^2 + 1} \right), \quad (43)$$

while the synchronous state is unstable if

$$\psi = -\arccos \left( \frac{2\nu/\epsilon - \sqrt{\beta^2 + 1 - (2\nu/\epsilon)^2}}{\beta^2 + 1} \right). \quad (44)$$

For $\frac{\epsilon}{2} \beta < \nu < \frac{\epsilon}{2} \sqrt{\beta^2 + 1}$ the synchronous oscillation is stable if

$$\psi = \arccos \left( \frac{2\nu/\epsilon + \sqrt{\beta^2 + 1 - (2\nu/\epsilon)^2}}{\beta^2 + 1} \right), \quad (45)$$

and unstable if

$$\psi = \arccos \left( \frac{2\nu/\epsilon - \sqrt{\beta^2 + 1 - (2\nu/\epsilon)^2}}{\beta^2 + 1} \right). \quad (46)$$

For $\nu = \pm \epsilon \sqrt{\beta^2 + 1}/2$ the two roots collide and the synchronous state vanish through a saddle–node bifurcation.

Fig. 4. Arnold tongues for 1 : $n$ resonances for the Stuart–Landau oscillator (up to $n = 6$).
5.2. Van der Pol oscillator

Let us consider the classical Van der Pol oscillator

$$\ddot{x}(t) + x(t) + \alpha \left( x^2(t) - 1 \right) \dot{x}(t) = 0.$$ (47)

Equations (14), (15) and (16) give

$$\begin{cases}
\dot{a}(t) = \frac{\alpha}{2} \left( a(t) - \frac{a(t)^3}{4} \right) \\
\Omega(t) = 0,
\end{cases}$$ (48)

which can be integrated giving

$$\begin{cases}
a(t) = \left( \frac{4a_0^2}{a_0^2 + (4 - a_0^2)e^{-\alpha t}} \right)^{\frac{1}{2}} \\
\Omega(t) = \Omega_0.
\end{cases}$$ (49)

We can see that for any initial condition \((a_0, \Omega_0)\) the trajectory converges to an oscillation with constant amplitude \(a_\infty = 2\) and constant angular frequency \(\dot{\theta}(t) = 1\). In this case the angular frequency is amplitude independent, and we can choose

$$\phi(t) = \theta(t) = -\arctan \frac{y(t)}{x(t)}.$$ (50)

Computing the partial derivatives we obtain the simple phase equation

$$\dot{\phi} = 1 - \frac{\epsilon}{2} \left( \sin \phi, -\cos \phi \right) \left( g_1(t), g_2(t) \right)$$ (51)

Now we apply the periodic forcing \(g(t) = (\cos(n \omega t), 0)\), and consider on 1 : \(n\) resonances. It is straightforward to obtain the averaged equation for the phase difference

$$\dot{\psi} = \nu - \frac{\epsilon}{4} \cos \psi$$ (52)

where \(\nu = 1 - n \omega\). The synchronization range is \(-\epsilon/4 \leq \nu \leq \epsilon/4\), and the synchronous states are \(\psi = \pm \arccos \left( \frac{4\nu}{\epsilon} \right)\). The positive root is always unstable, while the negative is stable. At \(\nu = \pm \frac{\epsilon}{4}\) the roots collide and vanish through a saddle–node bifurcation.

6. Conclusions

We have shown the reduction of a periodically driven nonlinear oscillator to the corresponding phase model. When considering structurally stable oscillators, this represents the ideal framework to investigate the synchronization of the oscillation with an external forcing. In fact the phase can already be modulated by a weak external forcing, whereas the amplitude remains almost constant.

The key point is the possibility to describe the limit cycle (as well as the trajectories in its vicinity) in terms of amplitude and phase. If we can find a proper
change of coordinates achieving this task, as in the Stuart–Landau example, the phase equation can be considered exact, at least within the perturbative framework represented by the phase model reduction. When the research of the change of coordinates fails, as it happens in almost all nonlinear oscillators, we can derive analytical approximations of both the limit cycle and the trajectories attracted by it by the method of averaging. This technique gives the solutions in the ideal form for phase model reduction, but in this case the phase equation can be considered reliable only in the weakly nonlinear limit.

As shown in the examples, the method is completely analytic and very simple to apply. The research of periodic oscillation in a second order, nonlinear, non–autonomous differential equation is reduced to the research of equilibrium points in a scalar differential equation. The phase equation can be easily investigated to determine synchronization boundaries (Arnold tongues), phase relations among the oscillator and the forcing, and the stability of the synchronous state by looking at the stability of the corresponding equilibrium point.

The method based on the phase equation gives a clear picture of the synchronization mechanism and possible phase locked states. We expect that this approach can also be applied to networks of coupled oscillators.

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