The transfer-function approach to travelling waves in path graphs

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Abstract

The paper presents a novel approach for the analysis and control of distributed systems with an underlying path-graph topology based on the so-called wave-based control perspective. With the help of irrational wave transfer functions, the approach allows us to identify and describe the travelling waves in heterogenous path graphs. The main advantage of this approach is that it shows how a change in a system’s state propagates along the graph and how it is affected by graph heterogeneities. A graph heterogeneity acts as a boundary for wave propagation. Undesired effects of the boundary, such as amplification/attenuation, long transient or string instability, can be compensated by the feedback controllers introduced in this paper. Two types of path-graph boundaries, the soft and hard boundaries, are mathematically investigated. The type of the boundary determines the character of wave propagation in a heterogenous path graph. Each boundary is described by four boundary transfer functions, expressed in terms of the wave transfer functions. The properties of the boundary transfer functions are influenced by those of the wave transfer functions, for instance, the boundedness of the DC gains.

Keywords: travelling waves, wave transfer function, path graph, graph boundaries, irrational transfer function, infinite dimensional system

1. INTRODUCTION

1.1. Motivation

The study of distributed systems, that is a group of agents interacting with each other, started in the 1960s. Such studies focus on analysing both local and overall system performances under different underlying topologies. Their ultimate goal is to reach a consensus on the states of the individual agents.

The simplest underlying topology is a path graph, where each agent, except of the first and last ones, interacts with two neighbouring agents. Such a topology serves, for instance, as a model of vehicular platoons [25], or discretized flexible structures [5]. The nodes and edges of a path graph represent the local dynamics of individual agents and the connections between them, respectively. The later can be either mechanical, e.g., a spring or dashpot in a mass-spring model, or virtual, e.g., a controller implemented onboard an agent. Note that some virtual connections, for instance, the unidirectional connection in the predecessor following algorithm used in vehicular platooning control, have no physical equivalent since they violate Third Newton’s law. The connection between agents on a path leads to the situation where a change in the state of even a remote agent on the path will affect all the other agents. The propagation of a change along a path can be described with the help of travelling waves.

The heterogeneities in a path graph, for instance, different dynamics of the agents, act as boundaries for the travelling waves and cause the waves to be fully or partially reflected. However, a wave reflection is usually an undesirable effect since it significantly prolongs the settling time. One way to avoid the reflection is to remove all system heterogeneities by forcing the agents to be identical, which is usually impractical, or even impossible. On the other hand, a wave description allows us to design a feedback controller for minimizing system heterogeneity effects by modifying the boundaries between agents without changing their dynamics. For example, we design a controller to prevent the reflections of a wave travelling along the graph, which significantly shortens the settling time.

In this paper, we aim to provide the mathematical description of the travelling waves propagating along a heterogenous path graph. The underlying questions are: How do the graph heterogeneities affect the travelling wave? How to describe mathematically this effect? How to compensate it?

1.2. Laplacian and transfer function approaches to path graphs

A popular tool for the convergent analysis of distributed systems comes from algebraic graph theory, for instance, the spectral properties of the Laplacian matrix, see [15], [19] and [21] for a thorough overview.

The path-graph analysis focuses on the transfer function between two consecutive nodes and the transfer function from the first to the last node. Scaling of the magnitude of frequency response with the increasing size of a path graph is analyzed, see e.g., [16], [10] and [11], as well as the locations of poles and zeros affecting the input-output behaviour of a path graph, see e.g., [2]. Another topic of interest is the scaling of the transient with the size of a path graph, see e.g., [26], [3] and [12].

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The path-graph analysis is related to vehicular platooning control, where the string stability is widely used as an analytic measure of system performance. Although the string stability usually refers to vehicle-following applications, see e.g., [6] and [20], it is defined for an arbitrary interconnected system [24].

1.3. Transfer-function approach to travelling waves

The description of travelling waves by transfer functions started in 1980’s using the Fourier-based spatial transfer functions [7]. The transfer-function approach has recently been revisited in a series of papers for lumped models (see [17] and [27]) and for continuous flexible structures (see [9] and [22]).

The transfer functions for travelling waves are expressed by irrational functions since they describe infinite dimensional system. The analyses of irrational and rational transfer functions differ in several aspects, see [4] for a thorough overview. For instance, to the many examples of the inverse Laplace transform of irrational transfer functions, see [1], it is difficult to find an exact impulse response for some of them.

This paper continues the research started in [13], where the wave propagation in a platoon of identical vehicles is described. The platoon can be viewed as a chain of identical mechanically-unconnected lumped models. The virtual interaction is created by the controllers onboard each vehicle. A natural extension of this model is to consider a chain of nonidentical lumped models. The first step in the treatment of such a model is given in [13], where the description is limited to a mass-spring model. We generalize it by considering an arbitrary lumped model that can be connected either mechanically or virtually. The preliminary results are presented in [14], where we introduce the soft boundary in a chain of lumped models. We follow this concept of boundaries in path graphs and introduce the second fundamental type of boundary, the hard boundary. Although the boundaries are virtual in nature, they principally affect the overall system behaviour. We present some fundamental properties of the boundaries and design wave-absorbing controllers for both types of boundaries.

2. Mathematical preliminaries

2.1. Local control law

The nth node in a path graph is modelled in the Laplace domain as

\[ X_n(s) = P_n(s)U_n(s), \]

where \( s \) is the Laplace variable, \( X_n(s) \) is a state of the node, \( P_n(s) \) is the open-loop transfer function of the node and \( U_n(s) \) is the input to the node. The input is generated either by the mechanical connection with the neighbouring nodes, or by a local controller of the node with the task of equalizing the state \( X_n(s) \) with the states of the neighbouring nodes.

In a path graph, the connections to the neighbouring front and rear nodes are represented by the transfer functions \( C_{fn}(s) \) and \( C_{rn}(s) \), respectively, as

\[ U_n(s) = C_{fn}(s)(X_{n-1}(s) - X_{n}(s)) + C_{rn}(s)(X_{n+1}(s) - X_{n}(s)). \]

We denote the front node transfer function (NTF) and rear NTF by \( M_{fn}(s) = P_n(s)C_{fn}(s) \) and \( M_{rn}(s) = P_n(s)C_{rn}(s) \), respectively. The resulting model of the nth node is

\[ X_n(s) = M_{fn}(s)(X_{n-1}(s) - X_{n}(s)) + M_{rn}(s)(X_{n+1}(s) - X_{n}(s)). \]

The first node \((n = 0)\), the so-called leader, is externally controlled and serves as a reference signal for the graph. The last node \((n = N)\) of the path graph is described as

\[ X_N(s) = M_{fn}(s)(X_{N-1}(s) - X_{N}(s)). \]

2.2. Wave transfer function

This paper continues the research started in [13], where the wave propagation along a platoon of identical vehicles is studied. The results are also applicable for any lumped system with a path-graph topology. We briefly summarize the main idea of [13] in terms of nodes and edges.

![Figure 1: Scheme of a homogenous path graph. The squares are nodes with local dynamics described by \( G(s) \). The edges between the nodes are illustrated by springs representing mechanical or virtual links.](image-url)

The state of the nth node in a path graph with identical nodes and edges is described by two components, \( A_n(s) \) and \( B_n(s) \), that represent two waves propagating along a graph in the forward and backward directions, respectively. The mathematical model of a path graph with infinitely number of nodes, see Fig. [1] is

\[ X_n(s) = A_n(s) + B_n(s), \]
\[ A_{n+1}(s) = G(s)A_n(s), \]
\[ B_n(s) = G^{-1}(s)B_{n-1}(s), \]

where the wave transfer function (WTF), \( G(s) \), describes the wave propagation between two consecutive nodes and \( G^{-1} = 1/G \),

\[ G(s) = \frac{1}{2} \alpha(s) - \frac{1}{2} \sqrt{\alpha^2(s) - 4}, \]
\[ G^{-1}(s) = \frac{1}{2} \alpha(s) + \frac{1}{2} \sqrt{\alpha^2(s) - 4}, \]

with \( \alpha(s) = 2 + 1/M(s) \), or, alternatively, \( \alpha(s) = G(s) + G^{-1}(s) \). For a homogenous path graph, the front and rear NTFs are identical and the same for each node, \( M_{fn}(s) = M_{rn}(s) = M(s) = \)
For example, if the last node of a path graph is a free-end boundary, located at a node with a bidirectional connection to one neighbour only. The controller absorbs the travelling wave by calculating the incident part of the wave and adding this part as a reference wave. The controller can be designed to prevent a reflection of the incident wave. The node absorbing the wave can be described in a similar way. This concludes the brief summary of Lemma 1. The stability of the wave transfer function is treated by the following Lemma.

**Lemma 1.** The wave transfer function $G(s)$ is exponentially stable if

$$
\frac{M(s)}{1 + \lambda M(s)}
$$

is exponentially stable for all $\lambda \in (0, 4)$ and $M(s)$ has no right-half plane poles.

**Proof.** The proof is given in Appendix A.

**Remark.** Lemma 1 can be generalized even for the NTF defined by $M(s) = \frac{1}{s^2}$, for which $M(s)/(1 + \lambda M(s))$ is only BIBO (bounded-input, bounded-output) stable. In this case, the Nyquist curve of NTF lies on the non-positive real axis ($-\infty, 0$). Since the curve crosses the interval $(-\infty, -1/4)$ only in the horizontal direction, the function $\sqrt{s^2 + 4s^2}$ is analytic. Hence, the wave transfer function $G(s)$, generated by $M(s) = \frac{1}{s^2}$, is exponentially stable.

We denote the WTFS of two nodes with different dynamics by $G(s)$ and $H(s)$. We call them $G$- and $H$-nodes and symbolize them by blue and red colors in the following figures, respectively.

### 2.3. Analyzed properties

The DC gain describing the steady-state amplification of a system, and $L_2$ string stability describing the amplification of disturbance in a system with the path-graph topology, are important analytical tools for system analysis. They are defined as follows.

**Definition 1.** The DC gain $\kappa_G$ of the transfer function $G(s)$ is defined as

$$
\kappa_G = \lim_{s \to 0} G(s).
$$

**Definition 2.** (From [6]) A system is called $L_2$ string stable if there is an upper bound on the $L_2$-induced system norm of $T_{0,n}$ that does not depend on the number of nodes, where $T_{0,n}$ is the transfer function from the state of the first node to the state of the $n$th node.

### 3. The soft boundary

We will call the boundary located between two nodes where the rear NTF of the $n$th node differs from the front NTF of the $(n + 1)$th node as the soft boundary. Such a boundary arises, for example, in a mass-spring model with identical springs and different masses, see Fig. 2. It is defined as follows.

**Definition 3.** The soft boundary is a virtual boundary between two nodes, indexed $n$ and $n + 1$, defined by

$$
X_n = M_{r,n}(X_{n+1} - X_n),
$$

$$
X_{n+1} = M_{l,n+1}(X_{n} - X_n) + M_{r,n+1}(X_{n+1} - X_n),
$$

where $M_{r,n} \neq M_{l,n+1}$.

![Figure 2: Scheme of a path graph with the soft boundary. The blue and red squares are nodes with the WTFS of $G(s)$ and $H(s)$, respectively. The edges between the nodes are illustrated by springs representing mechanical or virtual links. The blue-red spring is the soft boundary.](image)

The adjective ‘soft’ emphasizes the fact that the soft boundary is not located at a specific position in a graph, but exists at the edge between two nodes. However, it would be rather misleading to call it an edge boundary since the soft boundary is defined not only by edge dynamics, but also by node dynamics.
3.1. Mathematical description and properties

**Theorem 1.** A soft boundary is described by the following four boundary transfer functions (BTFs),

\[
T_{aa} = \frac{A_{n+1}}{A_n} = \frac{H - HG^2}{1 - HG}, \quad T_{bb} = \frac{B_{n+1}}{B_n} = \frac{HG - H^2}{1 - HG},
\]

\[
T_{ba} = \frac{B_n}{A_{n+1}} = \frac{G - H^2G}{1 - HG}, \quad T_{ab} = \frac{B_n}{A_n} = \frac{HG - G^2}{1 - HG},
\]

where

\[
G = \frac{1}{2}a_1 - \frac{1}{2} \sqrt{a_1^2 - 4}, \quad H = \frac{1}{2}a_2 - \sqrt{a_2^2 - 4},
\]

\[\alpha_1 = 2 + 1/M_{f,n} \text{ and } \alpha_2 = 2 + 1/M_{r,n+1}.\]

**Proof.** The proof is given in [13].

The interpretation of the theorem is as follows. If there is a wave travelling to the soft boundary from the left-hand side, the transfer functions \(T_{ba}\) and partially transmitted through the boundary (described by \(T_{ab}\)). Likewise, if the wave travels from the opposite side, then the transfer functions \(T_{ha}\) and \(T_{hb}\) represent the respective waves. Mathematically,

\[
X_n = G(1 + T_{ab})A_{n-1} + T_{bb}B_{n+1},
\]

\[
X_{n+1} = H(1 + T_{ha})B_{n+2} + T_{ab}A_n.
\]

For example, for the forced-end boundary \((G = 0)\), \(T_{aa} = H\), \(T_{ba} = -H^2\) and \(T_{bb} = T_{ab} = 0\), and [21] gives \(A_1 = HA_0 - H^2B_1\). Comparing it with [10], we can say that the forced-end boundary is a soft boundary. For a homogeneous path graph, \(G = H\), which gives \(T_{aa} = T_{bb} = G\) and \(T_{ab} = T_{ba} = 0\).

**Corollary 1.** The soft BTFs are mutually related as follows,

\[
T_{aa} + T_{bb} = G + H, \quad T_{ab} + T_{ba} = \frac{H}{G}, \quad T_{aa} = G^{-1}T_{ab} + G, \quad T_{bb} = H^{-1}T_{ab} + H.
\]

**Proof.** By a straightforward application of Theorem 1.

Usually, there is at least one integrator both in the front and rear NTFs allowing the node to follow the velocities of neighbouring nodes. In this case, it holds.

**Corollary 2.** If there is at least one integrator in \(M_{f,n}\) and at least one integrator in \(M_{r,n+1}\), then the DC gains of the soft BTFs are related as,

\[
k_{aa} + k_{bb} = 2, \quad k_{ab} + k_{ba} = 0, \quad k_{ab} - k_{ba} = 1, \quad k_{bb} - k_{ba} = 1,
\]

where \(k_{aa}, k_{ab}, k_{ba}\) and \(k_{bb}\) are the DC gains of \(T_{aa}, T_{ab}, T_{ba}\) and \(T_{bb}\), respectively.

**Proof.** Under the above assumptions, the DC gain of a WTF is equal to one, i.e. \(\lim G(s)_{s\rightarrow 0} = 1\) and \(\lim H(s)_{s\rightarrow 0} = 1\). Then, the proof is a straightforward application of Corollary 1.

The particular values of \(k_{aa}\) and \(k_{bb}\) are given by the following lemma.

**Lemma 2.** Let both \(M_{f,n}\) and \(M_{r,n+1}\) have at least one integrator. If \(M_{f,n}\) and \(M_{r,n+1}\) have the same number of integrators, then

\[
k_{aa} = \frac{2}{n_1(\omega_1) + 1}, \quad k_{bb} = \frac{2}{n_2(\omega_1) + 1}.
\]

If \(M_{f,n}\) has more integrators than \(M_{r,n+1}\), then

\[
k_{aa} = 0, \quad k_{bb} = 2.
\]

If \(M_{r,n+1}\) has more integrators than \(M_{f,n}\), then

\[
k_{aa} = 2, \quad k_{bb} = 0,
\]

where \(n_1(\omega_1) = \lim_{s \rightarrow 0} s^2 M_{f,n}, n_2(\omega_1) = \lim_{s \rightarrow 0} s^2 M_{r,n+1}\) and \(k\) is the number of integrators in \(M_{f,n}\) and \(M_{r,n+1}\).

**Proof.** The DC gain \(k_{aa}\) is derived in Appendix B, while the DC gain \(k_{bb}\) is from [24].

The DC gains of the soft BTFs for the various combinations of derivators and integrators in front and rear NTFs are summarized in Table 1.

3.2. The soft boundary controller

A soft-boundary controller can be designed for various purposes, for instance, to prevent or modify a wave’s transmission through the boundary. We will now design an absorbing controller that prevents the reflection of a wave from the soft boundary. The derivation will be shown only for the left side of the boundary, since the derivation for its right side is analogous.

First, we add one more input \(U_{c,s}\) to the \(n\)th node by expressing its state as

\[
X_n = M_{f,n}(U_{c,s} + (X_{n-1} - 2X_n + X_{n+1})),
\]

and introduce the transfer function \(T_{IL}\) from \(U_{c,s}\) to \(X_n\). In this step, we assume that the graph has an infinite number of identical nodes and edges, hence \(X_n = T_{IL} U_{c,s}\) and \(X_{n-1} = X_{n+1} = GT_{IL} U_{c,s}\). Substituting this into [29] yields

\[
T_{IL} U_{c,s} = M_{f,n}(U_{c,s} + 2GT_{IL} U_{c,s} - 2T_{IL} U_{c,s}),
\]

or,

\[
T_{IL}(s) = \frac{X_n(s)}{U_{c,s}(s)} = \frac{M_{f,n}(s)}{1 + 2M_{f,n}(s)(1 - G(s))} = \frac{M_{f,n}(s)}{\sqrt{1 + 4M_{f,n}(s)}}.
\]

Now, we relax the assumption that there is no boundary in the infinite graph and consider the soft boundary between the \(n\)th and \((n + 1)\)th nodes. The wave generated by the input \(U_{c,s}\) is reflected from the boundary and, hence, \(X_n = T_{IL}(1 + T_{ab})U_{c,s}\).

We are searching for an absorbing filter \(F_{IL}\) such that \(U_{c,s} = F_{IL} V_{IL}\), where \(V_{IL}\) is not-yet-specified filter input. The state
of the $n$th node in a finite path graph, described by (20), with the additional input $U_{LS}$ is
\[ X_n = (1 + T_{ab})A_n + T_{LS}F_{LS}(1 + T_{ab})V_{LS} + T_{bb}B_{n+1}. \] (32)

The reflection of the wave travelling from the left is described by term $T_{ab}A_n$. To prevent the reflection, we eliminate this term by the requirement that
\[ T_{ab}A_n + T_{LS}F_{LS}(1 + T_{ab})V_{LS} = 0. \] (33)

Since the input $V_{LS}$ has not been specified yet, we choose $V_{LS} = A_n$, and (33) gives
\[ F_{LS}(s) = \frac{-T_{ab}(s)}{T_{LS}(1 + T_{ab}(s))}. \] (34)

Substituting for $T_{ab}$ from (18) and for $T_{LS}$ from (31), the absorbing filter has a simple form
\[ F_{LS}(s) = G(s) - H(s). \] (35)

It remains to specify $A_n$, which represents a wave incident on the soft boundary from the left. By (5), (6) and (7), we have
\[ A_n = G(X_{n-1} - B_{n-1}) = GX_{n-1} - G^2(X_n - A_n), \]
\[ A_n = \frac{G}{1 - G^2}X_{n-1} - \frac{G^2}{1 - G^2}X_n, \] (36)

Therefore, the left-side absorbing controller $C_{LS}$ is described by the consequent application of the above equations as
\[ C_{LS} = M_{t,\rho}U_{c,\bar{S}} = M_{t,\rho}F_{LS}V_{LS} = M_{t,\rho}F_{LS}A_n \]
\[ = M_{t,\rho}(G - H) \frac{G}{1 - G^2} (X_{n-1} - GX_n), \]
\[ = \frac{G^2(G - H)}{(1 - G)^2(1 + G)} (X_{n-1} - GX_n), \] (37)

where $M_{t,\rho} = G/(1 - G^2)$, as follows from (5) and (6).

The state of the $n$th node with the left-side absorbing controller is then
\[ X_n = M_{t,\rho}(X_{n-1} - 2X_n + X_{n+1}) + \frac{G^2(G - H)}{(1 - G)^2(1 + G)} (X_{n-1} - GX_n). \] (38)

The state of the neighbouring ($n + 1$)th node after implementing the left-side absorbing controller to the $n$th node changes to
\[ X_{n+1} = T_{ba}(1 + F_{LS}T_{LS})A_n + H B_{n+2} = HA_n + H B_{n+2}. \] (39)

Comparing (39) with (20), we see that the left-side absorbing controller requires the transfer functions $T_{ba}$ and $T_{bb}$ to be equal to $T_{ba} = H$ and $T_{bb} = 0$.

Likewise, the right-side absorbing controller changes the state of the $n$th node as
\[ X_n = GA_{n-1} + GB_{n+1}. \] (40)

That is, $T_{bb} = G$ and $T_{ab} = 0$ for this controller.

The results can be summarized in the following theorem.

**Theorem 2.** The control law preventing any wave to be reflected from the soft boundary is
\[ X_n = M_{t,\rho}(X_{n-1} - 2X_n + X_{n+1}) + C_{LS}, \] (41)
\[ X_{n+1} = M_{t,\rho+1}(X_n - 2X_{n+1} + X_{n+2}) + C_{RS}, \] (42)

where
\[ C_{LS} = \frac{G^2(G - H)}{(1 - G)^2(1 + G)} (X_{n-1} - GX_n), \] (43)
\[ C_{RS} = \frac{H^2(G - H)}{(1 - H)^2(1 + H)} (X_{n+2} - HX_{n+1}). \] (44)

The stability of the control law is treated by the following Theorem.

**Theorem 3.** If $M_{t,\rho}/(1 + \lambda M_{t,\rho})$ and $M_{t,\rho+1}/(1 + \lambda M_{t,\rho+1})$ are both exponentially stable for $\lambda \in (0, 4)$, then the path graph with the control law of Theorem 2 is exponentially stable. Furthermore, the control law and the wave absorber, located at least on one of the path-graph ends, make the path graph $L_2$ string stable.

**Proof.** The proof is given in Appendix C.

3.3. Numerical simulations

As an example, we consider $M_{t,\rho} = (4s + 4)/(s^2(s + 4))$ and $M_{t,\rho+1} = (s + 1)/(s^2(s + 3))$ in (19). Therefore, $n_{1,0} = 4, d_{1,0} = 4, n_{2,0} = 1$ and $d_{2,0} = 3$ in (20).

3.3.1. Soft boundary performance

The effect of the soft boundary in the path graph of four consequent $G$-nodes followed by four consequent $H$-nodes is demonstrated in Fig. 3. The soft boundary is located between the 4th and 5th node.

Independent validation of the soft BTF approach is shown in Fig. 4. We can see that the steady-state value of the left-to-right travelling wave is $k_{ba} \approx 0.7321$, while $k_{bb} \approx 1.2679$ for the right-to-left travelling wave. These values are in agreement with those predicted by Lemma 1.

3.3.2. Control strategies

The performance of individual control strategies are shown in Fig. 5. Comparing the bottom-left and top-left panels, we can see that the soft-boundary absorber does not shorten the settling time if it is not combined with other absorbers on the first or last nodes. In fact, the settling time is even longer in this particular case. In the case of the absorber on the first node (top-middle panel), the wave keeps reflecting between the soft boundary and the non-absorbing last node which prolongs the transient to the settling time. The implementation of the soft-boundary absorber (bottom-middle panel) shortens the transient since it prevents the wave from being reflected back and forth. The absorbers implemented on both ends (top-right panel) cause a change of the steady-state value, as predicted by Lemma 2.

There are two possible ways to obtain a desired steady-state: a) overcompensate the input signal (see [4]), or b) implement the soft-boundary absorber (bottom-right panel).
Fig. 3 shows the comparison of the inputs to the fourth node for (i) a homogenous path graph, (ii) the heterogeneous path graph used in Fig. 3 without a soft-boundary absorber, and (iii) the path graph (ii) with the soft-boundary absorber. The wave absorbers on both path-graph ends are implemented in all cases. We can see that the inputs to the fourth node are the same for cases (i) and (iii).

4. The hard boundary

We will call the boundary located at the nth node where the front NTF differs from the rear NTF as the hard boundary. Such a boundary arises, for example, in a mass-spring model with identical masses and different springs (see Fig. 7). It is defined as follows.

Definition 4. The hard boundary is a virtual boundary located at the nth node, which is defined by

\[ X_n = M_{L,n}(X_{n-1} - X_n) + M_{R,n}(X_{n+1} - X_n), \]  

(45)

where \( M_{L,n} \neq M_{R,n} \).

The adjective ‘hard’ emphasizes the fact that the hard boundary is located at a node, which is in contrast to the soft boundary, located at an edge. Eq. (45) shows that the boundary is defined only by hard-boundary-node NTFs and, thus, its property is not affected by neighbouring nodes nor edges. Changing the state of the hard-boundary node initiates waves propagating in both directions with different dynamics. This is treated in Lemma 3.

4.1. Mathematical description

Let \( X_n \) be the state of the hard-boundary node. We decompose \( X_n \) into the hard-boundary wave components as

\[ X_n = A_{n,L} + B_{n,L} = A_{n,R} + B_{n,R}, \]  

(46)

where the indexes L and R denote the wave components that are next to the left and right side of the boundary, respectively. The decomposition (46) enables us to distinguish between the incident, transmitted and reflected waves. It holds.

Lemma 3. If there is no other boundary next to the hard-boundary node, then

\[ A_{n,L} = G A_{n-1}, \quad B_{n,L} = G^{-1} B_{n-1}, \]  

(47)

\[ A_{n,R} = H^{-1} A_{n+1}, \quad B_{n,R} = H B_{n+1}, \]  

(48)

where

\[ G = \frac{1}{2} \alpha_1 - \frac{1}{2} \sqrt{\alpha_1^2 - 4}, \quad H = \frac{1}{2} \alpha_2 - \sqrt{\alpha_2^2 - 4}, \]  

(49)

\[ \alpha_1 = 2 + 1/M_{f,n}, \quad \alpha_2 = 2 + 1/M_{e,n}. \]

PROOF. The proof is the same as that in Section 3.1 [13]. In this case, two different sets of continued fractions can be found, one converges to \( G \) and \( G^{-1} \), and the other to \( H \) and \( H^{-1} \).

Note that Lemma 3 is generalized in Section 6 for the case where a hard boundary is surrounded by a soft or another hard boundary. The description of the hard boundary is as follows.

Theorem 4. The BTFs describing the hard boundary are

\[ T_{AA} = \frac{A_{n,R}}{A_{n,L}} = \frac{(1 + G)(1 - H)}{1 - HG}, \quad T_{BA} = \frac{A_{n,R}}{B_{n,R}} = \frac{H - G}{1 - HG}, \]  

(50)

\[ T_{BB} = \frac{B_{n,L}}{B_{n,R}} = \frac{(1 + H)(1 - G)}{1 - HG}, \quad T_{AB} = \frac{B_{n,L}}{A_{n,L}} = \frac{G - H}{1 - HG}, \]  

(51)

where \( G \) and \( H \) are given by (49).
Figure 5: The performance comparison of individual path-graph control strategies with the soft boundary. The path graph is the same as in Fig. 3. The step responses of six individual control strategies are compared. The top-left: path graph with no absorber; top-middle: path graph with the absorber implemented on the first node; top-right: path graph with the absorbers implemented on the first and last nodes. In the bottom panels, the soft-boundary absorber between nodes 4 and 5 is additionally implemented.

Proof. The proof is given in Appendix D.

The interpretation of the theorem is as follows. A wave incident from the left side of the hard boundary (described by \( A_{n,L} \)) is partially reflected from the boundary (described by \( T_{AB} \)) and partially transmitted through the boundary (described by \( T_{AA} \)). For the wave incidenting from the opposite side (described by \( B_{n,R} \)), the transfer functions are \( T_{BA} \) and \( T_{BB} \), respectively. The state of the hard-boundary node can be expressed in two equivalent ways,

\[
X_n = G(1 + T_{AB})A_{n-1} + HT_{BB}B_{n+1},
\]

\[
X_n = GT_{AA}A_{n-1} + H(1 + T_{BA})B_{n+1}.
\]

For example, for the free-end boundary (\( H = 0 \)), \( T_{AB} = G \), which is in agreement with (1). For a homogenous path graph, \( G = H \), which gives \( T_{AA} = T_{BB} = G \) and \( T_{AB} = T_{BA} = 0 \).

**Corollary 3.** The hard BTFs are mutually related as follows,

\[
T_{AA} + T_{BA} = T_{BB} + T_{AB},
\]

\[
T_{AA} = 1 + T_{AB}, \quad T_{BB} = 1 + T_{BA},
\]

\[
T_{AA} + T_{BB} = 2, \quad T_{AB} + T_{BA} = 0.
\]

Proof. By a straightforward application of Theorem 4.

This leads to interesting conclusions: (i) The hard-boundary DC gains are mutually related in the same way as the soft-boundary DC gains in Corollary 2 and (ii) those relations hold even for the NTF without an integrator.

4.2. The hard-boundary controller

Controlling the hard boundary is similar to that of the soft boundary in Section 3.2. Here, we only provide a brief description of the absorbing-controller design.

The state of the \( n \)-th node controlled with additional input \( U_{e,H}(s) \) is

\[
X_n = M_{e,L}(X_{n-1} - X_n) + M_{e,R}(X_{n+1} - X_n) + M_{e,H}U_{e,H},
\]

where \( X_n = T_{H}U_{e,H} \) and \( X_{n+1} = T_{H}U_{e,H} \) in case of an infinite homogeneous graph. This leads to

\[
T_{H}(s) = \frac{X_n(s)}{U_{e,H}(s)} = \frac{M_{e,R}}{1 + M_{e,L} + M_{e,R} - M_{e,R}G - M_{e,H}}.
\]

Substituting \( U_{e,H}(s) = F_{H}(s)V_{L,H}(s) \), where \( F_{H}(s) \) is an absorbing filter and \( V_{L,H}(s) \) is its input, and assuming a finite graph with the hard boundary located at the \( n \)-th node, yields

\[
X_n = (1 + T_{AB})A_{n,L} + T_{H}F_{H}V_{L,H} + T_{BB}B_{n,R}.
\]

To absorb the wave travelling towards the hard boundary from the left, we set \( T_{AB}A_{n,L} = -T_{H}F_{H}V_{L,H} \) and \( V_{L,H} = A_{n,L} \). Hence,

\[
F_{H} = \frac{T_{AB}}{T_{H}} = \frac{(H - G)(1 - G)}{G(1 - H)}.
\]

The \( A_{n,L} \) term represents the wave travelling towards the hard boundary from left, which is again computed by (36),

\[
A_{n,L} = \frac{G}{1 - G^2}X_{n-1} - \frac{G^2}{1 - G^2}X_n.
\]

The left-side absorbing controller is then

\[
C_{L,H} = M_{e,R}F_{H}A_{n,L} = \left( \frac{1}{1 - H} - \frac{1}{1 - G} \right) \frac{G}{1 - G^2}(X_{n-1} - GX_n)
\]

\[
= \frac{G(H - G)}{(1 - G^2)(1 + G)(1 - H)}(X_{n-1} - GX_n).
\]

The design of the right-side absorbing controller is similar. When both controllers are implemented on the \( n \)-th node, we get

\[
X_n = T_{AA}A_{n,L} - T_{H}\frac{T_{AB}}{T_{H}}A_{n,L} + T_{BB}B_{n,R} - T_{H}\frac{T_{BA}}{T_{H}}B_{n,R}
\]

\[
= A_{n,L} + B_{n,R} = GA_{n,L} + HB_{n,R}.
\]
Figure 4: The comparison of the states of the 4th node (the last $G$-node) and the 5th node (the first $H$-node) simulated by the state-space approach using (3) (solid lines) and those computed by the soft BTF approach using (20) and (21) (crosses). The top panel shows the situation when the first $G$-node changes its state from 0 to 1 and initiates a wave travelling towards the soft boundary from the left. The reverse situation when the last $H$-node initiates the wave travelling towards the soft boundary from the right is shown in the bottom panel.

Figure 6: The comparison of the inputs to the fourth node for three different path graphs. The label ‘SB abs.’ stands for the soft-boundary absorber described by (37).

where (55) have been considered.

Combining (63) and (64) gives $A_{n,L} = A_{n,R}$ and $B_{n,L} = B_{n,R}$. In words, the hard boundary between the wave components indexed by L and R is removed at the $n$th node and the wave transmits through the node without being reflected.

The implementation of the absorbing controller shows yet another distinction between the soft and hard boundaries. The rheological property is symmetric for the two nodes neighbouring the soft boundary, but it is asymmetric for the hard boundary node. The results can be summarized in the following theorem.

**Theorem 5.** The control law that prevents any wave to be reflected from the hard boundary is

$$X_n = M_{f,0}(X_{n-1} + X_n) + M_{f,0}(X_{n+1} - X_n) + C_{L,H} + C_{R,H}. \quad (64)$$

Figure 7: The same as Fig. 2, but for the hard boundary represented by the red-blue square.

where

$$C_{L,H} = \left( G - H \right) \left( 1 - G \right)^2 \left( 1 + G \right) \left( 1 - H \right) \left( X_n - G X_n \right), \quad (65)$$

$$C_{R,H} = \left( H - G \right) \left( 1 - H \right)^2 \left( 1 + H \right) \left( 1 - G \right) \left( X_{n-1} - H X_n \right). \quad (66)$$

The stability of the control law is treated by the following Theorem.

**Theorem 6.** If $M_{f,0} (1 + \lambda M_{f,0})$ and $M_{f,0} (1 + \lambda M_{f,0})$ are both exponentially stable for $\lambda \in (0, 4)$, then a path graph with the control law of Theorem 5 is exponentially stable. Furthermore, the control law and the wave absorber, located at least on one of the path-graph ends, make the path graph $L_n$ string stable.

**Proof.** The proof is almost identical to the proof of Theorem 5 in Appendix C. The only difference is in the different powers of $G$ and $H$ in (C.1), (C.2), (C.3), (C.4), (C.7) and (C.8) which does not affect the exponential or string stability.

4.3. Numerical simulations

The WTFs defined in Section 3.3 are used for numerical simulations of the hard boundary. The methods of computing the BTFs are described in Appendix F.

4.3.1. Hard boundary performance

Fig. 8 shows how the hard boundary affects the travelling wave in a path graph, while an independent validation of the hard-boundary transfer-function approach is shown in Fig. 9.

We can see that the steady-state value of the left-to-right travelling wave $\kappa_{AA} \approx 1.2679$, while $\kappa_{BB} \approx 0.7321$ for the right-to-left travelling wave. These values are in agreement with those predicted by Corollary 5. Moreover, the identity (46) for the resulting state of the hard-boundary node is checked by calculating $A_L$ and $B_L$ (top panel), or, alternatively, by calculating $A_R$ and $B_R$ (bottom panel). We can see that the results are identical.

4.3.2. Control strategies

The performances of individual control strategies are shown in Fig. 10. Comparing the top-left panels in Figs. 5 and 10, we can see that the behaviour of the uncontrolled soft and hard boundaries are almost identical. The differences between the soft- and hard-boundary simulations are more pronounced with the absorber on the last and/or first node (top-right and top-middle panels, respectively). The bottom-middle and bottom-right panels show the effect of implementing the hard-boundary absorber. The absorber prevents any wave to be reflected from the boundary which makes the transient significantly faster.
Figure 8: The same as Fig. 3 but for the hard boundary located at the 4th node. Node indexes '4_L' and '4_R' stand for the left (A_L, B_L) and right (A_R, B_R) hard-boundary wave components, respectively.

Figure 10: The same as Fig. 5 but for the hard-boundary absorber located at the 4th node.

5. Common features of the boundaries

This section describes the relationships between the soft and hard boundaries. Although the definitions and physical interpretations of the boundaries are different, they have some common features. For instance, they have bounded DC gains, or there is an inverse-reciprocity relation indicating that \( T_{aa} \) is closely related to \( T_{BB} \) rather than to \( T_{AA} \), while \( T_{ab} \) is related to \( T_{BA} \), and so on. More specifications are as follows.

**Corollary 4.** The soft and hard BTFs are mutually related as

\[
T_{aa} = T_{BB} + G - 1, \quad T_{ba} = HT_{AB},
\]
\[
T_{bb} = T_{AA} + H - 1, \quad T_{ab} = GT_{BA}.
\]

**Proof.** By a straightforward application of Theorem 1 and Theorem 4.

**Corollary 5.** If there is at least one integrator in the front NTF and at least one integrator in the rear NTF, then

\[
\kappa_{aa} = \kappa_{BB}, \quad \kappa_{ba} = \kappa_{AB}, \quad \kappa_{bb} = \kappa_{AA}, \quad \kappa_{ab} = \kappa_{BA},
\]

where \( \kappa_{AA}, \kappa_{AB}, \kappa_{BA} \) and \( \kappa_{BB} \) are the DC gains of \( T_{AA}, T_{AB}, T_{BA} \) and \( T_{BB} \), respectively.

**Proof.** Under the assumptions, \( \lim_{s \to 0} G = \lim_{s \to 0} H = 1 \). Considering this in Corollary 4 results in (69) and (70).

**Lemma 4.** The DC gains of the BTFs are bounded as

\[-1 \leq \kappa_{aa}, \kappa_{bb}, \kappa_{AB}, \kappa_{BA} \leq 1.\]

If both the front and rear NTFs have neither an integrator, nor a derivator, then

\[-2 < \kappa_{aa}, \kappa_{bb} < 2.\]

In all other cases,

\[0 \leq \kappa_{aa}, \kappa_{bb}, \kappa_{AA}, \kappa_{BB} \leq 2.\]

**Proof.** The proof is given in Appendix E.

The overview of results proofed in Lemmas 2 and 4 and Corollary 4 is given in Table 1.
Table 1: The overview of the DC gains of the BTFs ($\kappa_G = \lim_{s \to 0} G(s)$ and $\kappa_H = \lim_{s \to 0} H(s)$).

<table>
<thead>
<tr>
<th>A derivator in the front NTF ($\kappa_G = 0$)</th>
<th>No derivator nor integrator in the front NTF ($-1 &lt; \kappa_G &lt; 1$)</th>
<th>An integrator in the front NTF ($\kappa_G = 1$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa_{aa} = 0 \quad \kappa_{AA} = 1$</td>
<td>$\kappa_{aa} = 0 \quad \kappa_{AA} = 1 + \kappa_G$</td>
<td>$\kappa_{aa} = 0 \quad \kappa_{AA} = 2$</td>
</tr>
<tr>
<td>$\kappa_{ab} = 0 \quad \kappa_{AB} = 0$</td>
<td>$\kappa_{ab} = -\kappa_G^2 \quad \kappa_{AB} = \kappa_G$</td>
<td>$\kappa_{ab} = -1 \quad \kappa_{AB} = 1$</td>
</tr>
<tr>
<td>$\kappa_{aa} = 1 \quad \kappa_{AA} = 0$</td>
<td>$\kappa_{aa} = 1 + \kappa_G \quad \kappa_{AA} = 0$</td>
<td>$0 \leq \kappa_{aa} \leq 2 \quad 0 \leq \kappa_{AA} \leq 2$</td>
</tr>
<tr>
<td>$\kappa_{ab} = 0 \quad \kappa_{AB} = -1$</td>
<td>$\kappa_{ab} = \kappa_G \quad \kappa_{AB} = -1$</td>
<td>$-1 \leq \kappa_{ab} \leq 1 \quad -1 \leq \kappa_{AB} \leq 1$</td>
</tr>
</tbody>
</table>

Figure 9: The comparison of the state of the hard-boundary node simulated by the state-space approach using (2) (solid lines) and that computed by the hard BTF approach using (52) (crosses). The top and bottom panels show the $A_L$ and $B_L$ and $A_R + B_R$ components, respectively. Two situations are shown: (i) the first G-node changes its state from 0 to 1 and initiates a wave travelling towards the hard boundary from the left (labeled by 'LR') and (ii) the reverse situation when the last H-node initiates the wave travelling towards the hard boundary from the right (labeled by 'RL').

6. A combination of soft and hard boundaries

The soft and hard boundaries are two special cases of path-graph asymmetries. The two boundaries can be combined to form any type of a complex boundary. The idea is to describe the transfer function of a complex boundary in terms of the soft- and hard-boundary BTFs. The combination of the two boundaries requires us to relax the assumption that there is no other boundary next to the hard or soft boundary, which is expressed by (47) and (48).

The approach is demonstrated for the complex boundary shown in Fig. 11. The boundary is formed by the combination of the soft boundary located between the nodes indexed 2 and 3 and the hard boundary located at node 3. This configuration violates the boundary conditions in (47) since now the wave travelling to $A_{3L}$ and the wave travelling from $B_{3L}$ transmit through the soft boundary. In view of Theorem 1 we have

$$A_{3L} = T_{aa}A_2 + T_{ba}B_{3L},$$

$$B_2 = T_{ab}A_2 + T_{bb}B_{3L}.$$  

The second part of the complex boundary is composed of the hard boundary, hence, by Theorem 2 we have

$$A_{3R} = T_{AA}A_{3L} + T_{BA}B_{3R},$$

$$B_{3L} = T_{AB}A_{3L} + T_{BB}B_{3R},$$

where the overlined transfer functions are different from those in (50) and (51). The original transfer function $T_{AA}$ describes the wave propagating from the blue to the red medium, while $T_{BA}$ describes the wave propagating in the opposite direction. Due to the same reasoning, $T_{AB} = T_{BB}, T_{AA} = T_{BA}, T_{AB} = T_{BA}$ and $T_{BB} = T_{AB}$ for this complex boundary.

The same procedure can be applied to the wave-absorbing controller. Since the wave-absorbing controllers are designed separately for the soft and hard boundaries, they can be combined to absorb the wave reflecting from boundaries of different complexity.

7. CONCLUSIONS

This paper deals with travelling waves propagating along a distributed system with an underlying path-graph topology. It mathematically describes two basic types of boundaries in a heterogeneous path graph. The wave description allows us to design a feedback controller to modify the effect of the boundaries. For instance, the waves reflecting from the boundaries can be absorbed, and thus the path-graph transient is shortened without changing local node dynamics. Moreover, such a controller makes a path graph string stable provided that the path-graph ends are equipped with the wave absorbers.
Note that certain features of the boundaries described in Section 5, for instance, the boundedness of the DC gains of transfer functions, are in agreement with knowledge from the basic wave physics, see e.g. [8].

8. ACKNOWLEDGEMENTS

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Appendix A. Proof of Lemma 1

The proof is based on [4] (Theorem A.2), which states: A linear system is stable if and only if its transfer function $T(s)$ is analytic in the right-half plane and $\|T\|_\infty < \infty$, where $\|T\|_\infty = \sup_{s \in \mathbb{C}^+} |T(s)|$.

It was proved in [28] that the WTF defined by [9] satisfies $|G(s)| \leq 1$. Hence, it remains to derive the condition when the WTF is analytic in the right-half plane.

First, we treat the square root function in [8]. From the complex function analysis, for instance [22], it is known that the square root function $f(z) = \sqrt{z}$ is analytic everywhere, except for the non-positive real axis. Therefore, $f_2(\alpha) = \sqrt{\alpha^2 - 4}$ in [8] is analytic everywhere, except for the closed interval $(-2, 2)^-\alpha$ on the real axis. Since $\alpha(s) = 2 + 1/M(s)$, we can say that $f_2(\alpha)$ is analytic everywhere, except for the interval $(\alpha(s), -1/4)$. This means that, if the NTF $M(s)$ crosses interval $(-\alpha(s), -1/4)$, then the WTF is unstable.

The Nyquist criterion of stability states that if $M(1 + \lambda M)$ is stable for $\lambda \in (0, 4)$, and if there are no right-half plane poles in the NTF, then its Nyquist curve does not encircle the point $[-1/4, 0]$. Hence, the NTF does not cross the interval $(-\alpha(s), -1/4)$ and $f_2(\alpha) = \sqrt{\alpha^2 - 4}$ is analytic.

The first part of the WTF, $f_1(\alpha) = \alpha = 2 + 1/M$, is a rational transfer function. Such a function is analytic in the right-half plane if and only if there is no singularity of $f_1(\alpha)$ in this plane. The only possible singularities of $f_1(\alpha)$ are zeros in the NTF in the right-half plane. Therefore, if the NTF has no zero in the right-half plane, then $f_1(\alpha)$ is analytic. Since the difference of two analytic functions is again analytic, then the WTF, $G = (f_1 - f_2)/2$, is analytic and exponentially stable under the above conditions.

In the case where the NTF has zero in the right-half plane, we express the WTF in terms of the NTF as

$$G = \frac{1 + 2M}{2M} (1 - \sqrt{1 + 4M}) = \frac{d_M + 2n_M}{2n_M} \left(1 - \sqrt{1 + 4n_M} \right),$$

where $n_M(s)$ and $d_M(s)$ are the numerator and the denominator of the NTF, respectively. Since $\lim_{s \to -n_M} n_M = 0$, where $z_0$ is the right-half plane zero in the NTF, then

$$\lim_{s \to -z_0} (s - z_0) G(s) = \lim_{s \to -z_0} \left(\frac{d_M(s)}{2n_M} (1 - \sqrt{1 + 4n_M})\right) = 0,$$

where $1/n_M = (s - z_0)/n_M$. Therefore, according to Riemann’s theorem on removable singularities ([23]), $z_0$ is a removable singularity and the WTF is analytically extendable over $z_0$. In other words, the WTF is analytic and exponentially stable also in the case where the NTF has a zero in the right-half plane.

Appendix B. Proof of Lemma 2

Let us denote $\alpha_1 = 2 + 1/M_{e,n} = 2 + d_i/n_1$ and $\alpha_2 = 2 + 1/M_{p,n+1} = 2 + d_2/n_2$. We will begin with deriving the DC gain $\kappa_{aa}$ of the $T_{aa}$ transfer function.

$$\kappa_{aa} = \lim_{s \to 0} T_{aa} = \lim_{s \to 0} \frac{H - H^2}{H - 1} = \lim_{s \to 0} \frac{1 - G^2}{H - 1 - G} = 0,$$

(B.1)

since $\lim_{s \to 0} G = \lim_{s \to 0} H = 1$ for at least one integrator in the rear and front NTFs. Applying the L’Hôpital’s rule to (B.1) gives

$$\kappa_{aa} = \lim_{s \to 0} \frac{2G}{H^2 - H(G')^{-1} + 1} = \lim_{s \to 0} \frac{2}{H^2 - H(G')^{-1} + 1},$$

(B.2)

where the symbol $'$ denotes the differentiation with respect to variable $s$. First, the differentiation of (13) yields

$$G' = \frac{1}{2\alpha_1} \frac{1}{2} \left(1 + \alpha_1^2\right).$$

(B.3)

The individual contributions to $\lim_{s \to 0} G'$ are

$$\lim_{s \to 0} \frac{d_i}{n_1} - \frac{d_i}{n_1} = \frac{\epsilon}{n_1},$$

where $k_1 > 0$ is the number of integrators in $M_{e,n}$, and

$$\lim_{s \to 0} \frac{\alpha_1}{\alpha_2} = \lim_{s \to 0} \frac{d_i/n_1 - d_i/n_1}{n_1/4 + d_i/n_1} = \lim_{s \to 0} \frac{s^{k_1/2}}{n_1/4 + d_i/n_1} = \lim_{s \to 0} \frac{s^{k_1/2}}{n_1/4 + d_i/n_1}$$

(B.5)

where $d_i = \lim_{s \to 0} s^{-k_1} d_1$ and $n_1,0 = \lim_{s \to 0} n_1$. Moreover, $\lim_{s \to 0} \alpha_1 = 2$. Similarly, $\lim_{s \to 0} H'$ can be evaluated. Then

$$\lim_{s \to 0} \frac{H'}{G'} = \lim_{s \to 0} \frac{s^{k_2-k_1}/2}{k_2} \frac{n_1/4d_2}{n_1/4d_1},$$

(B.6)

where $d_2 = \lim_{s \to 0} s^{-k_2} d_2$. Finally, substituting (B.6) into (B.2) yields

$$\kappa_{aa} = \lim_{s \to 0} \frac{2}{s^{k_2-k_1}/2 \frac{n_1/4d_2}{n_1/4d_1} + 1} = \frac{2}{s^{k_2-k_1}/2 \frac{n_1/4d_2}{n_1/4d_1} + 1}.$$

(B.7)

Therefore, if $M_{e,n}$ and $M_{p,n+1}$ have the same number of integrators ($k_1 = k_2$), then (B.7) simplifies to (26). If $k_2 > k_1$, i.e., $M_{p,n+1}$ have more integrators than $M_{e,n}$, then (B.6) converges to zero and $\kappa_{aa} = 2$. In the opposite case, (B.6) diverges and $\kappa_{aa} = 0$. \[\square\]
Appendix C. Proof of Theorem 3

In this proof, we assume that there is only one soft boundary in a path graph. However, the proof for the case with multiple soft and/or hard boundaries can be carried out analogously.

We consider a path graph with \( n \) G-nodes and \( m \) H-nodes. In front of the first G-node there is the 0th node acting as the reference signal. Hence, the 0th and \((n + m)\)th nodes represent the forced and free boundaries, respectively.

Appendix C.1. Wave absorber on the graph ends

If there is a wave absorber implemented on the first node of a path graph, then the combination of (3), (7) (39) and (40) gives

\[
\frac{X_p}{X_0} = G^p + G^2 \sum_{p=1}^{p=n} p H^{2m}, \quad \text{if } p \leq n, \tag{C.1}
\]

\[
\frac{X_p}{X_0} = G^p H^{p-n} + G^2 H^{2m+1+n-p}, \quad \text{if } p > n. \tag{C.2}
\]

In the alternative case of the wave absorbers implemented on both path-graph ends, the state of the \( p \)th node is described by

\[
\frac{X_p}{X_0} = G^p, \quad \text{if } p \leq n, \tag{C.3}
\]

\[
\frac{X_p}{X_0} = G^p H^{p-n}, \quad \text{if } p > n. \tag{C.4}
\]

Since \( G \) and \( H \) are exponentially stable, the transfer functions (C.1), (C.2), (C.3) and (C.4) are also exponentially stable.

Since \( |G|_{\infty} \leq 1 \) and \( |H|_{\infty} \leq 1 \), the \( H_{\infty} \) norm of (C.1) and (C.2) and the \( H_{\infty} \) norm of (C.3) and (C.4) are smaller or equal to two and one, respectively. This means that the \( H_{\infty} \) norms of (C.1), (C.2), (C.3) and (C.4) are limited, regardless of the number of nodes in a path graph. In view of Definition 2, we see that these two systems are \( L_2 \) string stable.

Appendix C.2. No wave absorber on the graph ends

First, we show a way to find the transfer function from \( X_0 \) to \( X_1 \) for the path graph with \( n = 3 \) and \( m = 5 \). In this case, \( A_1 = G X_0 - G^2 H^1 A_1 \) and \( B_1 = G H^1 X_0 - G^2 H^0 B_1 \), hence

\[
\frac{X_1}{X_0} = \frac{A_1 + B_1}{X_0} = \frac{G + G^2 H^1}{1 + G^2 H^1}. \tag{C.5}
\]

Similarly,

\[
\frac{X_2}{X_0} = \frac{G^2 + G^2 H^1}{1 + G^2 H^1}, \tag{C.6}
\]

and so on. For the \( p \)th node, we have

\[
\frac{X_p}{X_0} = \frac{G^p + G^2 \sum_{p=1}^{p=n} p H^{2m}}{1 + G^2 \sum_{p=1}^{p=n} p H^{2m}}, \quad \text{if } p \leq n, \tag{C.7}
\]

\[
\frac{X_p}{X_0} = \frac{G^p H^{p-n} + G^2 H^{2m+1+n-p}}{1 + G^2 \sum_{p=1}^{p=n} p H^{2m}}, \quad \text{if } p > n. \tag{C.8}
\]

Note that the transfer functions between two arbitrary nodes can be expressed similarly.

Due to \( \|G\|_{\infty} \), we have \( M(1 + 2M) = G(1 + G^2) \). In view of the Nyquist criterion of stability, we can say that if \( M(1 + 2M) \) is stable, that is if \( G(1 + G^2) \) is stable, and if \( |G|_{\infty} \leq 1 \) and \( |H|_{\infty} \leq 1 \), then the transfer function

\[
\frac{G}{1 + G^2 H^0 H^c}, \tag{C.9}
\]

is exponentially stable for \( q_1, q_2 \in \mathbb{N} \). Furthermore, since \( G \) and \( H \) are exponentially stable, then the transfer function

\[
\frac{G^q H^0 H^c}{1 + G^2 H^0 H^c}, \tag{C.10}
\]

is exponentially stable for \( q_1, q_2 \in \mathbb{N} \). Comparing (C.7) with (C.8), we can say that the transfer function between two arbitrary nodes in a path graph with no wave absorber on the graph ends and with the control law given by Theorem 2 is exponentially stable.

Appendix D. Proof of Theorem 4

The state of the hard-boundary node, defined by (45), can be rewritten as

\[
X_n = T_L X_{n-1} + T_R X_{n-1}, \tag{D.1}
\]

where \( T_L = M_{L,n}/(1 + M_{L,n} + M_{R,n}) \) and \( T_R = M_{R,n}/(1 + M_{L,n} + M_{R,n}) \). We combine (5), (47) and (48), and obtain

\[
X_{n-1} = A_{n-1} + B_{n-1} = G^{-1} A_{n,L} + G B_{n,L}, \tag{D.2}
\]

\[
X_{n+1} = A_{n+1} + B_{n+1} = H_{n,L} + H B_{n,R}. \tag{D.3}
\]

We substitute (D.2) and (D.3) into (D.1) and use (46) for \( X_n \),

\[
A_{n,L} + B_{n,L} = T_L (G^{-1} A_{n,L} + G B_{n,L}) + T_R (H_{n,L} + H B_{n,R}). \tag{D.4}
\]

Rearranging (D.4) with respect to the hard-boundary wave components gives

\[
A_{n,L} (1 - T_L G^{-1}) + B_{n,L} (1 - T_L G^{-1}) = B_{n,R} (T_R H^{-1}) + A_{n,R} T_R H. \tag{D.5}
\]

The four wave components are now reduced by substituting \( A_{n,R} = A_{n,L} + B_{n,L} - B_{R} \),

\[
B_{n,L} = B_{n,R} \frac{T_R H - T_R H^{-1}}{T_L G + T_R H - 1} + A_{n,L} \frac{1 - T_L G^{-1} - T_R H}{T_L G + T_R H - 1}, \tag{D.6}
\]

or, alternatively, by substituting \( B_{n,L} = A_{n,L} + B_{n,R} - A_{n,L} \),

\[
A_{n,R} = A_{n,L} \frac{T_R H - T_R H^{-1}}{T_L G + T_R H - 1} + B_{n,R} \frac{1 - T_L G^{-1} - T_R H^{-1}}{T_L G + T_R H - 1}. \tag{D.7}
\]

These formulas can be further simplified by expressing \( T_L \) and \( T_R \) in terms of \( G \) and \( H \). Specifically, \( M_{L,n} = (G + G^{-1} - 2)^{-1} \) and \( M_{R,n} = (H + H^{-1} - 2)^{-1} \), so

\[
T_L = \frac{(G + G^{-1} - 2)^{-1}}{1 + (G + G^{-1} - 2)^{-1} + (H + H^{-1} - 2)^{-1}}, \tag{D.8}
\]

\[
T_R = \frac{1 + (G + G^{-1} - 2)^{-1} + (H + H^{-1} - 2)^{-1}}{1 + (G + G^{-1} - 2)^{-1} + (H + H^{-1} - 2)^{-1}}. \tag{D.9}
\]
Finally, substituting (D.8) and (D.9) into (D.6) and (D.7) yields
\[ B_{n,R} = B_{n,R} \frac{(1 + H)(1 - G)}{1 - HG} + A_{n,R} \frac{G - H}{1 - HG}, \]
\[ A_{n,R} = A_{n,R} \frac{(1 + G)(1 - H)}{1 - HG} + B_{n,R} \frac{H - G}{1 - HG}, \]
which proves Theorem 4.

Appendix E. Proof of Lemma 4

Let us denote \( \kappa_G = \lim_{s \to 0} G(s) \) and \( \kappa_H = \lim_{s \to 0} H(s) \). As a matter of fact, the \( H_\infty \) norm of the WTFs is less than or equal to one \([13]\), that is, \(-1 \leq \kappa_G, \kappa_H \leq 1\).

We treat three possible configurations: (i) both front and rear NTFs have at least one integrator, (ii) either front or rear NTF has at least one derivator, and (iii) either front or rear NTF has neither an integrator nor a derivator.

Case (i). The soft-boundary DC gains are already treated by Lemma 2 and Corollary 2. The hard-boundary DC gains can be calculated using Corollary 5.

Case (ii). If there is a derivator in \( M_{f,r} \), then \( \kappa_f = 0 \). Therefore, we can directly substitute \( \kappa_G \) and \( \kappa_H \) into (50) and (51) and obtain, \( 0 \leq \kappa_{AA} = 1 - \kappa_H \leq 2, -1 \leq \kappa_{AB} = \kappa_H \leq 1 \), \( 0 \leq \kappa_{BB} = 1 + \kappa_H \leq 2 \) and \(-1 \leq \kappa_{BA} = \kappa_H \leq 1 \). In the case of a soft boundary, the DC gains can be calculated from Corollary 1. The DC gains for a derivator in \( M_{f,r} \) can be carried out analogously.

Case (iii). If there is no derivator or integrator in \( M_{f,r} \), then \(-1 < \kappa_G < 1\). Substituting \( \kappa_G \) into (50) and (51) yields \( \kappa_{AB} = (\kappa_G - \kappa_H)/(1 - \kappa_G \kappa_H) \). Since \(-1 \leq \kappa_H \leq 1\), then \(-1 \leq \kappa_{AB} \leq 1\). Substituting \( \kappa_{AB} \) into (55) and (56) gives \( 0 \leq \kappa_{AA} \leq 2 \) and \(-1 \leq \kappa_{BA} \leq 1\), respectively, and \( 0 \leq \kappa_{BB} \leq 2 \) by (55). The soft-boundary DC gains can be calculated from Corollary 4.

Appendix F. Numerical approximation of the BTFs

The soft and hard BTFs \([17], [18], [50] \) and (51) are composed of two different irrational WTFs. Since an analytical impulse response to the irrational BTFs is usually difficult to find, it is convenient to construct an approximation of BTF impulse responses. There are at least two ways to do this, both based on the approximations of \( G \) and \( H \) given in \([13]\).

First, the BTFs are approximated by a linear combination of rational approximations of \( G \) and \( H \) in the Laplace domain. To find the impulse response of a rational-function approximation is a routine procedure for the inverse-Laplace-transform solvers. However, depending on the oscillatory character of the system, only the first few seconds of the approximate impulse response are usually reliable. They are followed by numerical instabilities due to a high-order approximation of the transfer function and the limited precision of computational software. For the numerical examples presented in this paper, this method, however, provides satisfactory numerical results.

Second, a numerically more reliable way is to use approximated impulse responses of \( G \) and \( H \) in a feedback connection. For instance, the impulse response of \( T_{AB} = (G - H)/(1 - HG) \) is approximated by the feedback connection shown in Figure F.12. Other transfer functions can be approximated in a similar way.

Figure F.12: Feedback connection for approximating the impulse response of \( T_{AB} \).

References


