On colorings of bivariate random sequences

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Abstract—The ergodic sequences consisting of vectors \((\xi_n, \eta_n), n \geq 1\), over a finite alphabet \(A \times B\) are colored with \(e^{n\alpha}\) colors for \(A^n\) and \(e^{n\beta}\) colors for \(B^n\). Generic behavior of the colorings in terms of probabilities of monochromatic rectangles intersected with typical sets is examined. When \(n\) increases a big majority of pairs of colorings produces rectangles whose probabilities are bounded uniformly from above. Limiting rates of bounds are worked out in all regimes of the rates \(\alpha\) and \(\beta\) of colorings. As a consequence, generic behavior of the colorings in terms of Shannon entropies of the partitions into rectangles is described.

I. INTRODUCTION

Let \(N\) be a finite set and \((\xi_i)_{i \in N}\) a random vector taking a finite number of values. The collection of the Shannon entropies \(H(\xi_i, i \in I), I \subseteq N\), of all the subvectors of the vector can be interpreted as an entropic point of a Euclidean space. The last decade has seen renewed investigations of regions of the entropic points and closely related information theoretical inequalities [10], [4], [5], [6], [7]. Motivation has been drawn from numerous schemes of information theory [2], [9] and elsewhere.

Properties of regions of the entropic points and their limits were studied recently in [6] by taking independent identically distributed copies \((\xi_i^{(n)})_{i \in N}, n \geq 1\), of the vector and by randomly coloring the copies \(\xi_i^{(1)}, \ldots, \xi_i^{(n)}\) of a single distinguished variable \(\xi_i\) with \(e^{n\alpha}\) colors, \(\alpha_i \geq 0\). In this way new entropic points were constructed from old ones and their limits were described in [6, Theorem 3].

In this framework, a natural idea is to color independently the copies \(\xi_i^{(1)}, \ldots, \xi_i^{(n)}\) of each variable \(\xi_i\) with \(e^{n\alpha_i}\) colors, \(i \in N\), and to investigate the entropies of the partitions into monochromatic blocks. Thus, the \(N\)-tuples of colorings come into play and a majority of them is expected to have a similar, in some sense generic, behaviour in terms of entropies.

In this contribution, a restriction is made to bivariate vectors. The multivariate case likely differs only by additional technicalities and will be treated elsewhere. Instead of independent copies of a bivariate vector, an ergodic sequence consisting of vectors \((\xi_n, \eta_n), n \geq 1\), over a finite alphabet \(A \times B\) is considered. The more general assumption of ergodicity entails no additional technical complications.

A \(k\)-coloring of a set \(X\) is any mapping \(f\) of \(X\) into the set of colors \(\{1, \ldots, k\}\), to be denoted by \(\hat{k}\). The cardinality of \(X\) is denoted by \(|X|\).

In the bivariate sequence, first marginal sequence \(\xi_1, \ldots, \xi_n\) is colored with \(k_n = e^{n\alpha}\) colors and second one \(\eta_1, \ldots, \eta_n\) with \(\ell_n = e^{n\beta}\) colors, \(\alpha, \beta \geq 0\). If \(f_n: A^n \rightarrow \hat{k_n}\) and \(g_n: B^n \rightarrow \hat{\ell_n}\) are two such colorings, the product \(A^n \times B^n\) partitions into the \((f_n, g_n)\)-monochromatic rectangles

\[
f_n^{-1}(i) \times g_n^{-1}(j), \quad i \in \hat{k_n}, j \in \hat{\ell_n}.
\]

The main result, Theorem 2 in Section IV, deals with uniform upper bounds on probabilities of these rectangles intersected with special typical sets \(Z_n \subseteq A^n \times B^n\). The bounds decay exponentially with the rate

\[
h^{n\alpha\beta} = \min\{h_{\xi\eta} + h_\xi + h_\eta, h_\xi + \alpha + h_\eta + \beta\}
\]

where \(h_{\xi\eta}, h_\xi\) and \(h_\eta\) are the entropy rates of the ergodic sequence and their marginals, respectively. With increasing \(n\) the bounds become valid for more and more decisive majority of the pairs \((f_n, g_n)\) of colorings, as specified through a notion of a convergence to zero faster than exponentially (f.t.e.).

Using Theorem 2, the generic behaviour of the colorings in terms of entropies of partitions is described in Corollary 2: with increasing \(n\) for majority of the pairs \((f_n, g_n)\) of colorings the Shannon entropy of the partition into the \((f_n, g_n)\)-monochromatic rectangles is lower bounded by a sequence \(\alpha_n\) such that \(\alpha_n\) converges to \(h^{n\alpha\beta}_{\xi\eta}\). This rate is the highest possible, see Remark 2.

Main tools that underly proofs are collected in Section II. They include a couple of lemmas that go in the spirit back to [1], [3], and their dynamical versions. On the way to the main result, a general asymptotic scheme is singled out in Section III. It comprises crucial arguments towards the proof of Theorem 2 presented in Section IV.

II. PRELIMINARIES

The following lemma is a minor generalization and reformulation of [6, Lemma 6, p. 322]. The starting idea goes back to [1, Lemma 3.1, p. 230], see [6, Remark 4, p. 323]. Results of this sort have been of importance when studying common randomness and secrecy capacities, see [3, Appendix B].
Lemma 1. Let $k = e^q$ be a positive integer and $Q$ a finite set of measures on a finite set $X$ such that
\[ Q(x) \leq e^{-v} \cdot Q(X), \quad x \in X, Q \in Q, \] (1)
for some $v$. If $w \leq q$ then the number of those $k$-colorings $f$ of $X$ that violate
\[ Q(f^{-1}(i)) \leq e^{-w} \cdot Q(X), \quad i \in \hat{k}, Q \in Q, \] (2)
is at most
\[ |X|^k \cdot k \cdot |Q| \exp \left[ -\frac{1}{2} e^{-v-q} (e^{q-w} - 1)(q - w) \right]. \]

Proof: When $Q$ is empty or contains only the zero measure then (2) is never violated, and the assertion holds trivially. Otherwise, when $Q$ consists of probability measures this is a reformulation of [6, Lemma 6] where $\varepsilon = e^{q-w} - 1$ is nonnegative because $w \leq q$. The general case follows directly from this special one.

In Lemma 1, some colorings $f$ partition $X$ into monochromatic blocks $f^{-1}(i)$ such that their measures $Q(f^{-1}(i))$ are bounded from above, uniformly in $i$ and $Q$, while the number of the remaining colorings has an explicit upper bound. A symmetric counterpart of the lemma will be needed in the sequel. For readers convenience, its proof is worked out based on ideas of the proof of [6, Lemma 6].

Lemma 2. Under the assumptions of Lemma 1, if $w \geq q$ then the number of those $k$-colorings $f$ of $X$ that violate
\[ Q(f^{-1}(i)) \geq e^{-w} \cdot Q(X), \quad i \in \hat{k}, Q \in Q, \] (3)
is at most
\[ |X|^k \cdot k \cdot |Q| \exp \left[ -\frac{1}{2} e^{-v-q} (1 - e^{q-w}) \ln(2 - e^{q-w}) \right]. \]

Proof: Let the vector $(\eta_x)_{x \in X}$ consist of independent random variables, each one distributed uniformly on $\hat{k}$. Thus, a realization of the vector is a $k$-coloring of $X$. For $i \in \hat{k}$ and $Q \in Q$ let $\zeta_i$ be equal to $-(k-1)Q(x)$ if $\eta_x = i$ and to $Q(x)$ otherwise, $x \in X$. Then, these variables are independent and centered, that is their expectations are equal to zero. By (1), they are bounded in the absolute value by $e^{q-w}Q(X)$ and the sum of variances $(k - 1) \sum_{x \in X} Q(x)^2$ is majorized by $e^{q-w}Q(X)^2$. Hence, for $\varepsilon \geq 0$ the inequality
\[ P_r(\sum_{x \in X} \zeta_x > \varepsilon Q(X)) \leq \exp \left[ -\frac{\varepsilon}{2 e^{q-w}} \ln(1 + \varepsilon) \right] \]
follows from [6, Lemma 5] whenever $Q(X) > 0$, and holds trivially otherwise. This probability multiplied by $|X|^k$ is equal to the number of $k$-colorings $f$ that satisfy
\[ -\sum_{x \in f^{-1}(i)} (k-1)Q(x) + \sum_{x \in X \setminus f^{-1}(i)} Q(x) > \varepsilon Q(X). \]
This inequality rewrites to $kQ(f^{-1}(i)) < (1 - \varepsilon)Q(X)$. Since $w$ is at least $q$ the choice $\varepsilon = 1 - e^{q-w}$ is possible. Therefore, the number of those $k$-colorings $f$ that violate the inequality $Q(f^{-1}(i)) \geq e^{-w} \cdot Q(X)$ is at most
\[ |X|^k \cdot \exp \left[ -\frac{1}{2} e^{-v-q} (1 - e^{q-w}) \ln(2 - e^{q-w}) \right]. \]

This bound does not depend on $Q$ and $i$ whence the assertion follows.

Dynamical versions of Lemmas 1 and 2 are prepared below for later purposes.

Lemma 3. For $n \geq 1$ let $Q_n$ be a set of measures on a finite set $X_n$ such that
\[ Q(x) \leq e^{-nr_n} \cdot Q(X_n), \quad x \in X_n, Q \in Q_n, \] (4)
for a sequence $r_n$ that converges to a finite limit $h$. If $\alpha \geq 0$, $k_n = \lfloor e^{\alpha n} \rfloor$ and $s_n = \min \{r_n, \alpha, h \} - 2n^{-1/2}$ then the proportion of those $k_n$-colorings $f_n$ of $X_n$ that violate
\[ Q(f_n^{-1}(i)) \leq e^{-ns_n} \cdot Q(X_n), \quad i \in \hat{k_n}, Q \in Q_n, \] (5)
is at most
\[ e^{\alpha n} |Q_n| \exp \left[ -\frac{1}{2} e^{\alpha n r_n - qn} (e^{\alpha n} - s_n - 1) \sqrt{n} \right]. \]

Proof: If $q_n = \ln k_n$ then $\alpha n \geq q_n \geq \alpha n - 1$, and thus $q_n - ns_n \geq \sqrt{n}$. Lemma 1 implies that the proportion of those $k_n$-colorings $f_n$ that violate (5) is upper bounded by
\[ e^{\alpha n} |Q_n| \exp \left[ -\frac{1}{2} e^{\alpha n r_n - qn} (e^{\alpha n} - s_n - 1) \sqrt{n} \right]. \]
Omitting $\sqrt{n}$, the bracket is dominated by
\[ e^{\alpha n} \min \{r_n, \alpha, h \} - \alpha n (e^{\alpha n} - s_n - 1) \leq \frac{1}{2}(e^{\alpha n} - 1) \]
whence the assertion follows.

Let us say that a sequence of nonnegative numbers $p_n$ grows at most exponentially if the sequence $\frac{1}{n} \ln p_n$ is bounded from above. The sequence $p_n$ goes to zero faster than exponentially (f.t.e.) if $\frac{1}{n} \ln p_n$ tends to $-\infty$.

Lemma 3 is mostly used in a limiting version that only states existence of a sequence $s_n$.

Corollary 1. If a sequence $Q_n$ satisfies (4) with some $r_n \to h$, $|Q_n|$ grows at most exponentially, $\alpha \geq 0$ and $k_n = \lfloor e^{\alpha n} \rfloor$ then there exists a sequence $s_n$ converging to $\min \{\alpha, h \}$ such that the proportion of those $k_n$-colorings $f_n$ of $X_n$ that violate (5) goes to zero f.t.e.

A counterpart of this corollary is needed as well.

Lemma 4. Under the same assumptions as in Corollary 1, if $\alpha < h$ then there exists a sequence $t_n$ converging to $\alpha$ such that the proportion of those $k_n$-colorings $f_n$ of $X_n$ that violate
\[ Q_n(f_n^{-1}(i)) \geq e^{-nt_n} \cdot Q_n(X_n), \quad i \in \hat{k_n}, Q_n \in Q_n, \] (6)
goes to zero f.t.e.

Proof: Let $q_n = \ln k_n$ and $|Q_n| \leq e^{nu}$ for some $u$. If $w_n = \alpha n + 1$ then $w_n \geq q_n + 1$ and the sequence $t_n = \frac{w_n}{n}$ converges obviously to $\alpha$. Hence, Lemma 2 implies that the proportion of those $k_n$-colorings $f_n$ of $X_n$ that violate (6) is upper bounded by
\[ e^{\alpha n} e^{nu} \exp \left[ -\frac{1}{2} e^{\alpha n r_n - qn} (1 - \frac{1}{2}) \ln(2 - \frac{1}{2}) \right]. \]
This expression vanishes f.t.e.
III. MAIN ASYMPTOTIC SCHEME

In this section a general limiting scheme is presented that abstracts some typical situations encountered in bivariate ergodic sequences. This is believed to provide a better insight and, at the same time, to simplify and shorten proofs.

**Theorem 1.** For $n \geq 1$ let $Q_n$ be a finite measure on a finite set $X_n \times Y_n$ such that the cardinalities of the sets

$$\{y \in Y_n : Q_n(X_n \times \{y\}) \neq 0\}$$

grow at most exponentially. Let $h_X$, $h_Y$ and $h_{XY}$ be numbers such that $h_{XY} \leq h_X + h_Y$, the inequalities

$$Q_n(\{x\} \times Y_n) \leq e^{-n h_n} \cdot Q_n(X_n \times Y_n), \quad x \in X_n,$$

$$Q_n(X_n \times \{y\}) \leq e^{-n h_n} \cdot Q_n(X_n \times Y_n), \quad y \in Y_n,$$

hold with converging sequences $r_n \to h_X$, $s_n \to h_Y$, and

$$Q_n(x, y) \leq e^{-n h_n} \cdot Q_n(X_n \times \{y\}), \quad (x, y) \in X_n \times Y_n,$$

be satisfied with a converging sequence $t_n \to h_{XY} - h_Y$. Let further $\alpha, \beta \geq 0$, $k_n = \lfloor e^{n \alpha} \rfloor$, $\ell_n = \lfloor e^{n \beta} \rfloor$ and a pair $(f_n, g_n)$ consist of a $k_n$-coloring $f_n$ of $X_n$ and an $\ell_n$-coloring $g_n$ of $Y_n$.

If $h_X < h_Y$ or $h_{XY} \geq h_Y$ then there exists a sequence

$$w_n \to h_{XY}^\beta = \min \{h_{XY} + h_X, h_X + h_Y, h_{XY}, h_X + h_Y\}$$

such that the proportion of those pairs $(f_n, g_n)$ that violate

$$Q_n(f_n^{-1}(i) \times g_n^{-1}(j)) \leq e^{-n w_n} \cdot Q_n(X_n \times Y_n), \quad i \in \kappa_n, j \in \ell_n,$$

(11)


go to zero f.t.e.

**Proof:** Let $Q_n^X$ be the family consisting of the single measure on $X_n$ given by $x \mapsto Q_n(\{x\} \times Y_n)$. By (8), Corollary 1 applies to the sequence $Q_n^X$. There exists a sequence $u_n^+$ converging to $\min\{\alpha, h_X\}$ such that the proportion of those $k_n$-colorings $f_n$ of $X_n$ that violate

$$Q_n(f_n^{-1}(i) \times Y_n) \leq e^{-n u_n^+} \cdot Q_n(X_n \times Y_n), \quad i \in \kappa_n,$$

(12)

go to zero f.t.e.

Let us consider the set of measures $Q_n^{X \times Y}$ on $X_n \times Y_n$ given by

$$x \mapsto Q_n(x, y), \quad y \in Y_n,$$

that are nonzero. By (7), the cardinality of this set grows at most exponentially. Hence, Corollary 1 based on (10) implies existence of a sequence $p_n^+$ converging to $\min\{\alpha, h_{XY} - h_Y\}$ such that the proportion of those $k_n$-colorings $f_n$ that violate

$$Q_n(f_n^{-1}(i) \times \{y\}) \leq e^{-n p_n^+} \cdot Q_n(X_n \times \{y\}), \quad i \in \kappa_n, y \in Y_n,$$

(13)

go to zero f.t.e.

If a coloring $f_n$ satisfies (13) then by (9)

$$Q_n(f_n^{-1}(i) \times \{y\}) \leq e^{-n |s_n + p_n^+|} \cdot Q_n(X_n \times Y_n), \quad i \in \kappa_n, y \in Y_n,$$

(14)

where $s_n + p_n^+$ converges to $\min\{\alpha + h_Y, h_{XY}\}$.

1. Let us assume first that $\alpha < h_X$. On account of (8), Lemma 4 applies to the sequence $Q_n^Y$. There exists a sequence $u_n^-$ converging to $\alpha$ such that the proportion of those $k_n$-colorings $f_n$ that violate

$$Q_n(f_n^{-1}(i) \times Y_n) \geq e^{-n u_n^-} \cdot Q_n(X_n \times Y_n), \quad i \in \kappa_n,$$

(15)

go to zero f.t.e.

Let $F_n$ be the family of those $k_n$-colorings $f_n$ that violate an inequality in (12), (13), (14) or (15). By three above convergence statements, the proportion $\|F_n||X_n|^{-k_n}$ goes to zero f.t.e. If $f_n \notin F_n$ then (14) and (15) combine to

$$Q_n(f_n^{-1}(i) \times \{y\}) \leq e^{-n u_n^-} \cdot Q_n(f_n^{-1}(i) \times Y_n), \quad i \in \kappa_n, y \in Y_n,$$

(16)

where $v_n = s_n + p_n^+ - u_n^-$ converges to $\min\{h_Y, h_{XY} - \alpha\}$, denoted in the sequel by $h_v$.

Let $g_n = \min\{v_n, \beta, h_v\} - 2n^{-1/2}$ play the role of $s_n$ from Lemma 3 that is applied to the sets $Q_{f_n}$ of measures on $Y_n$ given by

$$y \mapsto Q_n(f_n^{-1}(i) \times \{y\}), \quad i \in \kappa_n,$$

for $k_n$-colorings $f_n \notin F_n$ and $n \geq n_0$. By (16), if $G_{f_n}$ denotes the family of those $\ell_n$-colorings $g_n$ of $Y_n$ that violate

$$Q_n(f_n^{-1}(i) \times g_n^{-1}(j)) \leq e^{-n v_n} \cdot Q_n(f_n^{-1}(i) \times Y_n), \quad i \in \kappa_n, j \in \ell_n,$$

(17)

then the proportion $\|G_{f_n}||Y_n|^{-\ell_n}$ is upper bounded by

$$c_n = e^{-n \beta} ||F_{f_n}| |X_n|^{-k_n} |Y_n|^{-\ell_n}$$

Here, obviously $|Q_{f_n}| \leq e^{n \alpha}$.

Therefore, the cardinality of the set

$$\mathcal{H}_n = \{(f_n, g_n) : f_n \notin F_n \text{ or } (f_n \notin F_n \text{ and } g_n \notin G_{f_n})\}$$

is upper bounded by

$$\|F_{f_n}||Y_n|^{-\ell_n} + \sum_{f_n \notin F_n} \|G_{f_n}||X_n|^{-k_n} |Y_n|^{-\ell_n}$$

and the proportion $\|\mathcal{H}_n||X_n|^{-k_n} |Y_n|^{-\ell_n}$ is at most the sum of $\|F_{f_n}||X_n|^{-k_n} |Y_n|^{-\ell_n}$ with $c_n$. It follows that this proportion goes to zero f.t.e.

Let $w_n = q_n + u_n^+$. This sequence converges to

$$\min\{\alpha + \beta, \alpha + h_Y, h_{XY}\}$$

which equals $h_{XY}^\alpha$ because $\alpha < h_X$. If a pair of colorings $(f_n, g_n)$ does not belong to $\mathcal{H}_n$, thus $f_n \notin F_n$ and $g_n \notin G_{f_n}$, then (12) and (17) hold. Since (11) is their consequence the proportion of those pairs $(f_n, g_n)$ that violate (11) is upper bounded by $\|\mathcal{H}_n||X_n|^{-k_n} |Y_n|^{-\ell_n}$, going to zero f.t.e.

2. It remains to consider $\alpha \geq h_X$ and $\beta \geq h_Y$. Let $Q_n^Y$ be the family consisting of the single measure on $Y_n$ given by $y \mapsto Q_n(X_n \times \{y\})$. Corollary 1 can be applied to the sequence $Q_n^Y$ and $\beta$ in the role of $\alpha$. By (9) and $\beta \geq h_Y$, there exists a sequence $v_n^+$ converging to $h_Y$ such that the proportion of those $\ell_n$-colorings $g_n$ of $Y_n$ that violate

$$Q_n(X_n \times g_n^{-1}(j)) \leq e^{-n v_n^+} \cdot Q_n(X_n \times Y_n), \quad j \in \ell_n,$$

(18)
by the entropy rate $h_{X|Y} = \lim \frac{1}{n} \ln H(P^n)$ of the bivariate sequence, and the entropies rates of the marginal sequences, $h_X = \lim \frac{1}{n} \ln H(P^n_X)$ and $h_Y = \lim \frac{1}{n} \ln H(P^n_Y)$, correspond to $h_X$ and $h_Y$, respectively.

The following theorem asserts rigorously what was earlier mentioned as the generic behavior of pairs of colorings in terms of probabilities of monochromatic rectangles intersected with typical sets.

Theorem 2. For $n \geq 1$ let $k_n = [e^{\alpha n}]$ and $\ell_n = [e^{\beta n}]$ where $\alpha, \beta \geq 0$. There exist a sequence of sets $Z_n \subseteq A^n \times B^n$ and a sequence of numbers $w_n$ such that $P_n(Z_n) \rightarrow 1$, $w_n \rightarrow h^0_{X|Y}$ and the proportion of those pairs $(f_n, g_n)$, consisting of a $k_n$-coloring $f_n$ of $A^n$ and an $\ell_n$-coloring $g_n$ of $B^n$, that violate

$P_n((f_n^{-1}(i) \times g_n^{-1}(j)) \cap Z_n) \leq e^{-wn_n}, \ i \in \widehat{k}_n, j \in \widehat{\ell}_n,$

(23)
goesto zero f.t.e.

Proof: By exchanging the coordinate variables $\xi_n$ and $\eta_n$ if necessary there is no loss of generality when assuming $\alpha < h_{\xi}$ or $\beta > h_Y$. The subset $Z_n$ is constructed below via the entropy-typical sets

$T^{\alpha}_{\xi,\delta} = \{ x \in A^n : P^n(x) \approx e^{-n[h_{\xi} \pm \delta]} \},$ $T^{\eta}_{\eta,\delta} = \{ y \in B^n : P^n(y) \approx e^{-n[h_{\eta} \pm \delta]} \},$ $T^{\alpha\beta}_{\xi,\eta,\delta} = \{(x, y) \in A^n \times B^n : P_n(x, y) \approx e^{-n[h_{\xi \eta} \pm \delta]} \}.$

Here, the symbol $\approx$ means that the number on the left is between the two numbers on the right.

Since the sequence $(\xi_n, \eta_n)$ and two marginal sequences are ergodic the Shannon-McMillan-Breiman theorem, known also as the asymptotic equipartition property [8, Sections 1.5 and 1.6], implies that for every positive $\delta$ the probabilities $P^{\alpha}_{\xi}(T^{\alpha}_{\xi,\delta})$, $P^{\eta}_{\eta}(T^{\eta}_{\eta,\delta})$ and $P_n(T^{\alpha\beta}_{\xi,\eta,\delta})$ tend to 1. By standard diagonal arguments, there exists a positive sequence $(\delta_n)_{n \geq 1}$ converging to zero, perhaps rather slowly, such that each of the sequences $P^{\alpha}_{\xi}(T^{\alpha}_{\xi,\delta_n})$, $P^{\eta}_{\eta}(T^{\eta}_{\eta,\delta_n})$ and $P_n(T^{\alpha\beta}_{\xi,\eta,\delta_n})$ converges to one.

Let $U_n$ denote the intersection of $T^{\alpha}_{\xi,\delta_n} \times B^n$, $A^n \times T^{\eta}_{\eta,\delta_n}$ and $T^{\alpha\beta}_{\xi,\eta,\delta_n}$. If $p_n = 1 - [1 - P_n(U_n)]^{1/2}$ and

$Y^{*}_n = \{ y \in B^n : P_n((A^n \times \{ y \}) \cap U_n) \geq p_n \cdot P^n_n(y) \}$

then the estimations

$p_n[1 - P^n_n(Y^{*}_n)] \geq \sum_{y \in B^n \setminus \varepsilon_n} p_n \cdot P^n_n(y) \geq \sum_{y \in B^n \setminus \varepsilon_n} P_n((A^n \times \{ y \}) \cap U_n) = P_n(U_n) - P_n((A^n \times Y^{*}_n) \cap U_n) \geq P_n(U_n) - P^n_n(Y^{*}_n)$

and $p_n < 1$ imply

$P^n_n(Y^{*}_n) \geq \frac{P_n(U_n) - p_n}{1 - p_n} = p_n.$

Obviously, $P^n_n(Y^{*}_n) = 1$ if $p_n = 1$.

Let $Z_n$ denote the intersection of $A^n \times Y^{*}_n$ with $U_n$. Then it is possible to conclude subsequently that the sequences $P_n(U_n)$, $p_n$, $P^n_n(Y^{*}_n)$ and $P_n(Z_n)$ converge to one. Let $Q_n$ denote the restriction of $P_n$ to $Z_n$. To apply Theorem 1, its assumptions are verified as follows.

The set in (7) is contained in $Y_n = B^n$ so that its cardinality grows at most exponential.

Let $r_n$ be equal to $h_{\xi} - \delta_n + \frac{1}{n} \ln P_n(Z_n)$ provided $P_n(Z_n)$ is positive. If $x \not\in T^{\alpha}_{\xi,\delta_n}$ then $Q_n((x) \times B^n)$ vanishes and the inequality in (8) holds trivially. Otherwise, if $x \in T^{\alpha}_{\xi,\delta_n}$ then

$Q_n((x) \times B^n) \leq P_n((x) \times B^n) = P^n_n(x) \leq e^{-n[h_{\xi} - \delta_n]} = e^{-n\alpha} \cdot P_n(Z_n) = e^{-n\alpha} \cdot Q_n(A^n \times B^n)$

IV. ERGODIC BIVARIATE SEQUENCES

Let $(\xi_n, \eta_n)$, $n \geq 1$, be a bivariate ergodic sequence with the states in a finite product $A \times B$. The distribution of the first $n$ vectors of the sequence is denoted by $P_n$ and its marginals to $A^n$ and $B^n$ by $P^{\alpha}_n$ and $P^{\eta}_n$, respectively. In this section Theorem 1 is applied to the restrictions of $P_n$ to certain subsets of $A^n \times B^n = A^n \times B^n$ that are constructed by a notion of entropic typicality. The role of $h_{X|Y}$ is played by the entropy rate $h_{X|Y} = \lim \frac{1}{n} \ln H(P^n)$ of the bivariate sequence, and the entropies rates of the marginal sequences, $h_X = \lim \frac{1}{n} \ln H(P^n_X)$ and $h_Y = \lim \frac{1}{n} \ln H(P^n_Y)$, correspond to $h_X$ and $h_Y$, respectively.

The following theorem asserts rigorously what was earlier mentioned as the generic behavior of pairs of colorings in terms of probabilities of monochromatic rectangles intersected with typical sets.
so that (8) is verified with a sequence \( r_n \) converging to \( h_\xi \).

For \( s_n \) defined through \( h_\xi - \delta_n + \frac{1}{n} \ln P_n(Z_n) \), a verification of (9) is analogous to that of (8) and omitted here.

Let \( t_n \) be equal to \( h_\xi - h_n - 2\delta_n + \frac{1}{n} \ln p_n \) provided \( p_n \) is positive. If \( (x, y) \not\in Z_n \) then \( Q_n(x, y) \) vanishes and the inequality in (10) holds trivially. Otherwise, \( (x, y) \in T^x_{\xi_n, \delta_n} \) and \( y \in T^y_{\eta_n, \delta_n} \) imply

\[
Q_n(x, y) = P_n(x, y) \leq e^{-n[h_\xi - \delta_n]} \leq e^{-n[h_\xi - \delta_n]} e^{n[h_\eta + \delta_n]} P_n(y).
\]

Using \( y \in Y^*_n \) and \( Z_n = (A^n \times Y^*_n) \cap U_n \),

\[
p_n \cdot P_n(y) \leq P_n((A^n \times \{y\}) \cap U_n) = Q_n(A^n \times \{y\}).
\]

Combining above estimations, it follows that (10) is verified with a sequence \( t_n \to h_\xi - h_n \). Obviously, \( h_\xi \leq h_\xi + h_n \).

Therefore, Theorem 1 implies existence of a sequence \( v_n \) converging to \( h_\xi^{\alpha\beta} \) such that the proportion of those pairs \( (f_n, g_n) \) that violate

\[
Q_n(f_n^{-1}(i) \times g_n^{-1}(j)) \leq e^{-n v_n} \cdot P_n(Z_n), \quad i \in \hat{k}_n, \ j \in \hat{c}_n,
\]

goes to zero f.t.e. Writing \( w_n = v_n - \frac{1}{n} \ln P_n(Z_n) \), the above inequalities coincide with (23) and \( w_n \to h_\xi^{\alpha\beta} \).

The following consequence of Theorem 2 describes what was alluded to as the generic behavior of the colorings in terms of Shannon entropies of the partitions into rectangles.

**Corollary 2.** If \( H(P_n|f_n, g_n) \) denotes the Shannon entropy under \( P_n \) of the partition of \( A^n \times B^n \) into the \( (f_n, g_n) \)-monochromatic rectangles \( f_n^{-1}(i) \times g_n^{-1}(j) \), \( i \in \hat{k}_n, \ j \in \hat{c}_n \), there exists a sequence \( a_n \) converging to \( h_\xi^{\alpha\beta} \) such that the proportion of those pairs \( (f_n, g_n) \) of colorings that violate

\[
\frac{1}{n} H(P_n|f_n, g_n) \geq a_n \to 0 \text{ f.t.e.}
\]

Proof: If \( Z_n = (A^n \times B^n) \setminus Z_n \) then

\[
\sum_D \left[ P_n(D \cap Z_n) \ln \frac{P_n(D \cap Z_n)}{P_n(D) P_n(Z_n)} + P_n(D \cap Z_n^c) \ln \frac{P_n(D \cap Z_n^c)}{P_n(D) P_n(Z_n^c)} \right]
\]

is nonnegative by convexity. Here, the summation runs over the \( (f_n, g_n) \)-monochromatic rectangles \( D \). This implies

\[
H(P_n|f_n, g_n) + \ln 2 \geq -\sum_D P_n(D \cap Z_n) \ln P_n(D \cap Z_n).
\]

By Theorem 2, for a sequence \( w_n \) converging to \( h_\xi^{\alpha\beta} \), the summand on the right is majorized by \( -n w_n P_n(D \cap Z_n) \). Taking \( a_n = w_n P_n(Z_n) - \frac{1}{n} \ln 2 \) the assertion follows.

**Remark.** If the assertion of Corollary 2 holds for a sequence \( a_n \) converging to some \( a \) instead of \( h_\xi^{\alpha\beta} \) then \( a \) cannot exceed this number. This follows from

\[
\limsup_{n \to \infty} \sup_{f_n, g_n} \frac{1}{n} H(P_n|f_n, g_n) \leq h_\xi^{\alpha\beta}.
\]

To prove the inequality, \( H(P_n|f_n, g_n) \) is majorized by the sum of \( H(P_n^x|f_n) \) and \( H(P_n^y|g_n) \), defined analogously. The former summand is dominated by \( H(P_n^x) \) and \( \alpha \) because the partition of \( A^n \) into \( f_n^{-1}(i), i \in \hat{k}_n \), has at most \( k_n \) blocks. Similarly, the latter summand is dominated by \( H(P_n^y) \) and \( \beta \).

It follows that the left-hand side of (24) is at most

\[
\min\{h_\xi, \alpha\} + \min\{h_\xi, \beta\}.
\]

This and \( H(P_n|f_n, g_n) \leq H(P_n) \) imply (24).

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