On region for fidelities of $1-3$ universal quantum cloner

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(Dated: January 31, 2012)

Keywords: quantum cloning; universal quantum cloning; qubit; Schur - Weyl duality

PACS numbers: 03.67.Dd, 03.65.Fd, 03.67.Hk

We analyze the region of fidelities for qubits which is obtained after performing quantum cloning, by means of $1-3$ universal quantum cloner. To obtain our result, we apply to our problem group representation theory, namely Schur - Weyl decomposition. We express the allowed region for fidelities in terms of overlaps of pure states with irreps of $S_n$. We then show that the pure states can be taken with real coefficients. Subsequently, the case $n = 4$, corresponding to $1-3$ cloner is studied in more detail. We first obtain a plot of possible range of fidelities for qubits related to a given irrep (irreducible representation) and then make a convex hull of it to obtain the allowed region of triples of fidelities. The method allows to construct the state giving rise to a given triple of fidelities.

I. INTRODUCTION

A. Historical overview: the no-cloning theorem

In 1982 Nick Herbert published the paper about faster than light communication, based on quantum correlations. He called this project FLASH, an acronym for ‘First Light Amplification Superluminal Hookup’ [1]. Of course, the idea was incorrect but it was the source for Wootters and Zurek [2] and independently Dieks [3] to establish the so-called no-cloning theorem. The proof [4, 5, 6] is as follows. Consider a cloning unitary operation $U_{cl}$ that is able to clone any given qubit. Before the copying process, we have states:

$$ |A\rangle = |x\rangle|0\rangle|M\rangle,$$
$$ |B\rangle = |x'\rangle|0\rangle|M\rangle, $$

where $|x\rangle$, $|x'\rangle$ denote ‘text’ to copy, $|0\rangle$ represents a ‘blank card’ and $|M\rangle$ is a cloning machine state ($M$ denotes the cloning machine). Acting $U_{cl}$ on (1), we should get:

$$ U_{cl}|A\rangle = |x\rangle|x\rangle|M'\rangle,$$
$$ U_{cl}|B\rangle = |x'\rangle|x'\rangle|M''\rangle, $$

where $|M'\rangle$, $|M''\rangle$ represent new states of the machine.

Now, let us consider a scalar product in both cases:

$$ \langle A|B \rangle = \langle x|x' \rangle, $$

because $\langle 0|0 \rangle = \langle M|M \rangle = 1$. From the other hand, for the second case, we have:

$$ \langle U_{cl}A|U_{cl}B \rangle = (\langle x|x' \rangle)^2 \langle M'|M'' \rangle. $$

Comparing (3) and (4), we get that in the case of $0 < |\langle x|x' \rangle| < 1$ scalar products do not match each other (but from the property of the unitary action, they should be the same). So, we have contradiction and it proofs that one is not able to clone initially unknown quantum states. In general, the no-cloning theorem comes from the linearity of quantum mechanics [7]. Of course the no-cloning theorem could be generalized [8] or even strengthen [9].

B. Beyond the no-cloning theorem: origins of quantum cloning

Although, the no-cloning theorem is fundamental for quantum physics, it is not of great use in practice. The no-cloning theorem states that we are not able to copy an arbitrary quantum state. However, it is well known that performing any ideal operations in physics and especially in quantum physics is nearly impossible. On the other hand, it is obvious that one can copy, perhaps, though with a very bad quality. Thus it is crucial to know the ultimate bounds for quality of copying.

Several years after Wootters and Zurek paper has been published, in 1996, Hillery and Bužek, published the paper called ‘Quantum copying: beyond the no-cloning theorem’ [10]. It was the first time, when the above question regarding imperfect cloning has been formulated. After this, the subject was a matter of wide research (see [8] and references therein for a comprehensive review). Here, we would like to point only a few (regarding the cloning of finite quantum states only). Quite soon, the Bužek-Hillery $1-2$ (qubits) Quantum Cloning Machine (QCM) (for all formal definitions of quantum cloning machines, we refer to [8]), was generalized to the case $N = M$; first for qubits by Bruß et al. in [11], and by Gisin and Massar in [12], and then for arbitrary-dimensional states by Werner in [13], and Keyl and Werner in [14]. We can conclude
from this that the family of symmetric universal quantum cloning machine (UCQM) is well known. What is more, nearly the same can be inferred for the asymmetric UCQM (1 − 1 + 1 case), let us mention here works of Braunstein et al. \cite{15}, Cerf \cite{16}, Fiurášek et al. \cite{17}, and Iblisdir et al. \cite{18, 19, 20}. Later, efforts were made to unify these two kinds of QCM \cite{21}. To emphasize the importance of all this works, let us mention that the optimal N − M1 + M2 universal asymmetric QCM is under serious studies, because of its possible usage in the security analysis of quantum cryptography protocols \cite{8}. Of course, a lot of questions and problems still need answers, for instance, there is a strong need for optimal state-dependent QCM (as an example, see \cite{21} or \cite{22} - unfortunately, not many results are known for this kind of QCM). There is also another ‘gap’ in quantum cloning: interestingly, up to our best knowledge, one is lacking a general result on fidelity region for universal quantum cloning machines; in our work, we want to make progress in this direction.

In this paper, we shall consider 1 − 3 universal quantum cloning machine (quantum cloner). We first point out that this problem could be related to fidelity with singlet states. Using this fact, it turns out that by the application of Schur - Weyl duality, our problem could be quite easily solved and it leads to plots of fidelities range for different partitions \( \lambda \) \cite{23} of the symmetric group \( S_4 \). The method of irreps (irreducible representations) is ‘nice’ here, because it makes calculations a lot easier. It allows us to decompose our initial Hilbert space into blocks of smaller dimensions, connected to a given partition \( \lambda \), and moreover, in our calculation, we can restrict ourself to a pure state only, linked to a given block. After taking convex hull of the figures corresponding to all partitions, we can obtain possible range of fidelities. Thus the method of irreps here is particularly useful, so that we can not only consider symmetric cloning machine but also the asymmetric one, even in the case when one want to perform 1 − 1 + 1 + 1 cloning operation (it means that all three fidelities can be different). We check that our results are compatible with the existing methods (for example, Keyl and Werner work \cite{14}), namely we obtain that in the case of optimal, symmetric UQCM, result of all three fidelities equal \( \frac{2}{3} \) is obtained. Finally, as a direct application of our method, we study a particular example of the asymmetric UQCM, namely the case when one has \( \max \{ F_1 + F_3 = 2F_2 \} \).

This work is organized as follows. In section \[ \text{II} \] we start with showing that the quantum cloning problem is equivalent to the entanglement sharing picture, so that the cloning machine is equivalently represented by a multipartite quantum state. Then, we formulate problem that we want to address in this paper, namely, to determine the region of allowed fidelities for our QCM. What is more (and crucial for us), we show a transformation between Bell’s states that allows to use Schur - Weyl duality. Section \[ \text{III} \] is devoted to a group’s representation theory, in particular, Schur - Weyl decomposition. We present the basic formalism that leads to Schur - Weyl duality. At the beginning, we introduce SWAP operators and then we show how they lead to Schur - Weyl decomposition. At the end of this section, we show how Schur - Weyl duality is connected with Young diagrams formalism. This section is strictly mathematically oriented, but as it will be shown in the next section, this formalism allows us in our case to simplify our problem a lot and it makes all calculations quite elementary. In section \[ \text{IV} \] which is the key section of our paper, we present our main lemmas and theorems, especially, we show which form of the multipartite state representing QCM can be used, and also how one can connect fidelity calculations with a Young diagrams formalism. We also prove that one can restrict attention only to real pure states, an allowed region for fidelities obtained using these stats is indeed the optimal one. What is important, all results are valid for \( n \)-tuples of fidelities. In section \[ \text{V} \] which is the case study of the 1 − 3 universal quantum cloner, we present how to obtain the allowed region for triples of fidelities. We also show that results for a symmetric cloning case, in our model, are in comparison with results predicted by the Werner’s formula \cite{13}. At the end, we present an application of our model, namely we show that for any given triple of fidelities from the allowed region, we could reconstruct a state that gives rise to that fidelities; so that strictly speaking, our QCM is ‘general’ in this sense.

\[ \text{II. STATEMENT OF THE PROBLEM} \]

Let us recast a question of cloning in an equivalent picture of entanglement sharing. Suppose that we have an initial state described by the Bell state: \( |\psi^+\rangle = \sqrt{\frac{1}{2}} (|00\rangle + |11\rangle) \). We also have a cloning machine \( M \), which action could be described by a (CPTP) map \( \Lambda \). As an output, we want to obtain three shares of our initial state. To this end, we extend formalism presented in \cite{24}, where similar problem were studied, but there, it was only one copy as an output. Our scheme is presented on Figure 1. Let us write state obtained after applying the cloning map to half of \( |\psi^+\rangle \)

\[
\rho_{1234} = \frac{1}{2} \left( I \otimes \Lambda \right) \left( |\psi^+\rangle\langle\psi^+| \right),
\]

where indexes 1, 2, 3 and 4 are connected to an initial state and clones, according to Figure 1. We want to calculate an allowed region for singlet fractions \( F_{ij} \) (between an initial state and one of the three clones
Note that:

where $\rho$ is one of the Pauli spin matrix. $|\psi_+\rangle$ and $|\psi_-\rangle$ are obtained after the action of $-i\sigma_y$ on $|\psi+\rangle$:

$$ |\psi^-\rangle = -i\sigma_y|\psi^+\rangle = -i\sigma_y\sqrt{\frac{1}{2}}(|00\rangle + |11\rangle) = \sqrt{\frac{1}{2}}(|01\rangle - |10\rangle), $$

where $\sigma_y$ is one of the Pauli spin matrix: $\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$.

Using (8) we can write that:

$$ |\psi^-_{11}\rangle = U \otimes I |\psi^+_{11}\rangle, $$

where $U = -i\sigma_y$. State $\tilde{\rho}_{1234}$ from Eq. (7) is obtained after the following transformation:

$$ \tilde{\rho}_{1234} = (I \otimes \tilde{A}) |\psi^-_{11}\rangle $$

$$ = (U \otimes I) \left( (I \otimes \tilde{A}) |\psi^+_{11}\rangle / |\psi^-_{11}\rangle \right) (U^\dagger \otimes I). $$

The fourpartites states $\tilde{\rho}_{1234}$, with the constraint $\tilde{\rho}_1 = I/2$, are in one-to-one correspondence with a cloning machine, and the cloning fidelities of a given machine is determined fidelities $F_{ij}$ of the corresponding state.

Note that the singlet fraction could be related to the average fidelity of transmission of an initial state by the following relation [24]:

$$ f = \frac{F_d + 1}{d + 1}, $$

where $d$ is the dimension of the Hilbert space $\mathcal{H} \cong \mathbb{C}^d$. Thus instead of cloning fidelities $f_i$, we can consider singlet fractions $F_i$, which we will further call simply fidelities, while the fidelity of cloning we will term ‘cloning fidelity’.

Werner [13] provided the following formula for optimal cloning fidelity of universal symmetric $N - M$ cloning machine:

$$ f_{NM}(d) = \frac{N}{M} + \frac{(M - N)(N + 1)}{M(N + d)}. $$

In our case, when $N = 1$, $M = 3$ and $d = 2$, we obtain that fidelity $f$ for a universal, symmetric cloning machine should be equal

$$ f = \frac{1}{3} + \frac{4}{9} = \frac{7}{9}. $$

We can now formulate our main question: which values of triples of cloning fidelities $(f_{12}, f_{13}, f_{14})$ are allowed for universal cloning machine? As said above, we shall address equivalent question: what values of triples of fidelities $(F_{12}, F_{13}, F_{14})$ are allowed for an arbitrary state of a maximally mixed first subsystem? In the next sections, an answer is presented.

III. MATHEMATICAL INTRODUCTION: SCHUR-WEYL DECOMPOSITION

In this section we introduce necessary mathematical tools from group theory. We are especially focused on Schur-Weyl duality [23].

Consider a unitary representation of permutation group $S_n$ acting on $n$-fold tensor product of complex spaces $\mathbb{C}^d$, so our full Hilbert space is $\mathcal{H} \cong (\mathbb{C}^d)^{\otimes n}$. For a fixed permutation $\pi \in S_n$ a unitary transformation $V_\pi$ is given by

$$ V_\pi (|i_1\rangle \otimes \ldots \otimes |i_n\rangle) = |i_{\pi(1)}\rangle \otimes \ldots \otimes |i_{\pi(n)}\rangle, $$

where $|i_1\rangle, \ldots, |i_n\rangle$ is a standard basis in $(\mathbb{C}^d)^{\otimes n}$. The space of rank-$n$ tensors can be also consider as a representation space for a general linear group $\text{GL}(d, \mathbb{C})$. Let $U \in \text{GL}(d, \mathbb{C})$, thus, this non-singular transformation induces in $(\mathbb{C}^d)^{\otimes n}$ another transformation

$$ U^{\otimes n} (|i_1\rangle \otimes \ldots \otimes |i_n\rangle) = U|i_1\rangle \otimes \ldots \otimes U|i_n\rangle. $$

A key property is that these two representations turn out to be each other commutants. Any operator on
Thanks to above-mentioned method we can decompose symmetric part density operators that are in commutant of $U$ in allowed region of $n$. This is contained in theorem 2. We further show in lemma 3 that one can restrict to pure states with real coefficients.

**Lemma 1.** Fidelity $F_{ik}$ as defined in (7) is of the form

$$F_{ik} = \sum_{\lambda} F_{ik}^{\lambda},$$

where

$$F_{ik}^{\lambda} = \frac{1}{2} - \frac{1}{2} \Tr\left(\rho^{\lambda} V_{(ik)}^{\lambda}\right),$$

The lower index $(ik)$ means a permutation that swaps $i$ and $k$, and $\rho^{\lambda}$'s are arbitrary unnormalized states satisfying $\Tr(\sum_{\lambda} \rho^{\lambda}) = 1$.

**Proof.** From the definition of fidelity we can write

$$F_{ik} = \langle \psi_{1k} | \rho_{1k} | \psi_{1k} \rangle = \Tr(\rho_{1k} | \psi_{1k}\rangle\langle \psi_{1k} |),$$

where $| \psi_{1k}\rangle\langle \psi_{1k} |$ is a permutation that swaps 1 and $k$. Expanding (22), we obtain:

$$F_{ik} = \Tr\left(\left(\frac{1}{2} \rho_{1k} (id_{1k} - V_{(ik)}^{\lambda})\right)\left(\frac{1}{2} \rho_{1k} - \frac{1}{2} \rho_{1k} V_{(ik)}^{\lambda}\right)\right) = \frac{1}{2} - \frac{1}{2} \Tr\left(V_{(ik)}^{\lambda} \rho_{1...n}\right).$$

Now, we can use Schur-Weyl decomposition to represent $V_{(ik)}^{\lambda}$ and $\rho_{1...n}$:

$$V_{(ik)}^{\lambda} = \bigoplus_{\lambda} \mathbb{I}_{r(\lambda)} \otimes \tilde{V}_{(ik)}^{\lambda}, \quad \rho_{1...n} = \bigoplus_{\lambda} \mathbb{I}_{r(\lambda)} \otimes \tilde{\rho}^{\lambda}.$$

Inserting (24) into (23), we have:

$$F_{ik} = \frac{1}{2} \Tr\left(\bigoplus_{\lambda} \mathbb{I}_{r(\lambda)} \otimes \tilde{\rho}^{\lambda}\right) - \frac{1}{2} \Tr\left(\left(\bigoplus_{\lambda} \mathbb{I}_{r(\lambda)} \otimes \tilde{\rho}^{\lambda}\right)\left(\bigoplus_{\mu} \mathbb{I}_{r(\mu)} \otimes \tilde{V}_{(ik)}^{\lambda}\right)\right)$$

Equation (25) could be rewritten as:

$$F_{ik} = \sum_{\lambda} F_{ik}^{\lambda},$$

where $F_{ik}^{\lambda} = \frac{1}{2} \Tr(\tilde{\rho}^{\lambda}) - \frac{1}{2} \Tr\left(\rho^{\lambda} \tilde{V}_{(ik)}^{\lambda}\right)$, and $\rho^{\lambda} = d_{\lambda} \tilde{\rho}^{\lambda}$ and $d_{\lambda}$ stands for the dimension of a given partition. One can also see that $\sum_{\lambda} \Tr(\tilde{\rho}^{\lambda}) = 1$, which is in fact our normalization relation. Note finally, that arbitrary states $\tilde{\rho}^{\lambda}$ satisfying the above normalization give rise to some state $\rho_{1...n}$. Moreover, any state of the form (19) has the subsystem 1. Therefore, an arbitrary $k$-tuple of fidelities $F_{ik}$ corresponds to some set of $\rho^{\lambda}$'s satisfying the above normalization.

**IV. EXPRESSION FOR AN ALLOWED REGION OF n-TUPLES OF FIDELITIES.**

In this section we provide general formula for allowed region of $n$-tuples of fidelities in terms of overlaps of pure states with irreducible representations of $S_n$.
Now we are in position to formulate the main theorem of this section:

**Theorem 2.** Let $\mathcal{F}$ be the set of admissible vectors of fidelities $\{F_{12}, \ldots, F_{1n}\}$, then it can be shown that

$$\mathcal{F} = \text{conv} \left( \bigcup_{\lambda} \mathcal{F}^\lambda \right),$$

where $\text{conv}$ stands for convex hull, the union runs over all irreps of $S_n$ and

$$\mathcal{F}^\lambda = \left\{ \left( F_{12}^\lambda, \ldots, F_{1n}^\lambda \right) : |\psi\rangle \in \mathbb{C}^{d_{\lambda}} \right\},$$

where $F_{1k}^\lambda$ are of the form: $F_{1k}^\lambda = \frac{1}{2} - \frac{1}{2} \langle \psi | \tilde{\mathcal{V}}_{(ik)}^\lambda | \psi \rangle$, and where $|\psi\rangle \langle \psi|$ is a pure state.

**Proof.** At the beginning, let us consider the following mapping:

$$\tilde{F} : P \to \mathcal{R}^n$$

which maps states $\rho_{1\ldots n} \in P$ ($P$ stands for a convex set of all states $\rho_{1\ldots n}$) into the $n$-tuples $(F_{12}, \ldots, F_{1n}) \in \mathcal{R}^n$. Explicitly, we have

$$\tilde{F}(\rho_{1\ldots n}) = \left[ F_{12}(\rho_{1\ldots n}), \ldots, F_{1n}(\rho_{1\ldots n}) \right].$$

This mapping is affine (Lemma 7), i.e., is of the form:

$$\tilde{F}(\rho_{1\ldots n}) = \tilde{F}(\rho_{1\ldots n}) + \bar{C},$$

where $\tilde{F} : P \to \mathcal{R}^n$ is linear. Indeed, one can see that RHS of (30) could be written as:

$$F_{12}(\rho_{1\ldots n}) = \frac{1}{2} \text{Tr} \left[ (\mathcal{V}_{(1)} - \mathcal{V}_{(12)}) \rho_{1\ldots n} \right] = \frac{1}{2} - \frac{1}{2} \text{Tr}(\mathcal{V}_{(12)} \rho_{1\ldots n}),$$

$$F_{13}(\rho_{1\ldots n}) = \frac{1}{2} \text{Tr} \left[ (\mathcal{V}_{(1)} - \mathcal{V}_{(13)}) \rho_{1\ldots n} \right] = \frac{1}{2} - \frac{1}{2} \text{Tr}(\mathcal{V}_{(13)} \rho_{1\ldots n}).$$

Comparing (31) and (32), we obtain that in our case $\bar{C} = \left[ \frac{1}{2}, \ldots, \frac{1}{2} \right]$ and $\tilde{F}(\rho_{1\ldots n}) = \text{Tr}(\mathcal{V}_{(12)} \rho_{1\ldots n}), \ldots, \text{Tr}(\mathcal{V}_{(1n)} \rho_{1\ldots n})$, which is obviously linear with respect to $\rho_{1\ldots n}$.

One can note that, in general, $\tilde{\rho}^\lambda$ from Eq. (24) is a mixed state, but according to Lemma 7, we can take the mapping of extreme points of $\rho_{1\ldots n}$ of the form:

$$\rho_{1\ldots n} = \bigoplus_{\lambda} \mathbb{I}_{\lambda} \otimes \rho^\lambda = \bigoplus_{\lambda} \frac{1}{d_{\lambda}} \text{Tr}(\lambda) \otimes d_{\lambda} \tilde{\rho}^\lambda$$

$$= \bigoplus_{\lambda} \frac{1}{d_{\lambda}} \mathbb{I}_{\lambda} \otimes \rho^\lambda = \bigoplus_{\lambda} P_{\lambda} \left( \frac{\text{Tr}(\lambda)}{d_{\lambda}} \otimes \tilde{\rho}^\lambda \right),$$

where $\tilde{\rho}^\lambda = \frac{\rho^\lambda}{\text{Tr} \rho^\lambda}$, so namely, it is normalized, and $P_{\lambda} = \text{Tr} \rho^\lambda$. We can now eigen-decompose $\tilde{\rho}^\lambda$:

$$\tilde{\rho}^\lambda = \sum_{i=1}^{n_{\lambda}} p_{\lambda i}^\lambda \left| \psi_i^\lambda \right\rangle \langle \psi_i^\lambda \right|,$$

where $\sum_{\lambda} p_{\lambda i}^\lambda = 1$.

Inserting Eq. (34) into Eq. (33), we get that:

$$\bigoplus_{\lambda} \sum_{i=1}^{n_{\lambda}} p_{\lambda i}^\lambda \left( \frac{\text{Tr}(\lambda)}{d_{\lambda}} \otimes \left| \psi_i^\lambda \right\rangle \langle \psi_i^\lambda \right| \right),$$

we see from (35) that extreme points are of the form:

$$\rho_{\text{extreme}} = \frac{\text{Tr}(\lambda)}{d_{\lambda}} \otimes \left| \psi^\lambda \right\rangle \langle \psi^\lambda \right|,$$

where $\lambda$ runs over all irreps from $S_n$ and $|\psi^\lambda\rangle$ is an arbitrary state of irrep linked to a given partition $\lambda$. Inserting Eq. (36) into Eq. (29), we obtain the desired result of our theorem.

We note that to determine the allowed region of fidelities, it is enough to consider only vectors of real coefficients.

**Lemma 3.** To generate convex hull of the allowed region of fidelities, it is sufficient to consider pure states of real coefficients.

**Proof.** We need to show that, in our case, the majorization of complex pure states by real ones occurs. To prove that, note that our operators $\tilde{\mathcal{V}}_{(1i)}^\lambda$ (43) are symmetric and real, so they could be written in general form as:

$$\tilde{\mathcal{V}}_{(1i)}^\lambda = \begin{bmatrix} v_{i1} & v_{i2} & \cdots & v_{ik} \\ v_{21} & v_{22} & \cdots & v_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ v_{kk} & v_{2k} & \cdots & v_{kk} \end{bmatrix},$$

where $k = \dim \mathcal{H}_{\lambda}^S$. Now, let us write density matrix of a pure state $|\psi\rangle = (a_1, a_2, \ldots, a_k)^T$ with complex values (letter $C$ corresponds to the word complex):

$$\rho_C^\lambda = \begin{bmatrix} |a_1|^2 & a_1 a_2 & \cdots & a_1 a_k \\ a_1 a_2 & |a_2|^2 & \cdots & a_2 a_k \\ \vdots & \vdots & \ddots & \vdots \\ a_1 a_k & a_2 a_k & \cdots & |a_k|^2 \end{bmatrix}.$$
What is more, let us rewrite $\text{Tr} \left( \rho^C (1) \right)$ from Lemma 1 using (38) and (37), as follows:

$$
\text{Tr} \left( \rho^C (1) \right) = \sum_{j=1}^{k} |a_j|^2 v_{jj} + \sum_{j,s=1}^{k} (a_j a_s + \bar{a}_j \bar{a}_s) v_{js} =
\sum_{j=1}^{k} |a_j|^2 v_{jj} + 2 \sum_{j,s=1}^{k} \text{Re} (a_j a_s) v_{js}.
$$

(39)

We see that calculating the trace (39) with the operator $\rho^C$ is equivalent to calculating the trace (39) with a following matrix $\rho^C$:

$$
\rho^C = \begin{bmatrix}
|a_1|^2 & \text{Re}(a_1 a_2) & \ldots & \text{Re}(a_1 a_k) \\
\text{Re}(a_1 a_2) & |a_2|^2 & \ldots & \text{Re}(a_2 a_k) \\
& \ddots & \ddots & \vdots \\
\text{Re}(a_1 a_k) & \text{Re}(a_2 a_k) & \ldots & |a_k|^2
\end{bmatrix},
$$

(40)

where the new index $R - C$ means that we are left only with real numbers.

Let us denote the total phase in this case by $e^{i\phi}$ for $j = 1, \ldots, k$, so we are left with $\text{Re}(e^{i\phi}) = \text{Re}(\cos \phi_j + i \sin \phi_j) = \cos \phi_j$. We then obtain

$$
\rho^C = \begin{bmatrix}
|a_1|^2 & |a_1 a_2| & \ldots & |a_1 a_k| \\
|a_2 a_1| & |a_2|^2 & \ldots & |a_2 a_k| \\
& \ddots & \ddots & \vdots \\
|a_k a_1| & |a_k a_2| & \ldots & |a_k|^2
\end{bmatrix} 
\begin{bmatrix}
\cos(\phi_1 - \phi_2) & \ldots & \cos(\phi_1 - \phi_k) \\
\cos(\phi_2 - \phi_1) & 1 & \ldots & \cos(\phi_2 - \phi_k) \\
& \ddots & \ddots & \vdots \\
\cos(\phi_k - \phi_1) & \ldots & \cos(\phi_k - \phi_2) & 1
\end{bmatrix}
\begin{bmatrix}
\cos(\phi_1 - \phi_2) & \ldots & \cos(\phi_1 - \phi_k) \\
\cos(\phi_2 - \phi_1) & 1 & \ldots & \cos(\phi_2 - \phi_k) \\
& \ddots & \ddots & \vdots \\
\cos(\phi_k - \phi_1) & \ldots & \cos(\phi_k - \phi_2) & 1
\end{bmatrix}
\begin{bmatrix}
|a_1|^2 & |a_1 a_2| & \ldots & |a_1 a_k| \\
|a_2 a_1| & |a_2|^2 & \ldots & |a_2 a_k| \\
& \ddots & \ddots & \vdots \\
|a_k a_1| & |a_k a_2| & \ldots & |a_k|^2
\end{bmatrix}.
$$

(41)

where $\phi_j$ are phases connected with a given $|a_j|$ and by $\bullet$ we denote a Hadamard product of two matrices. Now our goal is to show that the matrices $A$ and $C$ are positive semi-definite. First let us define a set of vectors: $|\omega_j\rangle = \cos \phi_j |0\rangle + \sin \phi_j |1\rangle$, for $j = 1, 2, \ldots, k$. (42)

It easy to show that matrix $C$ can be rewrite in terms of vectors $|\omega_j\rangle$ i.e. $C_{ij} = \langle \omega_i | \omega_j \rangle$ for $i, j = 1, 2, \ldots, k$. Thanks to these operations we can conclude that $C$ is a square Gramian matrix. From [30] we know that every square Gramian matrix is positive-semidefinite.

Matrix $A$ is of the form $|\phi\rangle \langle \phi|$, with $|\phi\rangle = \sum |a_i| |i\rangle$, so it is positive-semidefinite. Now using fact (8) we obtain that our operator $\rho^C$ is also positive-semidefinite.

Because $\rho^C$ is real and symmetric we get from fact (9) that matrix $\rho^C$ posses real eigenvectors, so indeed it is mixture of pure states.

V. CASE STUDY - REGION FOR TRIPLES OF FIDELITIES

A. Partitions and transpositions

In our case we have four particles ($n = 4$) which means that allowed partitions are $\lambda_1 = (4), \lambda_2 = (3, 1), \lambda_3 = (2, 2), \lambda_4 = (2, 1, 1)$ and $\lambda_5 = (1, 1, 1, 1)$. Because we want to consider only qubits here, in our case $d = 2$, and hence $\lambda$ runs over binary partitions only or, equivalently, over Young diagrams with two rows (for other diagrams the multiplicity space $\mathcal{H}_d^{(\lambda)}$ becomes zero-dimensional). Thanks to this we are left only with: $\lambda_1 = (4), \lambda_2 = (3, 1), \lambda_3 = (2, 2)$. For the partition $\lambda_2 = (3, 1)$ unitary representations of a transpositions $T(1, 2)$, $T(1, 3)$, $T(1, 4)$, which we shall need, are [31]:

$$
\widetilde{\rho}^{\lambda_2}_{(12)} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{bmatrix}, \quad \widetilde{\rho}^{\lambda_2}_{(13)} = \begin{bmatrix}
1 & 0 & 0 \\
0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\
0 & -\frac{\sqrt{3}}{2} & \frac{1}{2}
\end{bmatrix}, \quad \widetilde{\rho}^{\lambda_2}_{(14)} = \begin{bmatrix}
-\frac{1}{2} & -\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\
\frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{1}{2} \\
-\frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}.
$$

(43)

For the partition $\lambda_3 = (2, 2)$ unitary transposition representations $T(1, 2)$, $T(1, 3)$, $T(1, 4)$, which we shall need, are [31]:

$$
\widetilde{\rho}^{\lambda_3}_{(12)} = \begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}, \quad \widetilde{\rho}^{\lambda_3}_{(13)} = \begin{bmatrix}
-\frac{1}{2} & -\frac{\sqrt{3}}{2} \\
\frac{\sqrt{3}}{2} & \frac{1}{2}
\end{bmatrix}, \quad \widetilde{\rho}^{\lambda_3}_{(14)} = \begin{bmatrix}
-\frac{1}{2} & \frac{\sqrt{3}}{2} \\
\frac{\sqrt{3}}{2} & \frac{1}{2}
\end{bmatrix},
$$

(44)

Note that for the partition $\lambda_1$ we have a trivial representation, so it will not be reported here, explanation is provided later.

B. Partial result: a fidelity region for each partition

According to Lemma 1, it occurs that all extreme points are just the convex hull of the figure from Figure 2. It allows us to use pure states only. In our situation, we use states which are of the form $|\psi^{(2, 2)}\rangle = \begin{bmatrix}
a_1 \\
a_2
\end{bmatrix}$ and $|\psi^{(3, 1)}\rangle = \begin{bmatrix}
a_1 \\
a_2 \\
a_3
\end{bmatrix}$, so they are real pure states.

Each of this state generates pure state $\rho^{(2, 2)}$ of the form (from now lower indexes in the case of $\rho$ are omitted):

$$
\rho^{(2, 2)} = \begin{bmatrix}
a_1^2 & a_1 a_2 \\
a_1 a_2 & a_2^2
\end{bmatrix},
$$

where $a_1^2 + a_2^2 = 1$, for the partition $\lambda = (2, 2)$, and $\rho^{(3, 1)} = \begin{bmatrix}
a_1^2 & a_1 a_2 & a_1 a_3 \\
a_1 a_2 & a_2^2 & a_2 a_3 \\
a_1 a_3 & a_2 a_3 & a_3^2
\end{bmatrix}$
\( a_1^2 + a_2^2 + a_3^2 = 1 \), for \( \lambda = (3, 1) \) respectively. According to Lemma 3 we can assume that \( a_1, a_2 \) and \( a_3 \) are real numbers. Inserting \( \rho^{(2,2)} \) and \( \rho^{(3,1)} \) into Equation 20 from Lemma 1 we obtain the following plot\(^2\) presented on Figure 2 with values \( F_{12}, F_{13} \) and \( F_{14} \) on axes. Now, we want to make a point, why the partition \( \lambda = (4) \) is not included on figures. It can be shown that in this case, all fidelities are equal to 0, and since we are only interested in situations when fidelity is ‘not too bad’ (and because, we did not want to ‘blur’ the main message of this paper), we decided to exclude this partition. But, strictly speaking, we should take convex hull, in the next step, also with this point and it should lead us to a ‘larger’ possible region for fidelities.

C. The main result: an allowed region for fidelities

Since, we have two partitions \((2, 2)\) and \((3, 1)\) and corresponding pure states: \( \rho_{13}^{(2,2)} \) and \( \rho_{13}^{(3,1)} \), we obtain two figures on Figure 2. But, we are interested in situation, where we can obtain a general answer to our question from Section II namely in situation where we have a mixture of both partitions: \( \sum p_i F_{ij} \). To solve this task, we can construct a convex hull of figures from Figure 2. It is presented on Figure 5. One can also see that our Figures 2 and 3 are invariant under rotation of angle \( 2\pi/3 \) along straight line \( F_{12} = F_{13} = F_{14} \). This corresponds to three conjugacy classes\(^3\).

D. Special case: a symmetric cloning

It can also be shown that in the case of symmetric cloning, maximal possible value of triplet \((F_{12}, F_{13}, F_{14})\) is \((\frac{2}{3}, \frac{2}{3}, \frac{2}{3})\), so it is in accordance with the formula obtained by Keyl et al. [14] (first using Equation 11), namely we are able to find such a point \((F_{12}, F_{13}, F_{14})\) that corresponds to the case of optimal symmetric fidelity \( F \). To find the maximal possible fidelity (for triplet \((F_{12}, F_{13}, F_{14})\)) which corresponds to the case of symmetric cloning, one method is to find a plane equation which includes the ellipse (which corresponds to the partition \((2, 2)\)). It can be shown that it is given by the formula \( F_{12} + F_{13} + F_{14} = \frac{3}{2} \). It can be obtained, for example, by taking coordinates of three non-linear points from Figure 2 and calculate plane equation according to the following well-known formula:

\[
\begin{bmatrix}
 x - x_1 & y - y_1 & z - z_1 \\
 x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\
 x_3 - x_1 & y_3 - y_1 & z_3 - z_1
\end{bmatrix} = 0. \quad (45)
\]

\(^2\)All plots are obtained using Mathematica software.

\(^3\)Two permutation \( \pi \) and \( \pi' \) are conjugate iff: \( \pi = \sigma \cdot \pi' \cdot \sigma^{-1} \), where \( \sigma \) is also a permutation and \( \sigma, \pi, \pi' \in S_n \).
So, of course we have to identify $x, y, z$ with $F_{12}, F_{13}$ and $F_{14}$ first, and then insert to values of three non-linear points $P_1(F_{12}, F_{13}, F_{14})$, $P_2(F_{12}, F_{13}, F_{14})$ and $P_3(F_{12}, F_{13}, F_{14})$ from the ellipse and solve equation which is obtained.

### E. Applications

Above-mentioned model allows us to reconstruct states from every subspace $H_S^2$ which satisfies not only the symmetric case ($F_1 = F_2 = F_3$) but also more general relations between fidelities. In this section we reconstruct states belonging to the subspaces corresponding to partitions $\lambda_1 = (2, 2), \lambda_2 = (3, 1)$ and satisfying the following formula

$$\max_{F_1} (F_1 + F_3 = 2F_2).$$

#### a) Partition $\lambda_1 = (2, 2)$

In this case we have two constrains $F_1 + F_3 = 2F_2$ and $a_1^2 + a_2^2 = 1$, so after solving this system of equations we obtain four allowed pairs of solutions

$$\begin{align*}
(a_1, a_2)_{(1)} &= \left( \frac{1}{2} \sqrt{\frac{3}{4}} - \frac{1}{2} \sqrt{2 - \sqrt{3}} \right), \\
(a_1, a_2)_{(2)} &= \left( -\frac{1}{2} \sqrt{2 - \sqrt{3}}, \frac{1}{2} \sqrt{\frac{3}{4}} + \sqrt{2 - \sqrt{3}} \right), \\
(a_1, a_2)_{(3)} &= \left( -\frac{1}{2} \sqrt{2 - \sqrt{3}}, -\frac{1}{2} \sqrt{2 - \sqrt{3}}, \frac{1}{2} \sqrt{\frac{3}{4}} \right), \\
(a_1, a_2)_{(4)} &= \left( \frac{1}{2} \sqrt{2 - \sqrt{3}}, \frac{1}{2} \sqrt{\frac{3}{4}} \right).
\end{align*}$$

Fidelities corresponding to the above pairs are

$$\begin{align*}
(F_{11}^A, F_{22}^A, F_{33}^A)_{(1)} &= \left( \frac{1}{4}(2 - \sqrt{3}), \frac{1}{2} \sqrt{\frac{3}{4}}(2 + \sqrt{3}) \right), \\
(F_{11}^A, F_{22}^A, F_{33}^A)_{(2)} &= \left( \frac{1}{4}(2 + \sqrt{3}), \frac{1}{2} \sqrt{\frac{3}{4}}(2 - \sqrt{3}) \right), \\
(F_{11}^A, F_{22}^A, F_{33}^A)_{(3)} &= \left( \frac{1}{4}(2 - \sqrt{3}), \frac{1}{2} \sqrt{\frac{3}{4}}(2 + \sqrt{3}) \right), \\
(F_{11}^A, F_{22}^A, F_{33}^A)_{(4)} &= \left( \frac{1}{4}(2 + \sqrt{3}), \frac{1}{2} \sqrt{\frac{3}{4}}(2 - \sqrt{3}) \right).
\end{align*}$$

We can see that maximal fidelity $F_{11}^A$ can be obtained for pair (2) and (4), so our reconstructed states are the following

$$\begin{align*}
\rho_{(2)}^{A} &= \frac{1}{4} \begin{pmatrix}
2 - \sqrt{3} & -1 \\
-1 & 2 + \sqrt{3}
\end{pmatrix}, \\
\rho_{(4)}^{A} &= \frac{1}{4} \begin{pmatrix}
2 - \sqrt{3} & 1 \\
1 & 2 + \sqrt{3}
\end{pmatrix}.
\end{align*}$$
Now, note something more general. Thanks to Lemma 4, we have $F_{1}^{1} = a_{2}^{2}$ and then $a_{1}^{2} = 1 - a_{2}^{2} = 1 - F_{1}^{1}$, so we can express every states $\rho^{1} \in \mathcal{H}_{1}^{2}$ in term of fidelity $F_{1}^{1}$

$$\rho^{1} = \left( \frac{1 - F_{1}^{1}}{\pm \sqrt{F_{1}^{1}(1 - F_{1}^{1})}} \right).$$

(50)

For example sign ‘+’ in (50) corresponds to states obtained from pairs $(a_{4}, b_{4})$ and $(a_{3}, b_{3})$ while sign ‘−’ corresponds to states obtain from pairs $(a_{1}, b_{1}), (a_{2}, b_{2})$.

b) Partition $\lambda_{2} = (3, 1)$

For the partition $\lambda_{2} = (3, 1)$ we have more complicated situation. Here we have three parameters $a_{1}, a_{2}, a_{3}$ but only two constrains $F_{1} + F_{3} = 2F_{2}$ and $a_{1}^{2} + a_{2}^{2} + a_{3}^{2} = 1$, so we first eliminate parameter $a_{1}$ and then express parameter $a_{2}$ as a function of $a_{2}$. Numerically we find that fidelity $F_{1}$ is maximal for the following values of parameters $a_{1}, a_{2}, a_{3}$

$$(a_{1}, a_{2}, a_{3})_{(1)} = (0.114, 0.318, 0.941),$$

$$(a_{1}, a_{2}, a_{3})_{(2)} = (-0.114, -0.318, -0.941),$$

$$(a_{1}, a_{2}, a_{3})_{(3)} = (0.114, -0.318, -0.941),$$

$$(a_{1}, a_{2}, a_{3})_{(4)} = (-0.114, 0.318, 0.941).$$

(51)

Corresponding states are

$$\rho_{1}^{\lambda_{2}} = \rho_{2}^{\lambda_{2}} = \begin{pmatrix} 0.013 & 0.036 & 0.107 \\ 0.036 & 0.101 & 0.299 \\ 0.107 & 0.299 & 0.886 \end{pmatrix},$$

$$\rho_{3}^{\lambda_{2}} = \rho_{4}^{\lambda_{2}} = \begin{pmatrix} 0.013 & -0.036 & -0.107 \\ -0.036 & 0.101 & 0.299 \\ -0.107 & 0.299 & 0.886 \end{pmatrix}.$$  

(52)

Note that fidelities in this case are $(F_{1}^{\lambda_{2}}, F_{2}^{\lambda_{2}}, F_{3}^{\lambda_{2}}) = (0.886, 0.556, 0.220)$. The numbers for our numerical data are obtained by Wolfram Mathematica Alpha [32]. Most likely they are indeed optimal.

VI. CONCLUSIONS

We show that by using representation theory, especially Young diagrams, action of the universal quantum cloning machine could be described. The method of irreps is quite powerful, because it allows us to decompose (usually big) Hilbert space into block (linked with a given partition $\lambda$) of smaller dimensions which, of course, are easier to deal with. For example, in our case, the Hilbert space $C^{16}$ is decomposed into blocks of dimension 1, 2 and 3 respectively.

We also show that convex hull could be made to obtain full knowledge about our model, in the sense that it gives raise to the full possible range of fidelities. What is more, we show that by restricting ones attention only to real pure states in each of the block, the full answer to our initial question is obtained. We point out that our UQCM gives correct results for the case of the symmetric cloning and we also prove that the optimal value, in the symmetric case, could be obtained. What is more important, our approach allows to reconstruct any given state connected to a given $n$-tuple of fidelities.

Of course, in future, it would be interesting to extend our method in such a way that it could also describe qudits.

Acknowledgments: We would like to thank Paweł Horodecki for many valubles discussions and comments on this paper. M.S. also would like to thank Piotr Migdal for discussions. P. C., M. H. and M. S. are supported by Polish Ministry of Science and Higher Education grant N N202 231937. M. S. is also supported by the International PhD Project "Physics of future quantum-based information technologies": grant MPD/2009-3/4 from Foundation for Polish Science. M.H. is also supported by EC IP Q-ESSENCE and ERC grant QOLAPS. Part of this work was done in National Quantum Information Centre of Gdańsk.

VII. APPENDIX

To prove our results need the following definitions and lemmas:

Definition 4. A set $Z$ is convex iff $\forall_{x, y \in Z} \forall_{0 \leq \mu \leq 1} \mu x + (1 - \mu) y \in Z$.

Definition 5. $L(\cdot): x \rightarrow y$ is affine map iff $\exists_{a, b} L(x) = \tilde{L}(x) + a$, where $\tilde{L}$ is linear.

Now, let us present two useful lemmas

Lemma 6. Suppose that $y, y' \in L(\Omega)$, then:

$$a y + (1 - a) y' \in L(\Omega)$$  

(it means that $L(\Omega)$ is a convex set).

Proof. Let us write:

$$L(x) = y = L(\sum_{i} p_{i} x_{i}) = \sum_{i} p_{i} \tilde{L}(x_{i}) + a,$$

$$L(x') = y' = L(\sum_{i} q_{i} x'_{i}) = \sum_{i} q_{i} \tilde{L}(x'_{i}) + a,$$

(54)

where we use the fact that $L$ is affine and $x = \sum_{i} p_{i} x_{i}$, $x' = \sum_{i} q_{i} x'$ for some $x_{i}, x'_{i} \in E(\Omega)$. Inserting (54) into (53) we get:

$$a (\sum_{i} p_{i} \tilde{L}(x_{i}) + a) + (1 - a) (\sum_{i} q_{i} \tilde{L}(x'_{i}) + a)$$

$$= a a + (1 - a) a + a \sum_{i} \tilde{L}(p_{i} x_{i}) + (1 - a) \sum_{i} \tilde{L}(q_{i} x'_{i})$$

$$= a \tilde{L}(a x + (1 - a) x') = L(\tilde{L}(a x + (1 - a) x')) \in L(\Omega).$$

(55)
Which ends the proof of Lemma.

Lemma 7. Suppose that we have an affine map \( L : X \to Y \). If by the \( \Omega \), we denote a convex subset of \( X \), then \( L(\Omega) \) is a convex set and \( L(E(\Omega)) \) can reproduce set \( L(\Omega) \) after taking the convex hull, i.e.:

\[
L(\Omega) = \text{conv} L(E(\Omega)).
\]

\[ (56) \]

Proof. We can write \( L(E(\Omega)) \subseteq L(\Omega) \), because we know that \( E(\Omega) \subseteq \Omega \). This together with Lemma (6) ends our proof.

Fact 8. The Hadamard product of two positive-definite matrices is again positive-definite.

Fact 9. If \( A \) is a real symmetric matrix, then all of its eigenvalues are real, and eigenvectors can always be chosen to be real.

Proof. Here, we only prove the second part of Fact 9 since the first one is obvious. Let \( \lambda \) be an real eigenvalue of the matrix \( A \) with an associated eigenvector \( v \). Let us write \( v = \text{Re}(v) + i \text{Im}(v) \), so that \( A(\text{Re}(v) + i \text{Im}(v)) = \lambda(\text{Re}(v) + i \text{Im}(v)) \). It implies that \( A \text{Re}(v) = \lambda \text{Re}(v) \) and \( A \text{Im}(v) = \lambda \text{Im}(v) \). Now, if \( \text{Re}(v) \neq 0 \), then it is a real eigenvector of \( A \). When \( \text{Re}(v) = 0 \), then \( \text{Im}(v) \) is a real eigenvector.