FRAMEWORK FOR FINITE ELEMENT RESPONSE SENSITIVITY AND RELIABILITY ANALYSES OF STRUCTURAL AND GEOTECHNICAL SYSTEMS

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SUMMARY

This paper presents recent developments in response sensitivity, probabilistic response and reliability analyses of structural and geotechnical systems. These developments are integrated within general-purpose software frameworks for non-linear finite element response analysis and provide the structural engineers with analytical tools to propagate uncertainties through advanced large-scale non-linear simulations to obtain probabilistic estimates of the predicted system response performance. Necessary extensions of the Direct Differentiation Method (DDM) for accurate and efficient computation of finite element response sensitivities are presented. Examples of such response sensitivity computations are shown for structural and Soil-Foundation-Structure-Interaction (SFSI) systems. Uncertainty propagation is illustrated through examples of probabilistic response (First-Order Second-Moment approximation), time-invariant (First- and Second-Order Reliability Method) and time-variant (mean outcrossing rate computation) reliability analyses. These examples are based on two types of analysis, namely pushover analysis and time history analysis, which are used extensively in earthquake engineering.

1. INTRODUCTION

Designing structures to achieve specified performance objectives during future, random earthquakes is an important and very challenging task facing structural engineers. In order to tackle this task, the engineer must account correctly during the design process for all pertinent sources of aleatory and epistemic uncertainties. Thus, proper methods are required for propagating uncertainties from modeling parameters describing the geometry, the material behavior and the applied loading to structural response quantities (engineering demand parameters) used to define the structural performance objectives. These methods need also to be integrated with analysis methodologies already well-known to practitioners, such as the ubiquitous finite element (FE) method.

This paper presents recent developments in response sensitivity, probabilistic response and reliability analyses of structural and geotechnical systems within general-purpose software frameworks (FedelasLab [Filippou and Constantinides, 2004] and OpenSees [Mazzoni et al., 2005]) for non-linear FE response analysis. Current advances are highlighted which cover relevant gaps between response sensitivity computation using the Direct Differentiation Method (DDM) and state-of-the-art FE response analysis. Necessary extensions of the DDM are shown and applied for accurate and efficient computation of FE response sensitivities of structural and Soil-Foundation-Structure-Interaction (SFSI) systems. Application examples of response sensitivity analysis are presented, which show insight into the effect and relative importance of model parameters on the predicted system response. Examples of probabilistic response analysis using First-Order Second-Moment (FOSM) approximation, time-invariant (First- and Second-Order Reliability Method) and time-variant (mean outcrossing rate computation) reliability methods are provided to illustrate the methodology used and its capabilities. These probabilistic response and reliability analyses are based on two types of response analysis used extensively in earthquake engineering, namely pushover analysis and time-history analysis.

2. FINITE ELEMENT RESPONSE SENSITIVITY COMPUTATION

Finite element response sensitivities represent an essential ingredient for gradient-based optimization methods needed in various subfields of structural engineering such as structural optimization, structural reliability analysis, structural identification, and finite element model updating [Ditlevsen and Madsen, 1996; Kleiber et al., 1997]. In addition, FE response sensitivities are invaluable for gaining deeper insight into the effect and relative importance of system and loading parameters in regards to structural response behavior. The computation of FE

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response sensitivities to geometric, material and loading parameters requires extension of the FE algorithms for response-only computation. If \( r \) denotes a generic scalar response quantity, the sensitivity of \( r \) with respect to the geometric, material or loading parameter \( \theta \) is expressed, by definition, as the partial derivative of \( r \) with respect to the variable \( \theta \), considering both explicit and implicit dependencies, evaluated at \( \theta = \theta_0 \), where \( \theta_0 \) denotes the nominal value taken by the sensitivity parameter \( \theta \) for the FE response analysis.

Several methodologies are available for response sensitivity computation, such as the forward/backward/central Finite Difference Method (FDM) [Zhang and Der Kiureghian, 1993; Kleiber et al., 1997; Conte et al., 2003; Conte et al., 2004], the Adjoint Method (AM) [Kleiber et al., 1997], the Perturbation Method (PM) [Kleiber and Hien, 1992] and the Direct Differentiation Method (DDM) [Kleiber et al. 1997; Conte 2001; Conte et al., 2003; Conte et al., 2004; Barbato and Conte, 2005; Zona et al., 2005; Barbato and Conte, 2006; Barbato et al., 2006b]. The FDM is the simplest method for response sensitivity computation, but is also computationally prohibitive and can be negatively affected by numerical noise [Gu and Conte, 2003]. The AM is extremely efficient for linear and non-linear elastic structural models, but is not competitive with other methods for path-dependent problems (i.e., analysis models involving non-linear inelastic material constitutive models) [Kleiber et al., 1997]. The PM is computationally efficient but generally not very accurate. The DDM, on the other hand, is very general, efficient and accurate and is applicable to any material constitutive model. These advantages can be obtained at the one-time cost of differentiating analytically the space (finite element) and time (finite difference) discrete equations describing the structural response and implementing these “exact” derivatives in a FE code.

In the DDM, the consistent FE response sensitivities are computed at each time step, after convergence is achieved for the response computation. This requires the exact differentiation of the FE algorithm for the response calculation with respect to each sensitivity parameter \( \theta \). Consequently, the response sensitivity calculation algorithm affects the various hierarchical layers of FE response calculations, namely: (1) the structure level, (2) the element level, (3) the integration point (section for frame/truss elements) level, and (4) the material level. Details on the derivation of the DDM sensitivity equation at the structure level, sensitivity equations at the element level for classical displacement-based finite elements, specific software implementations, and efficiency properties and accuracy issues of the DDM can be found in [Kleiber et al., 1997; Conte 2001; Conte et al., 2003; Gu and Conte, 2003]. Hereafter, some newly developed algorithms and recent extensions are presented which close gaps between FE response sensitivity computation using the DDM and state-of-the-art FE response-only analysis.

2.1 Response sensitivity algorithm for force-based frame elements

Recent years have seen significant advances in the non-linear analysis of frame structures. Advances were led by the development and implementation of force-based elements [Spacone et al., 1996], which are superior to classical displacement-based elements in tracing material non-linearities such as those encountered in reinforced concrete beams and columns. In a classical displacement-based element, the cubic and linear Hermitian polynomials used to interpolate the transverse and axial frame element displacements, respectively, are only approximations of the actual displacement fields in the presence of non-uniform beam cross-section and/or non-linear material behavior. On the other hand, force-based frame element formulations stem from equilibrium between section and nodal forces, which can be enforced exactly in the case of a frame element. The exact flexibility matrix can be computed for an arbitrary variation of the cross-section along the length of the element and for any section constitutive law. Thus, force-based elements enable, at no significant additional computational costs, a drastic reduction in the number of elements required for a given level of accuracy in the simulated response of a FE model of frame structures.

The advantages of force-based elements in FE response-only analysis of frame structures suggest the extension of the DDM response sensitivity computation algorithm to these elements. The problem is conceptually more complicated than for displacement-based elements, since in a force-based formulation no simple direct relation between the section deformations and the element node deformations is available. In fact, while equilibrium is enforced in a strong form, compatibility is enforced only in a weak form over the element. Solution of this problem has been derived recently and presented in [Conte et al., 2004]. The algorithm developed requires solving, at the element level and at each time-step, a system of linear equations (the size of which depends on the number of integration points for each element) having as unknowns the sensitivities of section deformations and element nodal forces. These quantities are necessary for the solution of the sensitivity equation at the structure level. Another solution, by-passing the solution of the system of linear equations at the element level, has been developed and presented in [Scott et al., 2004].

The benefit of using force-based instead of displacement-based frame elements has been found even more conspicuous when accurate and efficient computation of structural response sensitivities to material and loading parameters is required in addition to response-only computations [Barbato and Conte, 2005]. This benefit in terms of improved accuracy at lower computational cost increases with the complexity of the structural system being analyzed. As application example, a statically indeterminate two-dimensional single-story single-bay steel frame (shown in the inset of Figure 1(a)) with distributed plasticity (modeled by using a Von Mises J\( _p \) plasticity constitutive law with linear kinematic hardening, see [Conte et al., 2003]) subjected to a horizontal force \( P \) at roof level is presented here. Details on the mechanical and geometric properties of the structure and on its modeling can be found in [Barbato and Conte, 2005]. For this simple structure, closed-form solutions are
available for horizontal roof displacement and its sensitivities to material parameters as functions of $P$. Figure 1(a) and (b) compare the horizontal force versus horizontal displacement relations obtained from FE analyses employing different meshes of force-based and displacement-based frame elements, respectively. Similarly, Figure 2(a) and (b) compare the normalized sensitivity to the kinematic hardening modulus of the horizontal displacement obtained from FE analyses employing different meshes of force-based and displacement-based frame elements, respectively. It is found that convergence of the FE response to the exact solutions is much faster when force-based elements are employed and this trend is even stronger for FE response sensitivities.

![Figure 1. Applied horizontal force versus horizontal roof displacement from different FE meshes: (a) using force-based frame elements and (b) using displacement-based frame elements.](image1)

![Figure 2. Sensitivities of roof displacement to kinematic hardening modulus from different FE meshes: (a) using force-based frame elements and (b) using displacement-based frame elements.](image2)

2.2 Response sensitivity algorithm for three-field mixed formulation finite elements

A large body of research has been devoted to mixed FE formulations in the last 30 years. Several finite elements based on different variational principles have been developed [Washizu 1975; Belytschko et al. 2000] and relationships among them have been established. Accuracy and performance have been thoroughly analyzed and improved and important properties have been recognized and explained, such as equivalence between various stress recovery techniques and ability to eliminate shear-locking effects for specific applications [Belytschko et al., 2000]. After more than three decades of research in the field, mixed finite elements are now well established and largely adopted tools in a wide range of structural mechanics applications. Therefore, the advantage of extending the DDM to finite elements based on a mixed formulation is evident.

The DDM algorithm for a three-field mixed formulation based on the Hu-Washizu functional [Washizu, 1975] has been derived and presented in [Barbato et al., 2006b]. This formulation stems from basic principles (equilibrium, compatibility and material constitutive model equations), considers both material and geometric non-linearities, is valid for both quasi-static and dynamic FE analysis and incorporates material, geometric and loading sensitivity parameters. This formulation has also been specialized to frame elements and linear geometry (small displacements and small strains) and been implemented in a Matlab-based general-purpose non-linear FE code (FedeasLab).

2.3 Extension of the DDM to steel-concrete composite frame structures

The last decade has seen a growing interest in FE modeling and analysis of steel-concrete composite structures, with applications to seismic resistant frames and bridges [Spacone and El-Tawil, 2004]. The behavior of composite beams, made of two components connected through shear connectors to form an interacting unit, is significantly influenced by the type of connection between the steel beam and the concrete slab. Flexible shear
connectors allow the development of partial composite action and, for accurate analytical response predictions, structural models of composite structures must account for the interlayer slip between the steel and concrete components. Thus a composite beam finite element able to capture the interface slip is an essential tool for model-based response simulation of steel-concrete composite structures.

Compared to common monolithic beams, composite beams with deformable shear connection present additional difficulties. Even in very simple structural systems (e.g., simply supported beams), complex distributions of the interface slip and force can develop. Different finite elements representing composite beams with deformable shear connection have been proposed in the literature [Dall'Asta and Zona, 2004; Spacone and El-Tawil, 2004]. In addition, suitable models describing the section deformations and computing the section force resultants of steel-concrete composite structures with prescribed shear-slip behavior at the interface between the two components (steel beams and reinforced concrete slab) are also necessary. This task also requires the use of realistic material constitutive models for beam steel, reinforcement steel, concrete and shear-slip behavior of the studs connecting the two structural components [Zona et al., 2005, 2006].

The DDM has been recently extended for response sensitivity computation of steel-concrete composite frame structures [Zona et al., 2005, 2006; Barbato et al., 2006b]. Thus, advanced finite elements incorporating the deformable shear-connection between the two structural components of steel-concrete composite structures can be used for efficient computation of both the response and response sensitivities. Figure 3(a) shows the geometry of a two-span asymmetric continuous steel-concrete composite beam for which experimental data are available. Figure 3(b) depicts the degrees of freedom of the frame element with deformable shear connection used in modeling the structure. Figure 3(c) presents a comparison between experimental results and the numerical simulation of the structural response. The agreement of the numerical results to the experimental records is very good.

Figure 3. Application example of steel-concrete composite structure: (a) geometry and loading, (b) FE degrees of freedom and (c) comparison of experimental and numerical results.

Figure 4(a) displays the normalized sensitivities (i.e., multiplied by the nominal value of the sensitivity parameter and divided by the response quantity itself) of the vertical uplift \( v_3 \) at midpoint of the non-loaded span to several material parameters as function of the normalized vertical uplift (i.e., ratio between the current vertical uplift and the maximum uplift which is reached at the failure of the structure). The normalized sensitivities can be used directly as importance measure of the sensitivity parameter for the considered response quantity, since they can be interpreted as percent change in the response due to one percent change in the parameter. In the case presented here, the yield strength of the beam steel, \( f_y \), is the parameter affecting most significantly the vertical uplift \( v_3 \). Figure 4(b) shows the normalized sensitivities of several response quantities to parameter \( f_y \) as function of the normalized vertical uplift. The effects of the parameter \( f_y \) are pronounced for \( v_3 \), but much less so for the beam rotations at the left and central supports, \( \phi_1 \) and \( \phi_2 \), respectively.

Figure 4. Normalized response sensitivities for steel-concrete composite beam: (a) sensitivities of vertical uplift at midpoint of non-loaded span to several material constitutive parameters and (b) sensitivities of several response quantities to yield strength of the beam steel.
2.4 Extension of the DDM to Soil-Foundation-Structure-Interaction (SFSI) systems

The seismic excitation experienced by structures (buildings, bridges, etc.) is a function of the earthquake source (fault rupture mechanism), travel path effects, local site effects, and SFSI effects. Irrespective of the presence of a structure, the local soil conditions (stratification of subsurface materials) may change significantly, through their dynamic filtering effects, the earthquake motion (seismic waves) from the bedrock level to the ground surface. The complex and still poorly understood interactions between subsurface materials, foundations, and the structure during the passage of seismic waves is further significantly complicated by clouds of uncertainties associated with the various components of a SFSI system as well as the seismic excitation. A comprehensive and efficient analytical methodology is necessary for studying the propagation of uncertainties in non-linear dynamic analysis of SFSI systems for performance-based earthquake engineering.

The DDM has been extended for the analysis of SFSI systems, with response sensitivity algorithms for 2-dimensional (quadrilateral) and 3-dimensional (brick) isoparametric finite elements, soil materials and handling of multipoint constraints required for connecting the foundations to the soil domain in the FE model. A benchmark example of SFSI system is shown in Figure 5. A detailed description of this benchmark problem can be found in [Barbato et al., 2006a].

![Figure 5. Geometry, input earthquake ground motion and soil material constitutive model for the benchmark example of SFSI system.](image)

Figure 6(a) displays the time histories of the soil interlayer drifts in the x-direction, while Figure 6(b) shows the normalized sensitivities of the first interstory drift in the x-direction, $\Delta_{1x}$, to the shear strength parameter of each of the four soil layers. In this case, the parameters affecting most significantly the response quantity $\Delta_{1x}$ are the shear strengths of the two deeper soil layers, since they control the seismic forces/energy transferred by the soil from the earthquake input applied at the base of the computational soil domain.

![Figure 6. Benchmark SFSI system: (a) time histories of the soil interlayer drifts and (b) sensitivities of the building first interstory drift to the shear strengths of the soil layers.](image)
3. FINITE ELEMENT PROBABILISTIC RESPONSE ANALYSIS

Probabilistic response analysis consists of computing the probabilistic characteristics of the response of a specific structure, given as input the probabilistic characterization of the material, geometric and loading parameters. An approximate method of probabilistic response analysis is the First-Order Second-Moment (FOSM) method, in which mean values (first-order statistical moments), variances and covariances (second-order statistical moments) of the response quantities of interest are estimated by using a mean-centered, first-order Taylor series expansion of the response quantities in terms of the modeling parameters modeled as random variables. Thus, this method requires only the knowledge of the first- and second-order statistical moments of the random parameters. It is noteworthy that often statistical information about the random parameters is limited to first and second moments and therefore probabilistic response analysis methods more advanced than FOSM analysis cannot be fully exploited.

Given the vector of n random parameters \( \theta \), the corresponding covariance matrix \( \Sigma_\theta \) is defined as
\[
\Sigma_\theta = \begin{bmatrix} \rho_{ij} \sigma_i \sigma_j \end{bmatrix} ; \quad i,j = 1, 2, \ldots, n
\]
where \( \rho_{ij} \) denotes the correlation coefficient of random parameters \( \theta_i \) and \( \theta_j \) (\( \rho_{ii} = 1 \); \( i = 1, 2, \ldots, n \)), and \( \sigma_i \) and \( \sigma_j \) are the standard deviations of random parameters \( \theta_i \) and \( \theta_j \), respectively. The vector \( \mathbf{r} \) of m response quantities of interest is approximated by a first-order truncation of its Taylor series expansion in the random parameters \( \theta \) about their mean values \( \mu_\theta \) as
\[
\mathbf{r}(\theta) \approx \mathbf{r}(\theta = \mu_\theta) + \nabla \mathbf{r}_{\mu_\theta} \cdot (\theta - \mu_\theta)
\]
(2)
The first- and second-order statistical moments of the response quantities \( \mathbf{r} \) are approximated by the corresponding moments of the linearized response quantities, i.e.,
\[
\mu_r \approx \mu_{r_l} = E[\mathbf{r}_l(\theta)] = \mu_{r_l} + \nabla \mathbf{r}_{\mu_\theta} \cdot E[\theta - \mu_\theta] = \mu_{r_l}
\]
\[
\Sigma_r \approx \Sigma_{r_l} = E[(\mathbf{r}_l(\theta) - \mu_{r_l})(\mathbf{r}_l(\theta) - \mu_{r_l})^\top] = \nabla \mathbf{r}_{\mu_\theta} \cdot \Sigma_{\theta} \cdot (\nabla \mathbf{r}_{\mu_\theta})^\top
\]
(4)
in which \( E[...] \) denotes the mathematical expectation operator.

The approximate response statistics computed using Eqs. (3) and (4) are extremely important in evaluating the variability of the response quantities of interest due to the intrinsic uncertainty of the modeling parameters and provide information on the statistical correlation between the different response quantities. It is noteworthy that these approximate first- and second-order response statistics can be readily obtained when response sensitivities evaluated at the mean values of the random parameters are available. Only a single FE analysis is needed in order to perform a FOSM probabilistic response analysis, when the FE response sensitivities are computed using the DDM. Probabilistic response analysis can also be performed using Monte Carlo simulation (MCS). In this study, MCS is used to assess the accuracy of the FOSM approximations in Eqs. (3) and (4) when applied to nonlinear FE response analysis of R/C building structures characterized with random/uncertain material parameters and subjected to quasi-static pushover. The MCS procedure requires:

1. Generation of \( N \) realizations of the n-dimensional random parameter vector \( \theta \) according to a given n-dimensional joint probability density function (PDF).
2. Computation by FE analysis of \( N \) pushover response curves for each component of the response vector \( \mathbf{r} \), corresponding to the \( N \) realizations of the random parameter vector \( \theta \).
3. Statistical estimation of specified marginal and joint moments of the components of response vector \( \mathbf{r} \) at each load step of the FE response analysis.

MCS is a general and robust methodology for probabilistic response analysis, but it suffers two significant limitations: (1) it requires knowledge of the joint PDF of the random parameters \( \theta \), which, in general, is only partially known and (2) it requires performing a usually large number of FE response analyses, which often is computationally prohibitive.

In this study, the Nataf model [Ditlevsen and Madsen, 1996] was used to generate realizations of the random parameters \( \theta \). It requires specification of the marginal PDFs of the random parameters \( \theta \) and their correlation coefficients. It is therefore able to reproduce the given first- and second-order statistical moments of the random parameters \( \theta \). The same three-dimensional three-story reinforced concrete building presented in Section 2.4, but on rigid supports, is considered as an example. Table 1 provides the characteristic of the marginal distributions of the material parameters modeled as random variables. Other details on the modeling of the structure and random parameters can be found in [Barbato et al., 2006a]. Figure 7(a) compares the estimates of the mean value ± 1 standard deviation of the roof displacement in the x-direction, \( u_{x_3} \), for a quasi-static pushover analysis with an upper-triangular pattern of applied horizontal forces, obtained using FOSM and MCS. Figure 7(b) provides the estimates of the standard deviation of \( u_{x_3} \), obtained by using MCS and FOSM with sensitivities computed through DDM and backward/forward finite differences (BFD and FFD, respectively) and their average. It is found that a DDM-based FOSM analysis can provide at a very low computational cost estimates of the first- and second-order response statistics which are in good agreement with more expensive MCS estimates when the structural systems experience low-to-moderate non-linearities.
Figure 7. Comparison of probabilistic response analysis results for $u_{3x}$ obtained from FOSM and MCS: (a) mean value ± 1 standard deviation and (b) standard deviation estimates.

Table 1. Marginal PDFs of material parameters (parameters for lognormal distribution: (1) $\lambda = \mu_{\ln(X)}$ (2) $\zeta = \sigma_{\ln(X)}$; for beta distribution: (1) $x_{\min}$ (2) $x_{\max}$ (3) $\alpha_1$, (4) $\alpha_2$)

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4. FINITE ELEMENT RELIABILITY ANALYSIS

In general, a structural reliability problem consists of computing the probability of failure $P_f$ of a given structure, which is defined as the probability of exceedence of a specified limit-state (or damage-state) when the loading(s) and/or structural properties and/or parameters in the limit-state functions are uncertain/random quantities modeled as random variables. We focus here on component reliability problems, i.e., we consider a single limit-state function $g = g(r, \theta)$, where $r$ denotes a vector of response quantities of interest and $\theta$ is the vector of all basic random variables used to define the structure. The limit-state function $g$ is chosen such that $g \leq 0$ defines the failure domain/region. Thus, the time-invariant component reliability problem takes the following mathematical form

$$P_f = P[g(r, \theta) \leq 0] = \int_{g(r, \theta) \leq 0} p_{\theta}(\theta) d\theta$$

(5)

where $p_{\theta}(\theta)$ denotes the joint probability density function (PDF) of random variables $\theta$. For time-variant reliability problems, an upper bound of the probability of failure, $P_f(T)$, over the time interval $[0, T]$, can be obtained as

$$P_f(T) \leq \int_0^T v_g(t) dt$$

(6)

where $v_g(t)$ denotes the mean down-crossing rate of level zero of the limit-state function $g$ and $t$ represents the time. An estimate of $v_g(t)$ can be obtained numerically from the limit form relation [Hagen and Tvedt, 1991]

$$v_g(t) = \lim_{\delta t \to 0} \frac{P[g(r(\theta(t), \theta) > 0 \cap g(r(\theta, t+\delta t), \theta) \leq 0]}{\delta t}$$

(7)

The numerical evaluation of the numerator of Eq. (7) reduces to a time-invariant two-component parallel system reliability analysis. It is clear that Eq. (5) represents the building block for the solution of both time-invariant and time-variant reliability problems [Der Kiureghian 1996].

The problem posed in Eq. (5) is extremely challenging for real-world structures and can be solved only in approximate ways. A well established methodology consists of introducing a one-to-one mapping/transformation between the physical space of variables $\theta$ and the standard normal space of variables $y$ [Ditlevsen and Madsen, 1996] and then computing the probability of failure $P_f$ as
\[ P_f = P\{G(y) \leq 0\} = \int_{G(y) \leq 0} \varphi_y(y) \, dy \quad (8) \]

where \( \varphi_y(y) \) denotes the standard normal joint PDF and \( G(y) = g(r(\theta(y)), \theta(y)) \) is the limit-state function in the standard normal space. Solving the integral in Eq. (8) remains a formidable task, but this new form of \( P_f \) is suitable for approximate solutions taking advantage of the rotational symmetry of the standard normal joint PDF and its exponential decay in both the radial and tangential directions. An optimum point at which to approximate the limit-state surface \( G(y) = 0 \) is the “design point” (DP), which is defined as the most likely failure point in the standard normal space, i.e., the point on the limit-state surface that is closest to the origin. Finding the design point is a crucial step of semi-analytical approximate methods to evaluate the integral in Eq. (8), such as FORM, SORM and importance sampling [Breitung 1984; Au et al., 1999]. The DP, \( y^* \), is found as solution of the following constrained optimization problem:

\[ y^* = \arg\min \left\{ 0.5 y^T y \mid G(y) = 0 \right\} \quad (9) \]

The most effective techniques for solving the constrained optimization problem in Eq. (9) are gradient-based optimization algorithms [Liu and Der Kiureghian, 1991] coupled with algorithms for accurate and efficient computation of the gradient of the constraint function \( G(y) \), requiring computation of the sensitivities of the response quantities \( r \) to parameters \( \theta \). In fact, using the chain rule of differentiation for multi-variable functions, we have

\[ \nabla G = \nabla g \cdot \nabla r + \nabla g \cdot \nabla \theta \quad (10) \]

where \( \nabla g \) and \( \nabla \theta \) are the gradients of limit-state function \( g \) with respect to its explicit dependency on quantities \( r \) and \( \theta \), respectively, which can usually be computed analytically; the term \( \nabla r \) denotes the sensitivities of response variables \( r \) to parameters \( \theta \); and \( \nabla \theta \) is the Jacobian matrix of the transformation from the standard normal space to the physical space. For probability distribution models defined analytically, the gradient \( \nabla r \theta \) can be derived analytically as well [Ditlevsen and Madsen, 1996].

For real-world problems, the response simulation (computation of \( r \) for given \( \theta \) is typically performed using advanced mechanics-based non-linear computational models developed based on the FE method. Finite element reliability analysis requires augmenting existing FE formulations for response-only calculation in order to compute the response sensitivities, \( \nabla r \theta \), to parameters \( \theta \). As already seen in Section 1, an accurate and efficient way to perform FE response sensitivity analysis is through the DDM.

### 4.1 Time-invariant reliability analysis

A time-invariant reliability analysis has been performed on the same three-story reinforced concrete building used in Section 3. The probabilistic characterization of the material constitutive parameters also remains the same as in Section 3. In addition, the maximum total applied horizontal force (equal to the maximum base shear) is modelled as lognormal random variable (see Figure 8(d)). A maximum roof displacement \( u_{x3} = 0.3 \text{m} \) (corresponding to a maximum roof drift ratio of 2.8%) has been considered as failure condition (or limit-state).

First, a DP search has been performed (see Figure 8(a)) and a FORM approximation of the probability of failure has been obtained. Then, using the DP found in the FORM analysis, a SORM estimate has been obtained by computing the first principal curvature at the DP of the limit-state surface and correcting the FORM result with the Breitung’s formula [Breitung 1984]. Finally, importance sampling (IS) has been performed using as sampling distribution a joint standard normal distribution centered at the DP. It is found that the three analysis results are in good agreement and that employing SORM with the largest-only principal curvature gives a considerable improvement in the estimate of the probability of failure, assuming that IS provides the best estimate of the probability of failure (Figure 8(b) and (c)).

![Figure 8. Time-invariant reliability analysis of 3-story R/C building: (a) DP and mean responses, (b) analysis results, (c) importance sampling parameters, and (d) probabilistic characterization of \( P_{tot} \).](image)
4.2 Time-variant reliability analysis

The presented methodology has been tested on simple structures for time-variant reliability analysis. Mean outcrossing rates are estimated by FORM, first for linear elastic SDOF and MDOF structures with at rest initial conditions and subjected to white noise excitation. Very good agreement was obtained with available closed-form solutions for the mean out-crossing rates (see Figure 9(a) and (b) for SDOF and MDOF, respectively) when a sufficiently small time-interval, \( dt \), is used in discretizing the white noise input process.

![Figure 9. Mean out-crossing computation for linear elastic structures subjected to white noise with at rest initial conditions: (a) SDOF system \( \omega = 20 \text{ rad/s}, \xi = 10\% \) and (b) 3-DOF steel building.](image)

The same methodology is used for inelastic SDOF systems modelled using the Menegotto-Pinto (MP) constitutive law with different yield strengths for the same displacement threshold (thus producing different levels of non-linearity in the response). In Figure 10(a), the mean out-crossing rate time histories are shown for two SDOF systems with the same (deterministic) initial stiffness \( E_0 = 400 \text{kN/m} \) and mass \( M = 10^3 \text{kg} \) and with (deterministic) yield strength \( F_y = 200 \text{kN} \) (high non-linearity, HNL) and \( 520 \text{kN} \) (low non-linearity, LNL), respectively, and a common deterministic displacement threshold \( \delta = 1.3 \text{m} \). Figure 10(a) shows the mean outcrossing rate time histories for the two SDOF systems. The corresponding stress-strain responses (for 2.5s of excitation) at the DP are provided in the insets of Figure 10(a). As expected, the mean out-crossing rate is higher for the SDOF system with lower yield strength. Figure 10(b) compares the estimates of the expected number of out-crossing events obtained using FORM and MCS (for which the mean ± 1 standard deviation interval of the estimate is also provided). For LNL, the results obtained using FORM are in excellent agreement with those obtained via MCS. On the other hand, a FORM approximation of the limit-state surface for HNL yields a very crude approximation of the mean out-crossing rate. New accurate and efficient methods are needed to better approximate the probability content of the very narrow and highly non-linear failure domains for mean outcrossing rate computation in the case of HNL systems.

![Figure 10. Time-variant reliability analysis of non-linear hysteretic SDOF systems: (a) mean out-crossing rates and (b) expected number of out-crossings.](image)

5. CONCLUSIONS

This paper presents recent advances in FE response sensitivity, probabilistic response and reliability analyses of structural and/or geotechnical systems. These advances are integrated into general-purpose software frameworks for non-linear FE response analysis. The objective is to extend analytical/numerical tools familiar to structural engineers for propagating uncertainties through advanced realistic non-linear response analyses of structures to obtain probabilistic estimates of structural performance. Extensions of the Direct Differentiation Method (DDM) for accurate and efficient computation of structural response sensitivities are first shown. Then, First-Order Second-Moment (FOSM) probabilistic response analysis, and time-invariant and time-variant reliability analysis are presented and illustrated with some examples. The methodology presented yields, at reasonable computational cost, probabilistic results that, up to moderate inelastic structural behavior, are sufficiently
accurate for engineering purposes. However, more accurate and efficient methods are still needed for computing the time-invariant and time-variant probability of failure for highly non-linear structural behavior.

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