Input-to-state stability of interconnected hybrid systems

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Abstract

We consider interconnections of arbitrary topology of several hybrid input-to-state stable (ISS) systems and study whether the ISS property is preserved by an interconnection. The small gain approach is used for this purpose. We show that if the small gain condition is satisfied then the whole network is ISS and show how an ISS-Lyapunov function can be constructed in this case. Since this construction is non-smooth the methods of non-smooth analysis are used.

Key words: Stability of hybrid systems; Lyapunov methods; Large-scale systems.

1 Introduction

In many modern applications one has to deal with a combination of both continuous and discontinuous types of behaviors in one model. Hence hybrid systems that allow for this combination are very useful in this case. Such behavior occurs for example in control systems that combine digital and analog devices, e.g. robotics [1], network control systems [34,26], reset systems [27], engineering systems or logistics networks. These systems often have a large scale interconnected structure and can be naturally modelled as interconnected hybrid systems. In this paper our main interest is in stability and robustness for such interconnections as these properties are certainly of great importance for applications. Many results were obtained in this field, see for example [11], [15], [25] and [28].

There are different approaches to study stability of interconnections of nonlinear systems as for example in the passivity framework or with the help of specializing a type of systems to a certain class like monotone ones. In this paper we will use the framework of input-to-state stability (ISS) that was first introduced for continuous systems in [32] and then extended to other types of systems including hybrid ones, see for example [21], [2], [26], [14], [16], [19] and [8].

The ISS property of interconnected systems is usually studied using small gain conditions that take the interconnection structure of the whole system into account. First small gain conditions for ISS systems were introduced for interconnections of two continuous systems in [18,17]. These results were extended to arbitrary number of interconnected systems in [8,9,21].

Interconnections of two hybrid ISS systems were considered in [23], [26], e.g. A stability condition of the small gain type was used in [26] for a construction of an ISS-Lyapunov function for their feedback connection. Interconnection of arbitrary number of sampled-data systems that are a special class of hybrid systems was considered in [21]. A small gain condition was given there in terms of vector Lyapunov functions.

In view of modelling networks of hybrid systems similar results for an interconnection of arbitrary number of such systems are obtained in this paper. To this end we use the methodology recently developed in [8], [9] for investigation of stability of general networks of ISS systems. In particular we use a small gain condition in the same form as developed in these papers and we use non-smooth ISS-Lyapunov functions in our considerations.

The main result of this paper extends the result of [26] for the case of interconnection of \( n \geq 2 \) hybrid systems and [21] for general type of hybrid systems. Moreover in contrast to [26] and [21] our results are derived not only for the definition of ISS with maximization of gains corresponding to different inputs but also for the case of their summation. In this case one can obtain less conservative gains in some applications. This leads to a slightly different small gain condition.

There are different ways to introduce ISS-Lyapunov functions for hybrid systems, see for example [3], [26]. In this paper we show the equivalence between their existence for a hybrid system. Using methods developed

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In [9] we provide a construction of an ISS-Lyapunov function for interconnected hybrid systems. In addition to the small gain results shown in terms of Lyapunov functions, we provide their counterparts in terms of trajectories.

The next section introduces all necessary notions and notation. In Section 3 we consider ISS-Lyapunov functions and show the equivalence mentioned above. Section 4 contains small gain results. Section 5 concludes the paper.

2 Preliminaries

2.1 Basic notation

Let \( \mathbb{R}_+ \) be the set of nonnegative real numbers, \( \mathbb{R}_+^n \) be the positive orthant \( \{ x \in \mathbb{R}^n : x \geq 0 \} \) and \( \mathbb{N}_+ := \{0,1,2,...\} \). \( x^T \) stands for the transpose of a vector \( x \in \mathbb{R}^n \). \( \mathbb{B} \) is the open unit ball centered at the origin in \( \mathbb{R}^n \) and \( \overline{\mathbb{B}} \) is its closure. By \( (\cdot,\cdot) \) we denote the standard scalar product in \( \mathbb{R}^n \). For \( x,y \in \mathbb{R}^n \), we use a partial order induced by the positive orthant. It is given by

\[
 x \geq y \iff x_i \geq y_i, \quad (x > y \iff x_i > y_i) \quad i = 1,\ldots,n. 
\]

We write \( x \not\geq y \iff \exists i \in \{1,\ldots,n\} : x_i < y_i \). For a nonempty index set \( J \subset \{1,\ldots,n\} \) let \( P_J \) denote the projection of \( \mathbb{R}_+^n \) onto \( \mathbb{R}_+^J \). \( M^n \) denotes the \( n \)-fold composition of a map \( M : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n \) denoted by \( M \circ \cdots \circ M \). A function \( \alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+^n \) with \( \alpha(0) = 0 \) and \( \alpha(t) > 0 \) for \( t > 0 \) is called positive definite. A function \( \gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is said to be of class \( K \) if it is continuous, strictly increasing and \( \gamma(0) = 0 \). It is of class \( K_\infty \) if, in addition, it is unbounded. Note that for \( \gamma \in K_\infty \) the inverse function \( \gamma^{-1} \) always exists and \( \gamma^{-1} \in K_\infty \). A function \( \beta : \mathbb{R}_+^+ \times \mathbb{R}_+^+ \rightarrow \mathbb{R}_+ \) is said to be of class \( K_\mathcal{L} \) if for each fixed \( t \), the function \( \beta(t,\cdot) \) is of class \( K \) and, for each fixed \( s \), the function \( \beta(s,\cdot) \) is non-increasing and tends to zero for \( t \rightarrow \infty \). A function \( \beta : \mathbb{R}_+^+ \times \mathbb{R}_+^+ \rightarrow \mathbb{R}_+ \) is said to be of class \( K_{\mathcal{L},\mathcal{L}} \) if, for each fixed \( r \geq 0 \), the function \( \beta(\cdot,\cdot,r) \in K_{\mathcal{L}} \) and \( \beta(\cdot,\cdot,r) \in K_{\mathcal{L},\mathcal{L}} \).

2.2 Interconnected hybrid systems

Consider a system that is an interconnection of \( n \) hybrid subsystems with states \( x_i \in \chi_i \subset \mathbb{R}^N_i \) of the subsystems and external inputs \( u_i \in U_i \subset \mathbb{R}^M_i \), \( i = 1,\ldots,n \). Dynamics of the \( i \)-th subsystem is given by

\[
 \dot{x}_i = f_i(x_1,\ldots,x_n,u_i), \quad (x_1,\ldots,x_n,u_i) \in C_i,
\]

\[
 x_i^+ = g_i(x_1,\ldots,x_n,u_i), \quad (x_1,\ldots,x_n,u_i) \in D_i, \tag{1}
\]

where \( f_i : C_i \rightarrow \mathbb{R}^N_i, g_i : D_i \rightarrow \chi_i \) and \( C_i, D_i \) are subsets of \( \chi_1 \times \cdots \times \chi_n \times \cup_1 \times \cdots \times U_n \).

Each hybrid subsystem is described by \( (f_i, g_i, C_i, D_i, \chi_i, U_i) \). Function \( f_i \) describes continuous dynamics defined on the set \( C_i \), function \( g_i \) describes instantaneous jumps defined on the set \( D_i \). Define \( \chi := \chi_1 \times \cdots \times \chi_n \), \( U := U_1 \times \cdots \times U_n \). For the existence of solutions we assume that the following basic regularity conditions [3], [12] hold throughout the paper:

1. \( \chi \) is open, \( U_i \) is closed, and \( C_i, D_i \subset \chi \times U \) are relatively closed in \( \chi \times U \);
2. \( f_i, g_i \) are continuous.

Note that these conditions do not guarantee the uniqueness of solutions. For discussion of uniqueness of solutions and their continuous dependence on initial conditions for hybrid systems we refer to [12] and [24].

The solutions are defined on hybrid time domains. Hybrid time domains are defined as follows, cf. [31], [10], [26]. A subset \( \mathbb{R}_+ \cup \mathbb{N}_+ \) is called a hybrid time domain denoted by \( dom \) if it is given as a union of finitely or infinitely many intervals \( \{t_0,t_{k+1}\} \times \{k\} \), where the numbers \( 0 = t_0, t_1, \ldots \) form a finite or infinite, nondecreasing sequence of real numbers. The "last" interval is allowed to be of the form \( \{t_K,T\} \times \{k\} \) with \( T \) finite or \( T = +\infty \). A hybrid signal is a function defined on a hybrid time domain. For the \( i \)-th subsystem the hybrid input

\[
 v_i := (x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n,u_i)^T, \tag{2}
\]

consists of hybrid signals \( u_i : dom u_i \rightarrow U_i \subset \mathbb{R}^M_i, x_j : dom x_j \rightarrow \mathbb{R}^N_j, j \neq i \) such that \( u_i(\cdot,k), x_j(\cdot,k) \) are Lebesgue measurable and locally essentially bounded for each \( k \). We call \( x_j \) internal input for subsystem \( i \) and \( u_i \) is called external input. For a signal \( u_i : dom u_i \rightarrow U_i \subset \mathbb{R}^M_i \) we define its restriction to the interval \( \{(t_1,j_1),(t_2,j_2)\} \in dom u_i \) by

\[
 u_{i(t_1,j_1),(t_2,j_2)}(t,k) = \begin{cases} u_i(t,k), & (t,j_1) \leq (t,k) \leq (t_2,j_2), \\ 0, & \text{otherwise}, \end{cases}
\]

where for elements of hybrid time domain we define that \( (s,l) \leq (t,k) \) means \( s+l \leq t+k \). For convenience, we denote \( u_i(t,k) := u_i((0,0),(t,k)) \).

A hybrid arc of subsystem \( i \) is such a hybrid signal \( x_i : dom x_i \rightarrow \chi_i \), that \( x_i(\cdot,k) \) is locally absolutely continuous for each \( k \). Define \( x := (x_1^T,\ldots,x_n^T)^T \in \chi \subset \mathbb{R}^N \), \( u := (u_1^T,\ldots,u_n^T)^T \in U \subset \mathbb{R}^M, N := \sum N_i, M := \sum M_i \). A hybrid arc and a hybrid input is a solution pair \( (x_i, v_i) \) of the \( i \)-th hybrid subsystem (1) if

(i) \( dom x_i = dom u_i = dom x_j, j \neq i \) and \( (x(0,0), u(0,0)) \in C_i \cup D_i \),

(ii) for all \( k \in \mathbb{N}_+ \) and almost all \( (t,k) \in dom x_i \), for \( (x(t,k), u(t,k)) \in C_i \), holds

\[
 \dot{x}_i(t,k) = f_i(x_1(t,k),\ldots,x_n(t,k),u_i(t,k)) \tag{3}
\]

(iii) for all \( (t,k) \in dom x_i \) such that \( (t,k+1) \in dom x_i \),
for \((x(t, k), u(t, k))\) holds

\[
x_i(t, k + 1) = g_i(x_1(t, k), ..., x_n(t, k), u_i(t, k)).
\]

(4)

Variable \(t\) denotes time and \(k\) is the number of jumps till this time instant.

To consider interconnection (1) as one large hybrid system

\[
\begin{align*}
\dot{x} &= f(x, u), \quad (x, u) \in C, \\
x^+ &= g(x, u), \quad (x, u) \in D,
\end{align*}
\]

(5)

with state \(x\) and input \(u\) defined above it seems to be natural to define \(C := \cap C_i, D := \cup D_i\), since a jump of any subsystem means a jump for the overall state \(x\), and to define function \(f : C \to \mathbb{R}^N\) by \(f := (f^1, ..., f^n)^T\) and function \(g : D \to \mathbb{R}\) as \(g := (g_i^1, ..., g_i^n)^T\), where

\[
\bar{g}_i(x, u) := \begin{cases} 
  g_i(x, u), & \text{if } (x, u) \in D_i, \\
  x_i, & \text{otherwise}.
\end{cases}
\]

Note that the solutions of (5) may have different hybrid time domains than the solutions of the individual systems (1), see [30]. The above choice of \(C\) and \(D\) was used also in [30] considering interconnections of two hybrid systems. However this choice has certain drawbacks, see Remark 4.6, see also Remark 4.3 in [30]. Let \(\| \cdot \|\) denote some norm in \(\mathbb{R}^n\). The supremum norm of a hybrid input \(u\) defined on \([0, 0), (t, k) \in \text{dom } u\) is defined by

\[
\|u\|(t, k) := \max \left\{ \sup_{(s, l) \in \text{dom } u} |u(s, l)|, \sup_{(s, l) \in \Phi(u)} |u(s, l)| \right\}
\]

where \(\Phi(u)\) is the set of all \((s, l) \in \text{dom } u\) such that \((s, l + 1) \in \text{dom } u\). If \(t + k \to \infty\), the \(\|u\|(t, k)\) is denoted by \(\|u\|_\infty\). The set of hybrid inputs in \(\mathbb{R}^M\) that have finite \(\| \cdot \|_\infty\) is denoted by \(\mathcal{L}_\infty^\infty\). A solution pair of hybrid system is maximal if it cannot be extended. It is complete if its hybrid time domain is unbounded. Let \(S_{\text{fin}}(x_0)\) be the set of all maximal solution pairs \((x, u)\) to (5) with \(x(0, 0) = x_0\).

2.3 Input-to-state stability

To study stability of interconnected hybrid systems we use the notion of input-to-state stability:

Definition 2.1 The \(i\)th subsystem (1) is called ISS, if there exist \(\beta_i \in \mathcal{KL}, \gamma_{ij}, \gamma_i \in \mathcal{K}_\infty \cup \{0\}\) such that for all initial values \(x_{i0}\) each solution pair \((x_i, v_i) \in S_{\text{fin}}(x_{i0})\) with \(v_i\) from (2) satisfies \(\forall (t, k) \in \text{dom } x_i\) the following:

\[
|x_i(t, k)| \leq \max \{\beta_i(|x_{i0}|, t, k), \max_{j \neq i} \gamma_{ij}(|x_j|, (t, k)), \gamma_i(|u_i|, (t, k))\}.
\]

(6)

Functions \(\gamma_{ij}, \gamma_i\) are called ISS nonlinear gains.

Condition (6) in the case of system (5) reads as follows

\[
|x(t, k)| \leq \max \{\beta(|x_0|, t, k), \gamma(|u|, (t, k))\}, \forall (t, k) \in \text{dom } x,
\]

where \(\beta \in \mathcal{KL}\) and \(\gamma \in \mathcal{K}_\infty \cup \{0\}\).

Note that the ISS property can be equivalently defined replacing maximizations in (7) and (6) by sums, i.e., (7), (6) can be replaced by

\[
|x(t, k)| \leq \beta(|x_0|, t, k) + \gamma(|u|, (t, k)), \forall (t, k) \in \text{dom } x,
\]

(8)

\[
|x_i(t, k)| \leq \beta_i(|x_{i0}|, t, k) + \sum_{j \neq i} \gamma_{ij}(|x_j|, (t, k)) + \gamma_i(|u_i|, (t, k)),
\]

(9)

\(\forall (t, k) \in \text{dom } x_i\), respectively. The equivalence follows from the fact that for any nonnegative \(a_1, ..., a_n\)

\[
\max\{a_1, ..., a_n\} \leq \sum_{i=1}^n a_i \leq n \max\{a_1, ..., a_n\}.
\]

(10)

However the functions \(\beta, \beta_i\), as well as the gain functions are in general different in these two types of definitions. Recall that different types of ISS definitions necessarily lead to different small gain type conditions for interconnections. More details on the relation between these two types of ISS definitions can be found in [6].

To prove one of the main results of this paper we borrow the following stability notions from [3]:

Definition 2.2 System (5) is called 0-input pre-stable, if for each \(\epsilon > 0\) there exists \(\delta > 0\) such that each solution pair \((x, 0) \in S_{\text{fin}}(x_0)\) with \(|x_0| \leq \delta\) satisfies \(|x(t, k)| \leq \epsilon\) for all \((t, k) \in \text{dom } x\).

Definition 2.3 System (1) is called globally pre-stable (pre-GS), if \(\exists \sigma_1, \hat{\gamma}_ij, \hat{\gamma}_i \in \mathcal{K}_\infty \cup \{0\}\) such that for all initial values \(x_{i0}\) all solution pairs \((x_i, v_i) \in S_{\text{fin}}(x_{i0})\) satisfy

\[
|x_i(t, k)| \leq \max \{\sigma(|x_{i0}|), \max_{j \neq i} \hat{\gamma}_{ij}(|x_j|, (t, k)), \hat{\gamma}_i(|u_i|, (t, k))\}
\]

(11)

Condition (11) in the case of system (5) reads as follows

\[
|x(t, k)| \leq \max \{\sigma(|x_0|), \hat{\gamma}(|u|, (t, k))\}, \forall (t, k) \in \text{dom } x.
\]

(12)

Definition 2.4 System (1) has the asymptotic gain property (AG), if there exist \(\hat{\gamma}_ij, \hat{\gamma}_i \in \mathcal{K}_\infty \cup \{0\}\) such that for all initial values \(x_{i0}\) all solution pairs \((x_i, v_i) \in S_{\text{fin}}(x_{i0})\) are bounded and, if complete, then satisfy

\[
\limsup_{(t, j) \to \infty} |x_i(t, k)| \leq \max \{\hat{\gamma}_{ij}(|x_j|, \infty), \hat{\gamma}_i(|u_i|, \infty)\}
\]

(13)

Condition (13) in the case of system (5) reads as follows

\[
\limsup_{(t, j) \to \infty} |x(t, k)| \leq \hat{\gamma}(|u|, \infty).
\]
Another notion useful for stability investigations of ISS systems is the notion of ISS-Lyapunov function. In the next section we show how this function can be defined in three different ways and prove that if a system has such a function introduced in one way then there exists an ISS-Lyapunov function introduced in another way. As well we recall results showing that existence of an ISS-Lyapunov function guarantees that the system is ISS.

3 Lyapunov characterization

We consider locally Lipschitz continuous functions $V : \chi^n \to \mathbb{R}_+$ that are differentiable almost everywhere by the Rademacher’s theorem. The set of such functions we denote by $\text{Lip}_{loc}$. To cover the points where such function is not differentiable we use the notion of Clarke generalized gradient, see [4], [9]. The set

$$\partial V(x) = \text{conv}\{z \in \mathbb{R}^n : \exists x_k \to x, \exists V(x_k) \to z \}$$

is called Clarke generalized gradient of $V$ at $x \in \chi$. If $V$ is differential at some point, then its Clarke generalized gradient coincides with the usual gradient at this point.

Definition 3.1 Function $V : \chi \to \mathbb{R}, V \in \text{Lip}_{loc}$ is called ISS-Lyapunov function for system (5) if

1) There exist functions $\psi_1, \psi_2 \in K_\infty$ s.t.:

$$\psi_1(|x|) \leq V(x) \leq \psi_2(|x|) \text{ for any } x \in \chi. \quad (16)$$

2) There exist function $\gamma \in K$, and continuous, positive definite functions $\alpha, \lambda$ with $\lambda(s) < s$ for all $s > 0$ s.t.:

$$V(x) \geq \gamma(|u|) \Rightarrow \forall \zeta \in \partial V(x) : \langle \zeta, f(x, u) \rangle \leq -\alpha(V(x)), (x, u) \in C, \quad (17)$$

$$V(x) \geq \gamma(|u|) \Rightarrow V(g(x, u)) \leq \lambda(V(x)), \quad (x, u) \in D. \quad (18)$$

If $V$ is differentiable at $x$, then (17) can be written as

$$V(x) \geq \gamma(|u|) \Rightarrow \nabla V(x) \cdot f(x, u) \leq -\alpha(V(x)), (x, u) \in C.$$

Consider a function $W : \chi \to \mathbb{R}, W \in \text{Lip}_{loc}$ that satisfies the following properties for (5):

1) There exist functions $\tilde{\psi}_1, \tilde{\psi}_2 \in K_\infty$ such that:

$$\tilde{\psi}_1(|x|) \leq W(x) \leq \tilde{\psi}_2(|x|) \text{ for any } x \in \chi. \quad (19)$$

2) There exist function $\tilde{\gamma} \in K$, continuous, positive definite function $\tilde{\alpha}_1$ and function $\tilde{\alpha}_2 \in K_\infty$ such that:

$$|x| \geq \tilde{\gamma}(|u|) \Rightarrow \forall \zeta \in \partial W(x) : \langle \zeta, f(x, u) \rangle \leq -\tilde{\alpha}_1(|x|), (x, u) \in C, \quad (20)$$

$$|x| \geq \tilde{\gamma}(|u|) \Rightarrow W(g(x, u)) - W(x) \leq -\tilde{\alpha}_2(|x|), (x, u) \in D. \quad (21)$$

In [3] the conditions (19)-(21) with $\tilde{\alpha}_1 \in K_\infty$ were used to define an ISS-Lyapunov function for (5) and it was shown that existence of such (smooth) function $W$ implies that (5) is ISS. The next assertion is proved in Appendix A.1.

Proposition 3.2 System (5) has an ISS-Lyapunov function $V$ satisfying (16)-(18) if and only if there exists $W \in \text{Lip}_{loc}$ satisfying (19)-(21).

The next proposition, proved Appendix A.2, shows another way to introduce an ISS Lyapunov function, used in [5], [26]

Proposition 3.3 System (5) has an ISS-Lyapunov function $V$ satisfying (16)-(18) if and only if there exists $\tilde{V} \in \text{Lip}_{loc}$ satisfying (16)-(17) and

$$\tilde{V}(g(x, u)) \leq \max\{\tilde{\lambda}(\tilde{V}(x)), \tilde{\gamma}(|u|)\}, (x, u) \in D. \quad (22)$$

Remark 3.4 Relations between the existence of a smooth ISS-Lyapunov function and the ISS property for hybrid systems were discussed in [3]: In particular, Proposition 2.7 in [3] shows that if a hybrid system has an ISS-Lyapunov function, then it is ISS. Example 3.4 in [3] shows that the converse is in general not true. In Theorem 3.1 [3] it was proved that if (5) is ISS with $\gamma$ such that the set $\{f(x, u) : u \in U \cap \mathbb{B}\}$ is convex $\forall s > 0$ and for any $x \in \chi$, then it has an ISS-Lyapunov function.

It turns out that smoothness of an ISS-Lyapunov function is not necessary to guarantee the ISS property as the following proposition shows.

Proposition 3.5 If system (5) has a locally Lipschitz continuous ISS-Lyapunov function, then it is ISS.

Idea of the proof. The proof follows the same steps as in the smooth case using ISS-Lyapunov function $V$ satisfying (19)-(21), see Proposition 2.7 in [3]. At the places where $V$ is not differentiable in the classical sense the Clarke generalized derivative should be applied. Note that in [3] $\tilde{\alpha}_1 \in K_\infty$ was assumed, however the proof of Proposition 2.7 does not change in case $\tilde{\alpha}_1$ is continuous and positive definite.

Remark 3.6 If an input $u$ is written as a composed vector of several ones, the norm $|u|$ can be written as corresponding norm over the norms of the components, e.g. in case of (1), instead of $\gamma(|u|)$ in (17) and (18) we should write $\gamma(|x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, u_i|) = \gamma(|[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, u_i]^T|)$. By an appropriate choice of $\gamma_{ij}, \gamma_i \in K_\infty$, that depends on the used norms, the last expression can be written as sum or maximum over $\gamma_{ij}(|x_j|), j \neq i$ and $\gamma_i(|u_i|)$. If we have an ISS Lyapunov function for each subsystem then $\gamma_{ij}(|x_j|)$ can be estimated from above and below by $\bar{\gamma}_{ij}(V_i(x_j))$ with an appropriate choice of $\bar{\gamma}_{ij} \in K_\infty$, this follows from (16).

Consider (5) as an interconnection of $n$ hybrid subsystems (1). Assume that each subsystem $i$ has an ISS-Lyapunov function $V_i$, i.e.:

1) There exist functions $\psi_{i1}, \psi_{i2} \in K_\infty$ s.t.:

$$\psi_{i1}(|x_i|) \leq V_i(x_i) \leq \psi_{i2}(|x_i|) \text{ for any } x_i \in \chi_i. \quad (23)$$
2) There exist $\gamma_{ij}, \gamma_i \in \mathcal{K}_\infty$ and continuous, positive definite functions $\alpha_i$, $\Lambda_i$, with $\lambda_i(s) < s$ for all $s > 0$ such that for all $(x,u) \in C_i$

$$V_i(x_i) \geq \max \{ \max_j \{ \gamma_{ij}(V_j(x_j)) \}, \gamma_i(|u_i|) \} \Rightarrow \forall \zeta_i \in \partial V_i(x_i) : \langle \zeta_i, f_i(x, u_i) \rangle \leq -\alpha_i(V_i(x_i))$$

(24)

and for all $(x,u) \in D_i$

$$V_i(x_i) \geq \max_j \{ \gamma_{ij}(V_j(x_j)) \}, \gamma_i(|u_i|) \Rightarrow V_i(g_i(x_1, \ldots, x_n, u_i)) \leq \lambda_i(V_i(x_i)).$$

(25)

Functions $\gamma_{ij}$ and $\gamma_i$ are called ISS Lyapunov gains corresponding to internal inputs $x_j$ and external input $u_j$ respectively. Note that $\gamma_{ij}$ are taken the same in (24) and (25). This can be always achieved by taking maximum of separately obtained $\gamma_{ij}$’s for continuous and discontinuous dynamics.

**Remark 3.7** Instead of functions $V_i$ we can consider the other both types of ISS-Lyapunov functions $W_i$ and $V_i$ as discussed above. We do not write their definitions for space reasons. Moreover the maximum over gains used in each of these three definitions can be replaced by their sum yielding six equivalent ways to define an ISS-Lyapunov function for (1).

Note that an interconnection can be unstable, i.e. not ISS, even if each its subsystem is ISS. In the following section we introduce conditions that guarantee stability for interconnections of ISS hybrid systems.

4 Main results

4.1 Gain operators

The main question of this paper is whether the interconnection (5) of the ISS subsystems (1) is ISS. To study this question we collect the gains $\gamma_{ij}$ of the subsystems in a matrix $\Gamma = (\gamma_{ij})_{n \times n}$, $i,j = 1, \ldots, n$ denoting $\gamma_{ii} = 0$, $i = 1, \ldots, n$ for completeness [8,29]. Note that functions $\gamma_{ij}$ can be taken from any of the definitions of Section 2 or from (24), (25). This choice depends on the context. The matrix $\Gamma$ describes in particular interconnection topology of the whole network, moreover it contains the information about the mutual influence between the subsystems. We also introduce the following two gain operators $\Gamma_{\text{max}} : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ and $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, see [8,29,21] : 

$$\Gamma_{\text{max}}(s) := \begin{pmatrix} \max \{ \gamma_{1,2}(s_2), \ldots, \gamma_{1,n}(s_n) \} \\ \vdots \\ \max \{ \gamma_{n,1}(s_1), \ldots, \gamma_{n,n-1}(s_{n-1}) \} \end{pmatrix}$$

(26)

$$\Gamma(s) := \begin{pmatrix} \gamma_{1,2}(s_2) + \ldots + \gamma_{1,n}(s_n) \\ \vdots \\ \gamma_{n,1}(s_1) + \ldots + \gamma_{n,n-1}(s_{n-1}) \end{pmatrix}.$$ 

(27)

We define the small gain condition for the case of maximization as follows

$$\Gamma_{\text{max}}(s) \preceq s, \ \forall s \in \mathbb{R}^n_+, \ s \neq 0.$$ 

(28)

For the case of summation the small gain condition is given by

$$\exists \alpha \in \mathcal{K}_\infty : \text{diag}_n(\text{id} + \alpha \circ \Gamma)(s) \not\preceq s, \forall s \in \mathbb{R}^n_+, \ s \neq 0 \tag{29}$$

where $\text{diag}_n(\text{id} + \alpha) : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is defined by

$$\text{diag}_n(\text{id} + \alpha)(s)^T := ((\text{id} + \alpha)(s_1), \ldots, (\text{id} + \alpha)(s_n))^T.$$ 

The small gain conditions (28) and (29) were introduced and studied in [7], [8] and [29]. In particular, in [29] and [6] it was shown how these conditions can be brought to the cycle condition:

$$\tilde{\gamma}_{k_1} \circ \tilde{\gamma}_{k_2} \circ \ldots \circ \tilde{\gamma}_{k_{p-1}} \not\preceq \text{id},$$

(30)

for all $(k_1, \ldots, k_p) \in \{1, \ldots, n\}^p$ with $k_1 = k_p$, where in the case of (28) $\tilde{\gamma}_{k,k-1} = \gamma_{k,k-1}$. As matrix $\Gamma$ describes interconnection structure of the network, these conditions impose some restriction on interconnection properties. We will see that this restriction guarantee stability of the network. The intuitive meaning of (30) is that a signal going through the network is not amplified. See also [8] for further interpretations of the small gain conditions (28) and (29).

4.2 Small gain theorems in terms of trajectories

The following small gain theorems extend the results of [23] to the case of arbitrary number of interconnected subsystems and [29,8] to the case of hybrid systems.

**Theorem 4.1** Consider interconnected system (5) and assume that all subsystems in (1) are pre-GS. If operator $\Gamma_{\text{max}}$ defined in (26) satisfies (28), then (5) is pre-GS.

**Theorem 4.2** Assume that each subsystem (1) has the AG property and that solutions of the system (5) exist and are bounded. Let the gain matrix $\Gamma$ be given by $\Gamma = (\tilde{\gamma}_{ij})_{n \times n}$. If the operator $\Gamma_{\text{max}}$ defined in (26) satisfies (28), then system (5) satisfies the AG property.

See Appendix B.1,B.2 for the proofs of both theorems.

**Remark 4.3** The existence and boundedness of solutions of (5) is essential, otherwise the assertion is not true, see Example 14 in [8].
Theorem 4.4 Consider interconnected system (5) and assume that the set \( \{f(x,u) : u \in U \cap \mathbb{B}\} \) is convex for each \( x \in \chi, \epsilon > 0 \).

(i) If all subsystems in (1) are ISS satisfying (6) and \( \Gamma_{\text{max}} \) defined in (26) satisfies (28), then the system (5) is ISS.

(ii) If all subsystems in (1) are ISS satisfying (9) and if \( \Gamma_{\Sigma} \) defined in (27) satisfies the small gain condition (29), then the system (5) is ISS.

Idea of the proof. The idea follows from the proof of a similar theorem for continuous systems in [8]. We describe it briefly for the case (i): By Theorem 3.1 in [3], since each subsystem is ISS, they are pre-GS and have the AG property. By Theorem 4.1 and Theorem 4.2 the whole interconnection (5) is pre-GS and has the AG property. From global pre-stability of (5) \( \text{0}-\text{input pre-stability follows}. \) ISS of (5) follows then by Theorem 3.1 in [3]. \( \square \)

4.3 Small gain theorems in terms of Lyapunov functions

In this section we show how the small gain condition allows to establish the ISS of an interconnection (5) and to construct an ISS-Lyapunov function for it.

Theorem 4.5 Consider system (5) as interconnection of subsystems (1) and assume that \( D_i = D_j, i,j = 1, \ldots, n, \) and that each subsystem is has an ISS Lyapunov function \( V_i \) satisfying (23)-(25) with corresponding ISS-Lyapunov gains. Let the corresponding gain operator \( \Gamma_{\text{max}} \), defined by (26) in terms of these gains, satisfy (28), then the hybrid system (5) has an ISS-Lyapunov function given by

\[
V(x) = \max_i \sigma_i^{-1}(V_i(x_i))
\]

where \( \sigma_i(r) := \max_i \{a_i r (\Gamma_{\text{max}}(ar))i, \ldots, (\Gamma_{\text{max}}(ar))i\} \), \( r \in \mathbb{R}_+ \) with an arbitrary positive vector \( a = (a_1, \ldots, a_n)^T \).

Note that according to the notation given in Section 2, \( \Gamma_{\text{max}}(ar) \) is the ith component of the vector \( \Gamma_{\text{max}}(ar) \) where \( \Gamma_{\text{max}} \) is the \( k \)-fold composition of \( \Gamma_{\text{max}} \).

Proof. Consider the map \( Q : \mathbb{R}_+^n \to \mathbb{R}_+^n \) defined by \( Q(x) := (Q_1(x), \ldots, Q_n(x))^T \) with \( Q_i(x) := \max \{x_i, \Gamma_{\text{max}}(x_i), \ldots, (\Gamma_{\text{max}}(x_i))\} \). In [22], Proposition 2.4, it was shown that if \( \Gamma_{\text{max}}(s) \geq s, \forall s \in \mathbb{R}_+^n, s \neq 0 \) holds, then the inequality \( \Gamma_{\text{max}}(Q(x)) \leq Q(x) \) holds for all \( x \geq 0 \). Fix any positive vector \( a > 0 \) and consider \( \sigma(t) := Q(at) \in \mathbb{R}_+^n \). Obviously,

\[
\Gamma_{\text{max}}(\sigma(t)) \leq \sigma(t)
\]

and by the definition of \( Q \) it follows that \( \sigma_i \in \mathcal{K}_{\infty} \) for all \( i = 1, \ldots, n \). Without loss of generality, the gains \( \gamma_{ij} \) can be assumed to be smooth, see Appendix B in [13].

Hence, \( \sigma_i \) satisfy:

(i) \( \sigma_i^{-1} \) is locally Lipschitz continuous on \((0, \infty)\);

(ii) for every compact set \( K \subset (0, \infty) \) there are finite constants \( 0 < K_1 < K_2 \) such that for all points of differentiability of \( \sigma_i^{-1} \) we have

\[
0 < K_1 \leq (\sigma_i^{-1})'(r) \leq K_2, \quad \forall r \in K.
\]

With \( \phi \in \mathcal{K}_{\infty} \) such that \( \phi(t) \leq \max_i \{\max_i \{\gamma_{ij}(\sigma_j(t)), \sigma_i(t)\}\} \) for all \( t \geq 0 \) and using (32) we obtain for each \( i \)

\[
\max_i \{\max_i \{\gamma_{ij}(\sigma_j(r)), \sigma_i(r)\}\} \leq \sigma_i(r), \forall r > 0.
\]

Taking \( \psi_1(|x|) := \min_{r=1,\ldots,n} \sigma_i^{-1}(\psi_1(|x|)) \) and \( \psi_2(|x|) := \max_{i=1,\ldots,n} \sigma_i^{-1}(\psi_2(|x|)) \) the condition (16) is satisfied. Consider \( x \neq 0 \), as the case \( x = 0 \) is obvious. Define

\[
I := \{i \in \{1, \ldots, n\} : V(x_i) = \sigma_i^{-1}(V_i(x_i)) \geq \max_j \sigma_j^{-1}(V_j(x_j))\}.
\]

Note that \( x_i \neq 0 \) for \( i \in I \). As \( V \) is obtained through maximization (31), by [4, p.83] we have that

\[
\partial V(x) \subset \text{conv}\left\{ \bigcup_{i \in I} \partial \sigma_i^{-1} \circ V_i \circ P_i(x) \right\}.
\]

Fix \( i \in I \). If \( V(x) \geq \max_j \{\sigma_j^{-1}(\psi_1(|x|))\} \) then \( \gamma_i(|x|) \leq \phi(V(x)) \) and from (34), (35) we have

\[
V_i(x_i) = \sigma_i(V(x)) \geq \max_j \sigma_j(V(x)) \geq \max_i \{\gamma_{ij}(V_j(x_j)), \gamma_i(|u|)\}.
\]

If \( (x,u) \in C \), we can apply (24) to obtain for all \( \zeta_i \in \partial V_i(x_i) \) that

\[
\langle \zeta_i, f_i(x,u) \rangle \leq -\alpha_i(V_i(x_i)).
\]

By the chain rule for Lipschitz continuous functions (see Theorem 2.5 in [4]), we have

\[
\partial(\sigma_i^{-1} \circ V_i(x)) \subset \{\zeta_i : c \in \partial \sigma_i^{-1}(y), y = V_i(x), \zeta_i \in \partial V_i(x_i)\}.
\]

For \( \rho > 0 \) define \( \tilde{a}_i(\rho) := c_{\rho,\alpha}(\rho) > 0 \), where the constant \( c_{\rho,\alpha} \) corresponds to the set \( K := \{x_i \in \chi_i : \rho/2 \leq \|x_i\| \leq 2\rho\} \) given by (33). For \( r > 0 \) define \( \tilde{a}(r) := \min_{\rho \in \mathbb{R}_+^n} \{\tilde{a}_i(\rho) : \|x_i\| \leq r, V(x) = \sigma_i^{-1}(V_i(x)) \} \geq 0 \). Thus using (37), (38) for all \( \zeta \in \partial \sigma_i^{-1} \circ V_i(x_i) \) we obtain

\[
\langle \zeta, f_i(x,u) \rangle \leq -\tilde{a}(\|x\|).
\]

The same argument applies for all \( i \in I \). Now for any \( \zeta \in \partial V(x) \) we have by (36) that \( \zeta = \sum_{i \in I} c_i \zeta_i \) for suitable \( \mu_i \geq 0, \sum_{i \in I} \mu_i = 1 \), and with \( \zeta_i \in \partial(\sigma_i \circ P_i)(x_i) \) and \( c_i \in \partial(\sigma_i^{-1})\circ \partial V_i(x_i) \). It follows then that

\[
\langle \zeta, f(x,u) \rangle = \sum_{i \in I} \mu_i \langle c_i \zeta_i, f(x,u) \rangle = \sum_{i \in I} \mu_i \langle c_i P_i(\zeta_i), f_i(x,u) \rangle
\]
\[
\leq -\sum_{i \in I} \mu_i \hat{\alpha}(\|x\|) \leq -\dot{\alpha}(\|x\|) \leq -\alpha \circ \psi_2^{-1} \circ V(x).
\]

Thus condition (17) is satisfied with \( \alpha := \alpha \circ \psi_2^{-1} \).

Consider now the case \((x, u) \in D\). Define
\[
\lambda(t) = \max\{\sigma_i^{-1} \circ \lambda_i \circ \sigma(t)\}, \quad t > 0.
\]

Note that \( \sigma^{-1} \circ \lambda \circ \sigma(t) < \sigma_i^{-1} \circ \sigma_i(t) = t \) for all \( t > 0 \)
as \( \lambda_i(t) < t \). Thus \( \lambda(t) < t \). Condition (25) for ISS-Lyapunov function of subsystem 0 implies for \((x, u) \in D\)
\[
V(g(x, u)) = \max_i \sigma_i^{-1} \circ V_i(g_i(x, u)) \leq \max_i \sigma_i^{-1} \circ \lambda_i V_i(x_i)) = \max_i \sigma_i^{-1} \circ \lambda_i \circ \sigma_i^{-1} (V_i(x_i)) = \lambda \circ \sigma_i^{-1} (V_i(x_i)) = \lambda(V(x)).
\]

Thus (18) is also satisfied and hence \( V \) is an ISS-Lyapunov function of the network (5).

Remark 4.6 If we drop the requirement \( D_i = D_j \) in Theorem 4.5 the assertion is not true, because otherwise it may happen for an \( x \in D \), that \( x \in D_i \) but \( x \notin D_j \) for some \( i \neq j \). According to our definition of the interconnection (5) this allows for a situation that one system undergoes jumps and another one is "frozen", in particular (25) will not be satisfied and (41) does not hold.

Theorem 4.5 provides a constructive method to derive a Lyapunov function for interconnected hybrid systems. A similar result can be shown for interconnections with ISS Lyapunov functions defined in other ways mentioned in the previous section.

Theorem 4.7 Consider system (5) that is interconnection of subsystems (1). Assume there are Lyapunov functions \( V_i \) corresponding to subsystems (1) defined in any of the possible ways described in Remark 3.7 with corresponding ISS-Lyapunov gains \( \gamma_{ij} \). Consider operator \( \Gamma \) defined by (26) (resp. (27)) in use of definition with a maximum (resp. sum) over the gains.

If the small gain condition (28) (resp. (29)) is satisfied, then the hybrid system (5) is ISS with the ISS-Lyapunov function given by (31) with some auxiliary \( K_{\infty} \) gains \( \sigma_i \) (their existence is guaranteed by (28) (resp. (29))).

The proof is similar to the previous one, see also [9,5]. However, there is no explicit construction of \( \sigma \) available in case a definition in terms of summations is used.

Remark 4.8 In the case \( D = \emptyset \) we obtain result from [9] as a particular case.

Remark 4.9 If the set \( C \) has a nonempty interior then it is enough to consider classical derivative of \( V \) at the points of differentiability in (17). This would simplify the previous considerations. But for the set \( C \) with an empty interior it is not enough in general to use classical derivative of \( V \), see Example 2.6 in [5].

Remark 4.10 In some applications one is interested in stability of only a part of the states, see for example [26], Consider the following system
\[
T x = f(x, u), \quad (x, u) \in C
\]
\[
\dot{t} = f_t(x, u), \quad (x, u) \in D
\]
\[
x_t^+ = g_t(x, u), \quad (x, u) \in D
\]
\[
x_t^- = g_t(x, x_t, u), \quad (x, u) \in D
\]

where \( x \) is the state that we study on stability. Then the ISS property for (42) can be defined by
\[
|x(s, k)| \leq \max\{\beta(|x|, s, k), \gamma(|x|, s, k)\}, \forall (s, k) \in \Omega
\]

Using the proof of Theorem 4.5 one can also show ISS of such system under the small gain condition (28). The same one can show for summation formulation of ISS.

5 Conclusions

We have shown that a large scale interconnection of ISS hybrid systems is again ISS if a small gain condition is satisfied. The results are provided in terms of trajectories and Lyapunov functions. Moreover an explicit construction of an ISS-Lyapunov function is given. These results extend the corresponding known theorems from [26] to the case of interconnection of more than two hybrid systems and [21] for general type of hybrid systems.

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A Equivalence between formulation of ISS-Lyapunov functions

A.1 Proof of Proposition 3.2

"⇒" Let \( V \) satisfy (16)-(18). Define
\[
\tilde{\gamma}(|u|) := \psi_1^{-1} \circ \gamma(|u|).
\]

Let \( |x| \geq \gamma(|u|) \). Then (16), (A.1) \( \Rightarrow \ V(x) \geq \gamma(|u|) \).

Applying (18) it follows that for all \( (x, u) \in D \)
\[
V(g(x, u)) \leq \lambda(V(x)) = V(x) - \tilde{\alpha}(x)
\]

\( \Rightarrow V(g(x, u)) - V(x) \leq -\tilde{\alpha}(x) \).

where \( \tilde{\alpha}(x) := \lambda(V(x)) - \lambda(V(x)) \) is a continuous, positive definite function. From (A.2) and Lemma 2.8 in [20] there exists \( \rho, \tilde{\alpha} \in K_{\infty} \) such that \( W := \rho \circ V \) satisfies (21) with \( \tilde{\gamma}(|u|) \) defined in (A.1).

As \( V(x) \) satisfies (17), then \( W := \rho \circ V \) satisfies (20) with \( \tilde{\alpha} := \tilde{\rho} \cdot \alpha \) that is continuous, positive definite function, where \( \tilde{\rho} \in K_{\infty}, \alpha = V(x) \).

\[7\]
Thus function $W$ satisfies (19), (20) and (21) with $\tilde{\psi}_1 := \rho \circ \tilde{\psi}_1$, $\tilde{\psi}_2 := \rho \circ \tilde{\psi}_2$, $\hat{\gamma}(\|u\|) := \tilde{\psi}_1^{-1} \circ \gamma(\|u\|)$ and $\tilde{\alpha}_1 := \tilde{\rho} \circ \alpha$. 

Assume now that function $W$ satisfies (19)-(21) and define $V := W$, $\tilde{\psi}_1 := \tilde{\psi}_1$ and $\tilde{\psi}_2 := \tilde{\psi}_2$. Then condition (16) is satisfied. Let

$$
\hat{\gamma}(\|u\|) := \tilde{\psi}_2 \circ \gamma(\|u\|). \quad (A.3)
$$

Consider $V(x) \geq \gamma(\|u\|)$. Then from (A.3), (19) $|x| \geq \hat{\gamma}(\|u\|)$. From (19), (20) for all $(x, u) \in C$

$$
\forall \zeta \in \partial W(x): (\zeta, f(x, u)) \leq -\hat{\alpha}_1(\|x\|) \leq -\hat{\alpha}_1 \circ \tilde{\psi}_2^{-1}(W(x)).
$$

Thus $V$ satisfies (17) with $\alpha := \hat{\alpha}_1 \circ \tilde{\psi}_2^{-1}$.

From Lemma B.1 in [19] for any $\hat{\alpha}_1 \circ \tilde{\psi}_2^{-1} \in \mathcal{K}_\infty$ there exists $\hat{\alpha} \in \mathcal{K}_\infty$ such that $\hat{\alpha} \leq \hat{\alpha}_1 \circ \tilde{\psi}_2^{-1}$ and id $- \hat{\alpha} \in \mathcal{K}$. Applying (19) and (21) we obtain

$$
W(g(x, u)) \leq W(x) - \hat{\alpha}_2(\|x\|) \leq W(x) - \hat{\alpha}_1 \circ \tilde{\psi}_2^{-1}(W(x)) \leq W(x) - \hat{\alpha}(W(x)) \leq \lambda(\hat{\alpha}(W(x)), \hat{\gamma}(\|u\|)) \leq \max\{\rho(\hat{\alpha}(W(x)), \hat{\gamma}(\|u\|))\}.
$$

Define $\gamma := \rho^{-1} \circ \hat{\gamma} > \gamma$ and $V := \tilde{V}$. If $V(x) \geq \gamma(\|u\|)$ then

$$
\rho(V(x)) \geq \gamma(\|u\|) \geq \gamma(\|u\|) \text{ (using (A.4))}
$$

Using (A.4) we have

$$
\tilde{V}(g(x, u)) \leq \max\{\rho(\hat{\alpha}(W(x)), \hat{\gamma}(\|u\|))\} = \rho(V(x)) = \rho(V(x)).
$$

Thus function $V$ satisfies condition (18) with $\hat{\lambda} := \hat{\alpha} \circ \mathrm{id}$. 

"⇒" From (18) for $(x, u) \in D$, if $V(x) > \gamma(\|u\|)$ then $V(g(x, u)) \leq \rho(V(x))$. Consider now $(x, u) \in D$ such that $V(x) \leq \gamma(\|u\|)$ and define $A(u) := \{(x, u) \in D : V(x) \leq \gamma(\|u\|)\}$. Let us take now $\hat{\gamma}(\|u\|) := \max\{\rho(V(x), \gamma(\|u\|))\}$. Then $V(g(x, u)) \leq \hat{\gamma}(\|u\|)$. Note that $\hat{\gamma}(0) = 0$ as $V(x) \geq 0 \geq \gamma(0)$. Furthermore, as function $V$ is nonnegative and $V \in Lip_{loc}$ and function $g$ is continuous, function $\hat{\gamma} \in Lip_{loc}$ is nonnegative. We can always majorize such function $\hat{\gamma}$ by a function $\tilde{\gamma} \in \mathcal{K}$ such that $\tilde{\gamma} \leq \hat{\gamma}$. Thus for $(x, u) \in D$ we have obtained that $V(g(x, u)) \leq \max\{\hat{\gamma}(\|u\|), \Lambda(V(x))\}$ and condition (22) is satisfied with $\tilde{V} := V$ and $\tilde{\gamma} := \max\{\hat{\gamma}, \gamma\}$. 

### B.2 Small gain theorems

To prove Theorem 4.1 and Theorem 4.2 we need first the following auxiliary lemmas from [8,29].

**Lemma B.1** Let the operator $\Gamma_{\max}$ defined in (26) satisfy (28). Then there exists $\phi \in \mathcal{K}_\infty$ s.t. for all $w, v \in \mathbb{R}_+$,

$$
\phi(\|w\|) \leq \max\{\phi(\|v\|), \max_{j \neq 1} \phi(\|v_j\|)\}.
$$

**Lemma B.2** Let $s : \mathbb{R}_+ \rightarrow \mathbb{R}_+^n$ defined on time domain dom$s$ be continuous and bounded. Then

$$
\lim_{(t, k) \rightarrow \infty} \sup_{s(t, k)} s(t, k) \leq \lim_{t \rightarrow \infty} \sup_{s(t, k)} \|s(t, k)\|.
$$

**Proof.** The proof goes along the lines of the proof of a similar result for continuous systems in Lemma 3.2 in [8] but instead of time $t$ we consider the points $(t, k)$ of the time domain.

**B.1 Proof of Theorem 4.1**

Let us take the supremum over $(\tau, l) \leq (t, k)$ on both sides of (11)

$$
\|x_{i(t, k)}\|_{(\bar{\tau}, \bar{l})} \leq \max\{\sigma_i(\|x_{10}\|), \max_{j \neq 1} \bar{\gamma}_{ij}(\|x_{j(t, k)}\|_{(\bar{\tau}, \bar{l})}), \hat{\gamma}_i(\|u\|_{(\bar{\tau}, \bar{l})})\},
$$

where $(\bar{\tau}, \bar{l}) := \max_{(t, k) \in \text{dom} x_i} (\tau, l)$.

Let us denote $\Gamma := (\hat{\gamma}_{ij})_{n \times n}$, $w := (\|x_{1(t, k)}\|_{(\bar{\tau}, \bar{l})}, \ldots, \|x_{n(t, k)}\|_{(\bar{\tau}, \bar{l})})^T$, and

$$
w := \left(\begin{array}{c}
\max\{\sigma_1(\|x_{10}\|), \hat{\gamma}_1(\|u\|_{(\bar{\tau}, \bar{l})})\} \\
\vdots \\
\max\{\sigma_n(\|x_{n0}\|), \hat{\gamma}_n(\|u\|_{(\bar{\tau}, \bar{l})})\}
\end{array}\right) = \max\{\sigma(\|x_0\|), \hat{\gamma}(\|u\|_{(\bar{\tau}, \bar{l})})\}.
$$

From (B.2) we obtain $w \leq \max\{\Gamma_{\max}(w), v\}$. Then by Lemma B.1 there exists $\phi \in \mathcal{K}_\infty$ such that

$$
\|x(t, k)\|_{(\bar{\tau}, \bar{l})} \leq \phi(\|w\|) \leq \max\{\phi(\|v\|), \phi(\|w\|)\}.
$$

for all $(t, k) \in \text{dom} x$. Hence for every initial condition and essentially bounded input $u$ the solution of the system (5) exists and is bounded, since the right-hand side of (B.3) does not depend on $t, k$. From the last line in (B.3) the estimate (12) for the pre-GS follows.

**B.2 Proof of Theorem 4.2**

Assume system (5) is complete. Let $(\tau, l)$ be an arbitrary initial point of the time domain. From the definition of the AG property we have
\[
\limsup_{(i,j) \in \text{dom } x, t \to \infty} |x_i(t, k)| \\
\leq \max \left\{ \max_{j,j \neq i} \tilde{\gamma}_{ij} (\|x_j[(\tau, t)]\|), \lim_{t \to \infty} \tilde{\gamma}_i(\|u\|) \right\}.
\]

Since all solutions of (1) are bounded the following holds by Lemma B.2:

\[
\limsup_{(i,j) \in \text{dom } x, t \to \infty} |x_i(t, k)| = \limsup_{(i,j) \in \text{dom } x, t \to \infty} (\|x_i(t, \tau, t)\|) =: l_i(x_i).
\]

By this property from (B.4) and Lemma II.1 in [33] follows

\[
l_i(x_i) \leq \max \left\{ \max_{j,j \neq i} \tilde{\gamma}_{ij} (l_j(x_j)), \tilde{\gamma}_i(\|u\|) \right\}.
\]

Using Lemma B.1 for \( \Gamma = (\tilde{\gamma}_{ij})_{n \times n} \), \( w_i = l_i(x_i) \) and \( v_i := \tilde{\gamma}_i(\|u\|) \) we conclude

\[
\limsup_{(i,j) \in \text{dom } x, t \to \infty} |x_i(t, k)| \leq \phi(\|u\|).
\]

for some \( \phi \) of class \( K \), which is the desired AG property.

References


