A new tractable combinatorial decomposition

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Abstract. This paper introduces the u-modules, a generalisation of modules for the homogeneous relations. We first present some properties of the umodule family, then show that, if the homogeneous relation fulfills some natural axioms, the umodule family has a unique decomposition tree. We show that this tree can be computed in polynomial time, under a certain size assumption. We apply this theory to a new tournament decomposition and a graph decomposition. In both cases, the decomposition tree computing time becomes linear. Moreover, we characterise the completely decomposable tournaments. Finally we give polynomial-time algorithms for the maximal umodules, the undecomposability of a relation, and its decomposition tree. We conclude with further applications of homogeneous relations.

1 Introduction

Many authors have worked around modular decomposition\textsuperscript{[7,3,14,6]} or its variations\textsuperscript{[4,11,15]}. In the companion paper\textsuperscript{[1]} was introduced a generalisation for modular decomposition based on the so-called homogeneous relations, combinatorial structures more general than graphs. We study here a decomposition scheme which generalises both modular decomposition\textsuperscript{[7,14]} and bi-join decomposition\textsuperscript{[15]} from this viewpoint, and leads to a new tournament decomposition.

Modular decomposition on graphs is based on subsets of vertices \(M\) called modules, which have no splitters. A vertex is called a splitter of a subset if it distinguishes some elements of the subset from others (it is for instance linked with only some vertices). The “outside” of a module constitutes therefore the same ordered partition for all vertices of the module (for instance, in an undirected graph modules, all vertices have the same neighbourhood). We consider here u-modules. The outside of a umodule constitutes, by definition, the same unordered partition for all vertices of the umodule (the \(u\) of “u-module” came from there). For undirected graphs, the same set \(N\) is either the neighbourhood, or the non-neighbourhood, of all vertices. See Figure 1 for examples on graphs.

Although the modules of homogeneous relations inherit many interesting properties from graph modules\textsuperscript{[9]}, they do not necessarily satisfy the following essential one. One can shrink a whole graph module \(M\) into one single vertex \(m\): if some vertex of \(M\) distinguishes two exterior vertices, then so does every vertex of \(M\) and so does \(m\). Actually, this property is the basis of many divide-and-conquer paradigms derived from the modular decomposition framework, such as the computations of weighted maximal stable set or clique set, and graph colouring\textsuperscript{[13,8]}. That is what motivates us to study the family of u-modules — subsets of elements that fulfil by definition the above property.

Our central notion, namely the umodules, generalises the “decomposition frame with the intersection and transitivity properties” of Cunningham\textsuperscript{[4]} (also known under the different formalisms of “bipartitive families”\textsuperscript{[3]} or “unrooted set families”\textsuperscript{[11]}) as soon as an axiom, namely the \textit{the four elements condition} is fulfilled. This has important structural consequences, both theoretical and algorithmic.

As an application of this new general decomposition scheme we exhibit a new decomposition of tournaments which broadly generalises modular decomposition, and moreover we propose a linear time algorithm to handle this decomposition. Another direct application is the bi-join decomposition\textsuperscript{[15]}. For some important cases of umodular decomposition (including the above bi-join decomposition and tournament decomposition) we prove that a nice transformation can be made in order to reduce the problem into a general modular decomposition as studied in\textsuperscript{[1]}. This transformation is known as \textit{local complementation} also called Seidel switch and it is an important non trivial transformation in graphs.

For the general umodular decomposition, \(O(|X|^5)\) time algorithms are proposed. More precisely, if the family has the \textit{self-complemented} property, then it has a compact representation as a tree with \(|X|\) leaves.
Throughout this section $X$ is a finite set. The family of all subsets of $X$ is denoted by $\mathcal{P}(X)$. A 
reflectless triple is $(x, y, z) \subseteq X^3$ with $x \neq y$ and $x \neq z$. Reflectless triples will be denoted $(x|yz)$ instead of $(x, y, z)$ since the first element does not play the same role. Let $H$ be a boolean relation over the reflectless triples of $X$. Given $s \in X$, the relation $H_s$ is a binary relation on $X$ defined as $H_s(x, y)$ if and only if $H(s|xy)$. In Fig. 1 are defined homogeneous relations along with their modules.

Definition 1. [Homogeneous relation and Modules] $H$ is a homogeneous relation on $X$ if, for all $s \in X$, $H_s$ is an equivalence relation on $X \setminus \{s\}$, $M \subseteq X$ is a module of $H$ if $\forall m, m' \in M \ \forall x \in X \setminus M \ H(x|mm')$. The family of modules of $H$ is denoted $\mathcal{M}_H$ or $\mathcal{M}$ if not ambiguous. $M$ is a trivial module if $|M| \leq 1$ or $M = X$. $H$ is prime w.r.t. modules, or M-prime, if $\mathcal{M}_H$ is reduced to the trivial modules.

If $\neg H(x|yz)$ we say that $x$ distinguishes $y$ from $z$. The notions of homogeneity and distinction can be applied to graphs. Indeed, there is a natural homogeneous relation associated to the case of (directed) graphs as follows.

Definition 2 (Standard homogeneous relation of digraphs). The homogeneous relation $H(G)$ of a directed graph $G = (X, E)$ is defined such that, for all $a, b, c \in X$, $H(a|bc)$ is true if and only if the following conditions hold: 1. either both $b$ and $c$ or none of them are in-neighbours of $a$, and 2. either both $b$ and $c$ or none of them are out-neighbours of $a$.

In other word, $H(a|bc)$ tells if $a$ “sees” $b$ and $c$ in the same way. Of course this definition also holds for undirected graphs, tournaments, and can also be extended to 2-structures $\mathbb{2}$.

Proposition 1. For a graph $G$, and $H$ its standard homogeneous relation, the modules of $H$ are the modules of $G$ in the usual sense $\mathbb{3}$.

Let us now introduce the central notion of this paper, the umodules, which, thanks to Proposition $\mathbb{4}$, can be seen as a proper generalisation of modules.

Definition 3 (umodules). A subset $U$ of $X$ is an umodule of $H$ if

$$\forall u, u' \in U \ \forall x, x' \in X \setminus U \ H(u|xx') \iff H(u'|xx')$$

The family of umodules of $H$ is denoted $\mathcal{U}_H$ or $\mathcal{U}$ if not ambiguous. $U$ is a trivial umodule if $|U| \leq 1$ or $|U| \geq |X| - 1$. $H$ is prime w.r.t. umodules, or U-prime, if $\mathcal{U}_H$ is reduced to the trivial umodules.

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The umodules are in some sense the dual of the modules and this can be seen in the following property.

**Proposition 2.** Let $H$ be a homogeneous relation. If $M$ is a module then $X \setminus M$ is a umodule.

**Proof.** Obvious, since no pair of element belonging to $X \setminus M$ can distinguish two elements of Module $M$.  

The following results link umodules to the 1-intersecting families framework as defined in [3].

**Proposition 3.** Let $U$ be the family of umodules w.r.t. to $H$. For any two umodules $U, U' \in U$, if $U \cap U' \neq \emptyset$ then $U \cup U' \in U$.

Unfortunately the umodule family of an arbitrary homogeneous relation has thus, as far as we know, few structural properties. We shall now survey some homogeneous relations with additional properties, making the umodule family behave in a more tractable manner. To avoid examples similar to that described Fig. 1 (right), let us consider a regularity condition on $H$.

**Definition 4.** ([Four Elements Condition]) The homogeneous relation $H$ fulfills the four elements condition if $\forall m, m', x, x' \in X \ (H(m|xx') \land H(m'|xx') \land H(x|mm') \Rightarrow H(x'|mm')) \land (\neg H(m|xx') \land \neg H(m'|xx') \land H(x|mm') \Rightarrow \neg H(x'|mm'))$.

This condition is a very important one since it allows to contract an umodule and therefore to apply the divide and conquer paradigm to solve optimisation problems.

**Definition 5 (Self-complementation).** A family $\mathcal{F}$ of subsets of $X$ is self-complemented if for every subset $A$, $A \in \mathcal{F}$ implies $X \setminus A \in \mathcal{F}$.

**Proposition 4.** If the relation $H$ fulfills the four elements condition then the family $\mathcal{U}$ of umodules of $H$ is self-complemented.

**Proof.** Let us assume that $U$ is a umodule and $X \setminus U$ is not, i.e. there are two elements $x$ and $x'$ of $X \setminus U$, and two elements $m$ and $m'$ of $X$ such that $H(x|mm')$ but $\neg H(x'|mm')$. As $U$ is a umodule, either both $m$ and $m'$ distinguish $x$ from $x'$ (i.e. $\neg H(m|xx')$ and $\neg H(m'|xx')$) or none of $m$ and $m'$ distinguish $x$ from $x'$ (i.e. $H(m|xx')$ and $H(m'|xx')$). The first case is forbidden by the first implication of the Four Elements property, and the second case is forbidden by the second implication.

Notice that this is just an implication, the characterisation of relations having a self-complemented umodules family by a local axiom like the Four Elements condition appears to be more difficult.

In the following, we first study self-complemented families and then focus on homogeneous relations $H$ having at most 2 equivalence classes for each element. A general correspondence between umodules of $H$ and modules of a local complementation of $H$ is then provided. Applying these results to tournaments provides a new interesting combinatorial decomposition together with a linear time algorithm. When applied to undirected graphs, it yields bi-join decomposition [13]. In the last section a polynomial algorithm time is provided to compute maximal umodules.

### 3 Self-complemented umodular Families

We have seen (Proposition 3) that umodular families are closed under union of intersecting subsets. Self-complemented umodular families have however stronger properties that we shall now survey. The two applications presented here, namely decompositions on graphs and on tournaments, are indeed based on self-complemented umodular families.
3.1 Unique Decomposition Tree property

We shall now describe a general decomposition framework which allows us to prove the existence of a unique decomposition tree for umodular decomposition.

\( \{X^1_i, X^2_i\} \) is a bipartition of \( X \) if \( X^1_i \cup X^2_i = X \) and \( X^1_i \cap X^2_i = \emptyset \). Two bipartitions \( \{X^1_i, X^2_i\} \) and \( \{X^1_j, X^2_j\} \) overlap if for all \( a, b = 1, 2 \) the four intersections \( X^a_i \cap X^b_j \) are not empty. Let \( B = \{ \{X^1_i, X^2_i\} \mid i = 1, \ldots, k \} \) be a family of \( k \) bipartitions of \( X \). The strong bipartitions of \( X \) do not overlap any other bipartition. A bipartition is trivial if one part has only one element. Clearly, the trivial bipartitions contained in any family of bipartitions are strong.

The following results on bipartitions can be found in [4] under the name of “decomposition framework with intersection and transitivity properties”, in [5] under the name of “bipartitive families” (the formalism used here) and in [4] under the name of “unrooted set families”.

**Proposition 5.** If \( B \) contains all trivial bipartitions of \( X \), then there exists a unique tree \( T(B) \)

- with \( |X| \) leaves, each leaf being labelled by an element of \( X \).
- such that each edge \( e \) of \( T(B) \) correspond to a strong bipartition of \( B \): the leaf labels of the two connected components of \( T - e \) are exactly the two parts of a strong bipartition, and the converse also holds.

Let \( N \) be a node of \( T(B) \) of degree \( k \). The labels of the leaves of the connected components of \( T - N \) form a partition \( X_1, \ldots, X_k \) of \( X \). For \( I \subset \{1, \ldots, k\} \) with \( 1 < |I| < k \), the bipartition \( B(I) \) is \( \{ \bigcup_{i \in I} X_i, X \setminus \bigcup_{i \in I} X_i \} \).

**Definition 6 (Bipartitive Family).** A family of bipartitions is a bipartitive family if it contains all the trivial bipartitions and if, for two overlapping bipartitions \( \{X^1_i, X^2_i\} \) and \( \{X^1_j, X^2_j\} \), the four bipartitions \( \{X^a_i \cup X^b_j, X \setminus (X^a_i \cup X^b_j)\} \) (for all \( a, b = 1, 2 \)) belong to \( B \).

**Theorem 1.** If \( B \) is a bipartitive family, the nodes of \( T(B) \) can be labelled complete, circular or prime, and the children of the circular nodes can be ordered in such a way that

- If \( N \) is a complete node, for any \( I \subset \{1, \ldots, k\} \) such that \( 1 < |I| < k \), \( B(I) \in B \)
- If \( N \) is a circular node, for any interval \( I = [a, \ldots, b] \) of \( \{1, \ldots, k\} \) such that \( 1 < |b - a| < k \), \( B(I) \in B \)
- If \( N \) is a prime node, for any element \( I = \{a\} \) of \( \{1, \ldots, k\} \) \( B(I) \in B \)
- There are no more bipartitions in \( B \) than the ones described above.

For a bipartitive family \( B \), the labelled tree \( T(B) \) is thus an \( O(|X|) \)-sized representation of \( B \), while the family \( B \) can have up to \( 2^{|X| - 1} - 1 \) bipartitions of \( |X| \) elements each. Furthermore, it allows to perform some algorithmic operations efficiently on \( B \).

The self-complemented umodular families fit in this formalism. Indeed let us consider the family \( \mathcal{U}'(H) = \{ \{U, X \setminus U\} \mid U \text{ is a umodule of } H \} \).

**Theorem 2.** \( \mathcal{U}'(H) \) is a bipartitive family.

**Proof.** As \( \mathcal{U}'(H) \) is self-complemented each part of a bipartition is a umodule. Furthermore if the bipartitions \( \{U, X \setminus U\} \) and \( \{V, X \setminus V\} \) overlap (in the bipartition sense) then \( U \) and \( V \) overlap (in the set sense). According to Proposition 3, if \( U \) and \( V \) overlap \( U \cup V \) is a umodule, and therefore \( \{U \cup V, X \setminus (U \cup V) \in \mathcal{U}'(H) \} \). Self-complementation gives the results needed for the three other bipartitions.

**Corollary 1.** There exists a unique \( O(|X|) \)-sized decomposition tree that gives a description of all possible umodules of a homogeneous relation \( H \).

This tree is henceforth called umodular decomposition tree. Notice that it is an unrooted tree, unlike the modular decomposition tree.

Using results developed in section 5 for the general case, the computation of this unique tree is polynomial, but it would be nice to find a better algorithm in this particular case.
3.2 Self-Complemented Relations of Local Congruency

Now we have seen that the self-complemented unmodal families are well structured, let us now focus on a more specific class of homogeneous relation, namely local congruency 2 homogeneous relations, where there is a nice transformation from the umodules to the modules of another relation.

**Definition 7 (Local congruency).** Let $H$ be a homogeneous relation on $X$. For $x \in X$, the congruency of $x$ is the maximal number of elements that $x$ pairwise distinguishes. In other words, it is the number of equivalence classes of $H_x$. The local congruency of $H$ is the maximum congruency of the elements of $X$.

It is not hard to be convinced that

**Proposition 6.** The homogeneous relation of a directed graph (resp. partial order, undirected graph, tournament) has local congruency 4 (resp. 3, 2 and 2).

The Local complementation also called Seidel switch at a given vertex $x$ was defined in [16] for graphs as exchanging (or complementing) edges and non-edges between the neighbourhood of $x$ and its non-neighbourhood. This can be extended to local congruency 2 homogeneous relations. For convenience, if $H$ is a homogeneous relation on $X$ and $s \subseteq X$, we also refer to the equivalence classes of $H_s$ as $H_s^1, \ldots, H_s^k$.

**Definition 8 (Local complementation).** Let $H$ be a homogeneous relation of local congruency 2 on $X$, and $s$ an element of $X$. The local complementation at $s$ transforms $H$ into the homogeneous relation $H(s)$ on $X \setminus \{s\}$ defined as follows.

$$
\forall x \in X \setminus \{s\}, H(s)_1^i = (H_1^i \Delta H_1^i) \setminus \{s\} \text{ and } H(s)_2^i = (H_2^i \Delta H_1^i) \setminus \{s\}
$$

where $A \Delta B$ denotes the symmetric difference of $A$ and $B$.

**Theorem 3.** Let $H$ be a homogeneous relation of local congruency 2 on $X$ such that $\cup_H$ is self-complemented. Let $s$ be a member of $X$, and $U \subseteq X$ a subset containing $s$. Then, $U$ is a umodule of $H$ if and only if $M = X \setminus U$ is a module of local complementation $H(s)$.

**Proof.** Let $C = H_1^1 \cap M$ and $D = H_2^l \cap M$. Since $H$ is of local congruency 2, $\{C, D\}$ is a partition of $M$. Let $a \in U \setminus \{s\}$. Suppose that $U$ is a umodule of $H$. Then, for all $y, z \notin U$, $H(a|yz)$ if and only if $H(s|yz)$. In other words, $C$ is included in one class among $H_1^i$ and $H_2^i$, while $D$ is included in the other class. As $C \subseteq H_1^1$ and $D \cap H_1^1 = \emptyset$, $C \cup D$ is included in one among the two classes $H(s)_1^i = H_1^i \Delta H_1^i$ ($i \in \{1, 2\}$). Hence, $M = C \cup D$ is a module of $H(s)$.

Conversely, if $M$ is a module of $H(s)$, then $C \cup D$ is included in either $H(s)_1^1$ or $H(s)_2^2$. Moreover, the definition of the local complementation can also be written as $H_2^i = H(s)_1^i \Delta H_1^i$ for $i \in \{1, 2\}$. Therefore, $C$ is included in one class among $H_1^i$ and $H_2^i$, while $D$ is included in the other class. In other words, for all $a \in U \setminus \{s\}$, and $y, z \notin U$, $H(a|yz)$ if and only if $H(s|yz)$. This implies for all $a, b \in U$, and $y, z \notin U$, $H(a|yz)$ if and only if $H(b|yz)$ and $U$ is therefore a umodule. □

Modular decomposition trees have well-known properties [3,11]. They are rooted trees whose leaves are in one-to-one correspondence with elements of $X$. A node of the modular decomposition tree is exactly a strong module, a module that overlap (in the set sense) no other modules. For a node $N$ let $F_1, \ldots, F_k$ be the leaf-sets of its $k$ children in the tree. When the family of umodules of $H$ is bipartite as it is the case in Theorem 3, the family of modules of any local complementation of $H$ is a partitive set family [8], also known as rooted set family [11]. The following theorem from 3 describe the structure of partitive set families.

**Theorem 4.** The nodes of a modular decomposition tree $T$ can be labelled complete, linear or prime, and the children of the linear nodes can be ordered in such a way that

- If $N$ is a complete node, for any $I \subseteq \{1, \ldots, k\}$ such that $1 < |I| < k$, $\bigcup_{i \in I} F_i$ is a module
- If $N$ is a linear node, for any interval $I = [a, b]$ of $\{1, \ldots, k\}$ such that $1 < |b - a| < k$, $\bigcup_{i \in I} F_i$ is a module
- If $N$ is a prime node, for any element $I = \{a\}$ of $\{1, \ldots, k\}$ $\bigcup_{i \in I} F_i$ is a module
- There are no more modules than the ones described above
The relationships between the umodular decomposition tree of $H$ and the modular decomposition tree of $H(v)$ are very tight:

**Proposition 7.** Let $H$ be a homogeneous relation on $X$ and $s$ an element of $X$. The umodular decomposition tree $T_H$ of $H$ and the modular decomposition tree $T_{H(s)}$ of the local complementation $H(s)$ of $H$ at $s$ have the following properties:

- The two trees are exactly the same (same nodes and edges) except that the leaf with label $s$ is missing in $T_{H(s)}$ but present in $T_H$.
- The node of $T_H$ that is adjacent to the leaf $s$ corresponds to the root of $T_{H(s)}$ (while $T_{H(s)}$ is unrooted).
- A circular node of $T_U$ corresponds to a linear node of $T_M(s)$. The orderings of the children are the same.
- The prime and complete nodes are the same in both trees.

**Proof.** This is a consequence of Theorem 3. Every strong module of $H(s)$ gives a strong bipartition of $T_H$, and the converse is true. Then for a node $N$ of the modular decomposition tree, for any union $\bigcup_{i \in I} F_i$ of leaf-sets of children there is a bipartition $\{\bigcup_{i \in I} F_i, X \setminus (\bigcup_{i \in I} F_i)\} = B(I)$ using the notations defined above. For each bipartition of umodules of $H$, the part that contain $s$ is dropped and the other part is put in the family of modules of $H(s)$. \[\Box\]

A proof similar (and more detailed) with this result can be found in [11]. That article indeed describes the relationship between the consecutive-ones ordering and the circular-ones ordering of a boolean matrix, but the results (described in [11] as the transformation of a PQ-tree into a PC-tree) are the same. Notice that the modular decomposition tree of $H$ can be trivial, while the one of its local complementation at $s$ may be not.

**Corollary 2.** The umodular decomposition tree of a self-complemented homogeneous relation of local congruency 2 on $X$ can be computed in $O(|X|^2)$ time.

**Proof.** This can be done using a local complementation and then the $O(|X|^2)$-time modular decomposition algorithm for homogeneous relations of [2].

### 4 Applications to graphs

#### 4.1 A New Tournament Decomposition

In this section we present a new tournament decomposition: the umodular decomposition. It is indeed the umodular decomposition of the standard homogeneous relation of the tournament. Actually this decomposition is more powerful than the general modular decomposition of [1], because every module of a tournament is a umodule, while umodular decomposition is able to decompose M-prime tournaments (those having no nontrivial modules).

Given a graph decomposition, a graph is **totally decomposable** if every “large enough” induced subgraph contains a nontrivial decomposition set. For modular (resp. umodular) decomposition, every induced graph with at least three (resp. four) vertices contains a nontrivial module (resp. umodule). Clearly, this is an hereditary property. Totally decomposable relations may also be defined. It is well-known that

**Proposition 8.** – If $T$ is an $M$-prime tournament then $T$ contains an induced cycle with 3 vertices

– $T$ is totally decomposable w.r.t. modular decomposition iff it contains no induced cycle with 3 vertices (it is a transitive tournament).

**Proposition 9.** If $T$ is an $U$-prime tournament then $T$ contains one of the induced subgraph described in Figure 2. $T$ is totally decomposable w.r.t. umodular decomposition iff it is free of these induced subgraphs.

**Proof.** Thanks to Theorem 3, $T$ is $U$-prime iff for any vertex $v$ a local complementation at $v$ gives an $M$-prime tournament. Thanks to Proposition 8 one just has to check all the four-vertices tournament where a local complementation on a vertex produces the cycle with 3 vertices. It is tedious but no hard. \[\Box\]
We can deduce from Proposition 7 some very interesting properties of the umodular decomposition of tournaments.

**Corollary 3.** The umodular decomposition tree of a tournament has no complete node.

**Proof.** The modular decomposition tree of a tournament has no complete node. ☐

**Corollary 4.** There exists a circular permutation of the vertices of the tournament such that every umodule of the tournament is a factor (interval) of this circular permutation.

**Proof.** Any traversal of the umodular decomposition tree, respecting the order of the sons of a circular node, orders the leaf labels into the desired circular permutation. ☐

This result was already known for modular decomposition [5]: there exists a (not circular) permutation of the vertices whose every module of the tournament is a factor. It is called factorising permutation.

**Proposition 10.** The umodular decomposition tree of a tournament can be computed in $O(|X|^2)$ time.

**Proof.** Again Theorem 3 says that one just has to perform a local complementation on a arbitrarily chosen vertex, then to compute the modular decomposition of the tournament. This can be done in linear (in fact $O(|X|^2)$) time using the algorithm from [12]. Proposition 7 tells how to cast the modular decomposition tree into the umodular one. ☐

**Proposition 11.** A tournament $T$ is completely decomposable w.r.t. umodular decomposition iff for each vertex $x \in V(T)$, $N^+(x)$ and $N^-(x)$ are transitive tournaments.

**Proof.** Let us assume that there exists a vertex $x \in V(T)$ such that w.l.o.g. $N^+(x)$ is not a transitive tournament, so there exists at least one directed cycle with three elements, consequently, this cycle plus $x$ form a forbidden configuration, thus the tournament is not totally decomposable. Conversely, if $T$ is not a tournament totally decomposable, then $T$ contains at least one of the two forbidden configuration. In any of them we can find a vertex that does not fullfills the condition. ☐

This can be checked in $O(|X|^3)$ time, but the following test is faster.

**Proposition 12.** Let $T$ be a tournament and $x$ a vertex. $T$ is totally decomposable w.r.t. umodular decomposition iff

1. $T[N^+(x)]$ and $T[N^-(x)]$ are transitive tournaments, and
2. if a vertex $a \in N^+(x)$ has an out-neighbour $b \in N^-(x)$ and an in-neighbour $c \in N^-(x)$ then $(b, c) \in T$
3. if a vertex $a \in N^-(x)$ has an out-neighbour $b \in N^+(x)$ and an in-neighbour $c \in N^+(x)$ then $(b, c) \in T$

**Proof.** Let us suppose $T$ is totally decomposable. According to Proposition 11, Point 1 holds. If Point 2 does not hold for some vertices $a$, $b$ and $c$, i.e. if there is an arc $(c, b)$ instead of $(b, c)$, then $\{a, b, c, x\}$ induce a forbidden configuration of Figure 2. Same if Point 3 does not hold. Conversely let us suppose that the three points are true. We shall prove that for every vertex, its in- and out-neighbourhoods are transitive. Then Proposition 11 tells $T$ is totally decomposable w.r.t. umodular decomposition. For $x$, this is true thanks to Point 1. Let $t$ be a vertex of $N^+(x)$. If $T[N^+(t)]$ is not transitive then it contains a cycle $(u, v, w)$ (with arcs, w.l.o.g., $(u, v)$ and $(v, w)$ and $(w, u)$.) As both $T[N^+(x)]$ and

![Fig. 2. Minimal U-Prime Configurations in tournaments = forbidden subgraphs of a tournament totally decomposable w.r.t. umodular decomposition](image-url)
To modular decomposition. For modular decomposition there exists indeed only one tournament totally decomposable with \( n \) w.r.t. umodular decomposition is strictly larger than the number of tournaments totally decomposable w.r.t. modular decomposition. T created a tournament leading to an \( O(N^m) \) manner to proceed is to take a transitive tournament \( T \).

**Theorem 6 (Umodular tree for tournament totally decomposable).** Tournament totally decomposable w.r.t. umodular decomposition have only one single node. Moreover this node is a circular node.

**Proof.** According to Theorem 3 for any \( T \) totally decomposable w.r.t. umodular decomposition gives a tournament \( T(x) \) totally decomposable w.r.t. modular decomposition. According to Proposition 6 \( T(x) \) is transitive: its modular decomposition tree has a single linear node. According to Proposition 6 the umodular decomposition tree of \( T \) only has a circular node.

**Generating all totally decomposable tournaments** We now present an algorithm to generate (at least once) every tournament totally decomposable w.r.t. umodular decomposition. To generate a tournament with \( n \) vertices, we have to check that it does not violate the conditions described in Proposition 12. A very simple manner to proceed is to take a transitive tournament \( T_{n-1} \) with \( n - 1 \) vertices, and then choose a subset \( S \) of the vertices of \( T_{n-1} \). We add a vertex \( x \), and decide that \( N^+(x) = S \) and \( N^-(x) = V(T_{n-1}) \setminus S \). Once we have \( N^+(x) \) and \( N^-(x) \) it suffices to reverse every arc between \( N^+(x) \) and \( N^-(x) \). In this way we have created a tournament \( T_n \) which is totally decomposable w.r.t. umodular decomposition.

Thanks to this construction we are able to see that the number of tournaments totally decomposable w.r.t. umodular decomposition is strictly larger than the number of tournaments totally decomposable w.r.t. to modular decomposition. For modular decomposition there exists indeed only one tournament totally decomposable with \( n \) vertices!

**Fig. 3.** An example of a M-prime tournament who is not U-prime. The umodular decomposition tree is drawn on the right.
4.2 Bi-join decomposition of undirected graphs

We have investigated in Section 4.1 the umodules of tournaments, and seen that they lead to a nice decomposition. The same is true for undirected graphs. The decomposition was already published in [15]. We summarize here the main results of that paper and establish the link with umodules.

**Definition 9 (bi-join).** A bi-join of a graph \( G = (X, E) \) is a bipartition \( \{X^1, X^2\} \) of the vertex-set such that the edges between \( X^1 \) and \( X^2 \) form at most two disjoint complete bipartite graphs, and that for each \( i, j = 1, 2 \) every vertex of \( X^i \) is adjacent to a vertex of \( X^j \).

![Forbidden arc configuration](image1)

**Fig. 4** Forbidden arc configuration between neighbourhoods

**Fig. 5** Example of a bi-join of a graph

**Proposition 13.** If \( \{X^1, X^2\} \) is a bi-join of a graph then both \( X^1 \) and \( X^2 \) are umodules of \( G \)

**Proof.** Let \( A \) (resp. \( C \)) be the vertices of \( X^1 \) (resp. \( X^2 \)) incident with the first complete bipartite graph, and \( B \) (resp. \( D \)) be the other vertices of \( X^1 \) (resp. \( X^2 \)). Any vertex of \( X^1 \) distinguishes a vertex of \( C \) from a vertex of \( D \), but cannot distinguish two vertices from \( C \), nor two vertices from \( D \). \( X^1 \) is thus a umodule, and a similar proof holds for \( X^2 \). \( \Box \)

In [15] the local complementation was used to derive most of the properties:

**Proposition 14.** Let \( G \) be a graph. \( \{X^1, X^2\} \) is a bi-join of \( G \) iff for every \( v \in X^1 \) (resp. \( X^2 \)) \( X^2 \) (resp. \( X^1 \)) is a module of the local complementation \( G(v) \).

It may be used to prove the converse of Proposition 13:

**Proposition 15.** If \( U \) is a umodule of a graph \( G = (X, E) \) then \( \{M, X \setminus U\} \) is a bi-join of \( G \).

This is because the homogeneous relation of a tournament has local congruency 2 and is self-complemented (see Section 3.2).

**Corollary 5.** The umodular decomposition of a graph equipped with its standard homogeneous relation is exactly its bi-join decomposition

Among the consequences exposed in [15] are that the bi-join (thus umodular) decomposition tree has no circular nodes (but has complete ones), a characterisation of totally decomposable graphs in term of forbidden induced subgraphs (they are exactly the \( \{C_5, bull, gem, co-gem\} \)-free graphs), and a proof that computing the bi-join (thus umodular) decomposition tree can be performed in linear time, since modular decomposition can be computed in linear time.

5 Algorithms for the general case

We have seen that linear-time algorithms exist for computing the umodular decomposition of a undirected graph (see Section 4.2) or of a tournament (see Section 4.1), and an \( O(|X|^3) \)-time algorithm for the more general case when the relation is self-complemented and of local congruency 2 (see Section 3.2). We shall now give algorithms for the general case. If the umodules family is self-complemented, then we compute the umodular decomposition tree. If not, we have (as far as we know) no decomposition tree to compute. The only valuable objects to compute seem then to be the maximal umodules and the primality test.
5.1 Computation of maximal umodules

Let $S$ be a subset of $X$. As $\mathcal{U}$ is closed under the union of overlapping elements, then the maximal (w.r.t. inclusion) umodules included either in $S$ or in $X \setminus S$ are a partition of $X$, denoted $MU(S)$. Notice that $MU(S) = MU(X \setminus S)$. $MU(S)$ gives an indication on how the umodules are structured with respect to $S$, since a umodule either is included in a umodule of $MU(S)$ or properly intersects $S$ or properly intersects $X \setminus S$ or is trivial.

Definition 10. Let $H$ be a homogeneous relation over $X$ and $E \subseteq X$. The binary relation $R_E$ between two elements of $E$ is defined as

\[ R_E(x, y) \iff \forall a, b \in (X \setminus E) \quad H(x)ab \iff H(y)ab \]

This relation clearly is an equivalence relation on $E$. Furthermore, $E$ is a umodule iff $R_E$ contains only one equivalence class.

Definition 11. Let $\mathcal{P}$ be a partition of $X$ and $C$ a part of $\mathcal{P}$. Let $C_1, \ldots, C_k$ be the equivalence classes of $R_C$. $Refine(\mathcal{P}, C)$ is the partition $\mathcal{P}$ where the part $C$ is replaced with parts $C_1, \ldots, C_k$.

Definition 12. A partition is refinable if for every part $C$ of $\mathcal{P}$, we have $\mathcal{P} = Refine(\mathcal{P}, C)$. A refinement is proper if $\mathcal{P} \neq Refine(\mathcal{P}, C)$. Let $\mathcal{P}_0$ be a partition. A refinement sequence is a sequence $\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_k$ of partitions where $\mathcal{P}_{i+1} = Refine(\mathcal{P}_i, C_1)$ and $C_1$ is a part of $\mathcal{P}_i$.

Lemma 1. In an un refinable partition, all parts are umodules.

The proof is immediate since $C$ is an umodule iff $\mathcal{P} = Refine(\mathcal{P}, C)$. \[\square\] proves (for a more general case) that

Lemma 2. Let $S$ be a subset of $X$. Any sequence of proper refinements, starting at partition $P_0 = \{S, X \setminus S\}$, ends at the (unique un refinable) partition $MU(S)$.

This lemma suggests a polynomial-time algorithm for computing $MU(S)$.

Theorem 7. It is possible to compute $MU(S)$ in $O(|X|^3)$ time.

Proof. We first show how to test for $R_C(x, y)$, then how to implement the $Refine(\mathcal{P}, C)$ procedure in $O(|X|^2)$ time, and then how to implement the whole algorithm in $O(|X|^3)$ time. Let us consider a partition $\mathcal{P}_i$ and a class $C$ of $\mathcal{P}_i$.

First compute, for every element $x$ of $C$, a partition $H(x, C) = \{P_x^1, \ldots, P_x^{k(x)}\}$ of $X \setminus C$. It is the restriction of $H_x$ to $X \setminus C$, i.e $P_1 \subseteq H_x \setminus C$. It is easy to build in $O(|X|)$ time for each element of $C$. Then we have $R_C(x, y)$ if $H(x, C)$ is exactly the same partition than $H(y, C)$. It can be tested in $O(|X|)$ time, but performing this for each couple of elements of $C$ would lead to an $O(|X|^3)$ time implementation of $Refine(\mathcal{P}, C)$.

Instead, we will use sorting in order to cluster the elements having the same partitions. Let $L$ be the list of all parts of $H(x, C)$ (i.e. $L = \{P_x^1, \ldots, P_x^{k(x)}\}$, $P_x^1, \ldots, P_x^{k(x)}$). Let $L_1, \ldots, L_k$ be the partition of $L$ into duplicates classes (two parts $P_x^i$ and $P_x^j$ are equal if they are in the same $L_a$). This partition can be computed by bucket sorting $L$ using the first element of $X \setminus C$, then the second, and so on. This takes $O(|X|^2)$ time.

Let $Z(s)$ be the partition of $C$ such that $x$ and $y$ are in the same part iff the parts $P_x^i$ of $H(x, C)$ containing $s$ and the part $P_y^j$ of $H(y, C)$ containing $s$ are equal. Using the duplicate classes of $\{L_1, \ldots, L_k\}$ this equality can be tested in $O(1)$ time and thus $Z(s)$ can be computed in $O(|X|)$ time.

Now we have $R_C(x, y)$ iff, for every $s$ of $X \setminus C$, $x$ and $y$ are in the same class of $Z(s)$. The partition of $C$ into $C_1, \ldots, C_k$ computed by $Refine(\mathcal{P}, C)$ is thus the coarsest partition compatible with every partition $Z(s)$. In other word, for each $s$ each $C_i$ is included in one part of $Z(s)$, and $C_i$ is maximal for this property. Using classical partition refining techniques $\square$, this can be computed in time linear in the size of the partitions, i.e. in $O(|X|^2)$ time. We just have computed $Refine(\mathcal{P}, C)$ in $O(|X|^2)$ time.

If $\mathcal{P} = Refine(\mathcal{P}, C)$ then $C$ is a umodule and the computation of $Refine(\mathcal{P}, C)$ is no more performed. This case occurs $|X|$ times at most. Else, $C$ is split in new classes. As they are at most $|X|$ parts in the final partition, this case also occurs at most $|X|$ times. The whole algorithm thus runs in $O(|X|^3)$ time. \qed
5.2 Testing for U-primality

By definition, if \( MU(S) \) contains only trivial umodule, then all nontrivial umodules of \( X \) intersect both \( S \) and \( X \setminus S \). This allow to test for U-primality of a homogeneous relation in polynomial time:

**Theorem 8.** It is possible to check in \( O(|X|^3) \) time if a homogeneous relation \( H \) is U-prime and, if not, to output a nontrivial umodule.

**Proof.** Just test if \( MU(\{x,y\}) \) is trivial for all pair of elements \( \{x,y\} \).

Notice this is hard to do better than \( O(|X|^3) \) using the \( MU \) algorithm as toolbox, since there exists homogeneous relations whose sole umodule is \( X \setminus \{x,y\} \).

5.3 Umodular decomposition tree of a self complemented family

Let \( H \) be a self-complemented homogeneous relation, \( T(H) \) be its umodular decomposition tree and \( U \) be a nontrivial strong umodule (if any). Let us examine some consequences of Theorem 8. Notice that two umodules overlap iff they are incident to the same node of \( T(H) \). As \( H \) is self-complemented the union of two overlapping umodules is a umodule (Proposition 3) but also their intersection. The strong umodule \( U \) is an edge in \( T(H) \) incident with two nodes \( A \) and \( B \).

\begin{itemize}
  \item If one of them, say \( A \), is labelled prime then for any \( x, y \not\in U \) such that the least common ancestor of them in \( T(H) \) is \( A \), then \( U \in MU(\{x,y\}) \).
  \item If one of them, say \( A \), is labelled circular then for any \( x \) belonging to the subtree rooted in the successor of \( U \) in the ordered circular list of \( A \), and for any \( y \) belonging to the subtree rooted in the predecessor of \( U \), then \( U \in MU(\{x,y\}) \).
  \item If one of them, say \( A \), is labelled complete then the intersection, for all \( x, y \not\in U \) whose least common ancestor is \( A \), the intersection of all parts of \( MU(\{x,y\}) \) containing \( U \) is exactly \( U \).
\end{itemize}

As a consequence:

**Theorem 9.** There exists an \( O(|X|^3) \) algorithm to compute the unique decomposition tree for a self complemented umodule family.

**Proof.** For every pair \( \{x,y\} \) compute \( MU(\{x,y\}) \) in \( O(|X|^3) \) time (Theorem 8). That gives a family of at most \( |X|^3 \) umodules. Greedily compute the intersection of overlapping umodules of the family. It is possible in \( O(|X|^3) \) time: for each triple \( (a,b,c) \) look for the umodules containing exactly two of them, they overlap. Then you have all strong umodules. You just have to order them into a tree and to label its nodes, an easy task.

6 Extensions and further developments

We have presented the umodules and homogeneous relations focusing on graph theory field. But umodules may be found in many other objects. Let us briefly present an example.

6.1 Homogeneous relation based on a binary function

Let \( f \) be a binary function \( X \times X \to Y \). The homogeneous relation based on \( f \), written \( H_f \), is defined as \( H_f(s,a) \) iff \( f(s,a) = f(s,b) \) and \( f(a,s) = f(b,s) \).

For instance on graphs \( f \) is the existence of an edge. On directed graph is the existence of an arc. And on a 2-structures \( f(x,y) \) is the number of equivalence class of the couple \( (x,y) \). It can also be seen as a colouring of the edge \( (x,y) \).

Notice that weaker homogeneous relations can be defined from a binary function: the left homogeneous relation based on \( f \), \( H^l_f \), is defined as \( H^l_f(s,a) \) iff \( f(s,a) = f(s,b) \). And the right homogeneous relation based on \( f \), \( H^r_f \), is defined as \( H^r_f(s,a) \) iff \( f(a,s) = f(b,s) \). But these relation do not have the quotient properties, and have not the same umodules. We have:
Proposition 16. If \( M \) is a umodule for \( H^r_f \) and for \( H^l_f \) then is a umodule for \( H^l_f \)

The proof is immediate from definition. Notice that the converse is not true. For instance for \( X = \{a, b, c, d\} \) if \( f(a, c) = f(a, d), f(b, c) = f(b, d) \) and all other couples have pairwise different values, then \( \{a, b\} \) is a umodule for \( H^l_f \) but neither for \( H^r_f \) nor for \( H^l_f \). If \( f \) is a symmetric function, then the three homogeneous relations of course are the same. This is true for graphs and for symmetric 2-structures, for instance.

Proposition 17. The principal ideals of a ring are umodules (w.r.t. its multiplication homogeneous relation)

6.2 Further work

The umodular decomposition of directed graphs and 2-structures seems more difficult to study since it is not a self-complemented umodular family, but it is the next challenge. It is not known for instance if a compact representation (like a tree) of this exponentially-sized family is possible. If not, this is a limitation on umodular algorithmics.

The U-primality test presented here is polynomial, but its asymptotic complexity can surely be reduced, especially when applied to particular combinatorial objects. Same remark for the umodular decomposition algorithm of self-complemented homogeneous relation of local congruency greater than two. One can expect better than \( O(|X|^5) \).

References