Trees with Large Neighborhood Total Domination Number

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Abstract

In this paper, we continue the study of neighborhood total domination in graphs first studied by Arumugam and Sivagananam [Opuscula Math. 31 (2011), 519–531]. A neighborhood total dominating set, abbreviated NTD-set, in a graph $G$ is a dominating set $S$ in $G$ with the property that the subgraph induced by the open neighborhood of the set $S$ has no isolated vertex. The neighborhood total domination number, denoted by $\gamma_{nt}(G)$, is the minimum cardinality of a NTD-set of $G$. Every total dominating set is a NTD-set, implying that $\gamma(G) \leq \gamma_{nt}(G) \leq \gamma_t(G)$, where $\gamma(G)$ and $\gamma_t(G)$ denote the domination and total domination numbers of $G$, respectively. Arumugam and Sivagananam posed the problem of characterizing the connected graphs $G$ of order $n \geq 3$ achieving the largest possible neighborhood total domination number, namely $\gamma_{nt}(G) = \lceil n/2 \rceil$. A partial solution to this problem was presented by Henning and Rad [Discrete Applied Mathematics 161 (2013), 2460–2466] who showed that 5-cycles and subdivided stars are the only such graphs achieving equality in the bound when $n$ is odd. In this paper, we characterize the extremal trees achieving equality in the bound when $n$ is even. As a consequence of this tree characterization, a characterization of the connected graphs achieving equality in the bound when $n$ is even can be obtained noting that every spanning tree of such a graph belongs to our family of extremal trees.

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1 Introduction

In this paper we continue the study of a parameter, called the neighborhood total domination number, that is squeezed between arguably the two most important domination parameters, namely the domination number and the total domination number. A dominating set in a graph $G$ is a set $S$ of vertices of $G$ such that every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in $S$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$. A total dominating set, abbreviated a TD-set, of a graph $G$ with no isolated vertex is a set $S$ of vertices of $G$ such that every vertex in $V(G)$ is adjacent to at least one vertex in $S$. The total domination number of $G$, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of $G$. The literature on the subject of domination parameters in graphs up to the year 1997 has been surveyed and detailed in the two books [5, 6]. Total domination is now well studied in graph theory. For a recent book on the topic, see [10]. A survey of total domination in graphs can also be found in [7].

Arumugam and Sivagnanam [1] introduced and studied the concept of neighborhood total domination in graphs. A neighbor of a vertex $v$ is a vertex different from $v$ that is adjacent to $v$. The neighborhood of a set $S$ is the set of all neighbors of vertices in $S$. A neighborhood total dominating set, abbreviated NTD-set, in a graph $G$ is a dominating set $S$ in $G$ with the property that the subgraph induced by the open neighborhood of the set $S$ has no isolated vertex. The neighborhood total domination number of $G$, denoted by $\gamma_{nt}(G)$, is the minimum cardinality of a NTD-set of $G$. A NTD-set of $G$ of cardinality $\gamma_{nt}(G)$ is called a $\gamma_{nt}(G)$-set.

Every TD-set is a NTD-set, while every NTD-set is a dominating set. Hence the neighborhood total domination number is bounded below by the domination number and above by the total domination number as first observed by Arumugam and Sivagnanam in [1].

Observation 1 ([1, 5]) If $G$ is a graph with no isolated vertex, then $\gamma(G) \leq \gamma_{nt}(G) \leq \gamma_t(G)$.

1.1 Terminology and Notation

For notation and graph theory terminology not defined herein, we refer the reader to [5]. Let $G$ be a graph with vertex set $V(G)$ of order $n = |V(G)|$ and edge set $E(G)$ of size $m = |E(G)|$, and let $v$ be a vertex in $V$. We denote the degree of $v$ in $G$ by $d_G(v)$. The minimum degree among the vertices of $G$ is denoted by $\delta(G)$. A vertex of degree one is called a leaf and its neighbor a support vertex. We denote the set of leaves in $G$ by $L(G)$, and the set of support vertices by $S(G)$. A support vertex adjacent to two or more leaves is a strong support vertex. For a set $S \subseteq V$, the subgraph induced by $S$ is denoted by $G[S]$. A 2-packing in $G$ is a set of vertices that are pairwise at distance at least 3 apart in $G$.

A cycle and path on $n$ vertices are denoted by $C_n$ and $P_n$, respectively. A star on $n \geq 2$ vertices is a tree with a vertex of degree $n - 1$ and is denoted by $K_{1,n-1}$. A double star is a...
tree containing exactly two vertices that are not leaves (which are necessarily adjacent). A subdivided star is a graph obtained from a star on at least two vertices by subdividing each edge exactly once. The subdivided star obtained from a star $K_{1,4}$, for example, is shown in Figure 1. We note that the smallest two subdivided stars are the paths $P_3$ and $P_5$. Let $\mathcal{F}$ be the family of all subdivided stars. Let $F \in \mathcal{F}$. If $F = P_3$, we select a leaf of $F$ and call it the link vertex of $F$, while if $F \neq P_3$, the link vertex of $F$ is the central vertex of $F$.

![Figure 1: A subdivided star.]

The open neighborhood of $v$ is the set $N_G(v) = \{u \in V \mid uv \in E\}$ and the closed neighborhood of $v$ is $N_G[v] = \{v\} \cup N_G(v)$. For a set $S \subseteq V$, its open neighborhood is the set $N_G(S) = \bigcup_{v \in S} N_G(v)$, and its closed neighborhood is the set $N_G[S] = N_G(S) \cup S$. If the graph $G$ is clear from the context, we simply write $d(v)$, $N(v)$, $N[v]$, $N(S)$ and $N[S]$ rather than $d_G(v)$, $N_G(v)$, $N_G[v]$, $N_G(S)$ and $N_G[S]$, respectively. As observed in [8] a NTD-set in $G$ is a set $S$ of vertices such that $N[S] = V$ and $G[N(S)]$ contains no isolated vertex.

A rooted tree distinguishes one vertex $r$ called the root. For each vertex $v \neq r$ of $T$, the parent of $v$ is the neighbor of $v$ on the unique $(r,v)$-path, while a child of $v$ is any other neighbor of $v$. A descendant of $v$ is a vertex $u$ such that the unique $(r,u)$-path contains $v$. Let $D(v)$ denote the set of children and descendants, respectively, of $v$, and let $D[v] = D(v) \cup \{v\}$. The maximal subtree at $v$ is the subtree of $T$ induced by $D[v]$, and is denoted by $T_v$.

## 2 Known Results

The following upper bound on the neighborhood total domination number of a connected graph in terms of its order is established in [8].

**Theorem 2** ([8]) If $G$ is a connected graph of order $n \geq 3$, then $\gamma_{nt}(G) \leq (n + 1)/2$.

In this paper we consider the following problem posed by Arumugam and Sivagnanam [1] to characterize the connected graphs of largest possible neighborhood total domination number.

**Problem 1** ([1]) Characterize the connected graphs $G$ of order $n$ for which $\gamma_{nt}(G) = \lceil n/2 \rceil$.

A partial solution to this problem was presented by Henning and Rad [8] who provided the following characterization in the case when $n$ is odd.
Theorem 3 (§) Let $G \neq C_5$ be a connected graph of order $n \geq 3$. If $\gamma_{nt}(G) = (n + 1)/2$, then $G \in \mathcal{F}$.

As first observed in §, a characterization in the case when $n$ is even and the minimum degree is at least 2 follows readily from a result on the restrained domination number of a graph due to Domke, Hattingh, Henning and Markus §. Let $B_1, B_2, \ldots, B_5$ be the five graphs shown in Figure 2.

![Figure 2: The five graphs $B_1, B_2, \ldots, B_5$.]

Theorem 4 (§) Let $G \neq C_5$ be a connected graph of order $n \geq 4$ with $\delta(G) \geq 2$. If $\gamma_{nt}(G) = n/2$, then $G \in \{B_1, B_2, B_3, B_4, B_5\}$.

3 The Family $\mathcal{T}$ of Trees

In this section we define a family of trees $\mathcal{T}$ as follows. Let $T_0$ be an arbitrary tree. Let $T_1$ be the tree obtained from $T_0$ by the following operation: for each vertex $x \in V(T_0)$, either add a new vertex and an edge joining it to $x$ or add a new path $P_3$ and an edge joining its central vertex to $x$. Let $\mathcal{T}$ be the family of all trees $T$ that can be obtained from $T_1$ by performing the following operation:

- Choose a set of leaves, $L_1$, in $T_1$, that form a 2-packing (possibly $L_1 = \emptyset$). For each vertex $v \in L_1$, add $k \geq 0$ vertex-disjoint copies of $P_2$ and join $v$ to exactly one end of each added copy of $P_2$. We refer to these $k$ added copies of $P_2$ as appended $P_2$s associated with $x$.

![Figure 3: A tree in the family $\mathcal{T}$.]

A tree in the family $\mathcal{T}$ is illustrated in Figure 3. For ease of reference, we introduce some terminology for a tree $T \in \mathcal{T}$. We use the standard notation $[k] = \{1, 2, \ldots, k\}$. First note that given a tree $T \in \mathcal{T}$, a tree $T_0$ used to construct the tree $T$ may not be unique. That
is, in some cases we may be able to choose two distinct trees \( T_0 \) and \( T'_0 \) such that \( T \) is obtained from either \( T_0 \) or \( T'_0 \) by performing different combinations of the above operations. Therefore, we refer to \( T_0 \) as an underlying tree of \( T \), and we refer to \( T_1 \) as the corresponding base tree of \( T_0 \).

The vertex set \( V(T_1) \) of \( T_1 \) can be partitioned into sets \( V_1, \ldots, V_\ell \) such that each \( V_i \) contains exactly one vertex of \( T_0 \) and \( T[V_i] \in \{P_2, K_{1,3}\} \) for each \( i \in [\ell] \). We say that \( T[V_i] \) is a \( P_2 \)-unit of \( T_1 \) if \( T[V_i] = P_2 \) and \( T[V_i] \) is a star-unit otherwise.

If \( x \in L_1 \) belongs to a \( P_2 \)-unit of \( T_1 \), then an appended \( P_2 \) associated with \( x \) we call a Type-1 appended \( P_2 \) while if \( x \in L_1 \) belongs to a star-unit of \( T_1 \), then an appended \( P_2 \) associated with \( x \) we call a Type-2 appended \( P_2 \).

For each vertex \( v \) that is the central vertex of a star-unit of \( T_1 \), we denote the two leaf neighbors of \( v \) in \( T_1 \) that do not belong to the underlying tree \( T_0 \) by \( a_v \) and \( b_v \). If \( a_v \in L_1 \), then \( a_v \) has appended \( P_2 \)'s in \( T \) and, by construction, \( b_v \) remains a leaf in \( T \) (since \( L_1 \) is a 2-packing). In this case, we say that \( v \) and \( b_v \) are blocked vertices. Similarly, if \( b_v \) has appended \( P_2 \)'s in \( T \), then \( v \) and \( a_v \) are blocked vertices. If \( a_v \) and \( b_v \) are both leaves of \( T \), then only \( v \) is a blocked vertex. We shall adopt the convention that if one of \( a_v \) or \( b_v \) is a blocked vertex of \( T_1 \), then renaming vertices if necessary, \( b_v \) is the blocked vertex.

## 4 Main Result

By Theorem 2 every connected graph \( G \) of order \( n \geq 3 \) satisfies \( \gamma_{nt}(G) \leq (n + 1)/2 \). If \( T \) is a tree of order \( n \geq 3 \) and \( \gamma_{nt}(T) = (n + 1)/2 \), then by Theorem 3 \( T \) is a subdivided star. Our aim in this paper is to characterize the trees \( T \) of order \( n \geq 4 \) satisfying \( \gamma_{nt}(T) = n/2 \). We shall prove the following result.

**Theorem 5** Let \( T \) be a tree of order \( n \geq 4 \). If \( \gamma_{nt}(T) = n/2 \), then \( T \in \mathcal{T} \).

**Proof.** We proceed by induction on the order \( n \geq 4 \) of a tree \( T \) satisfying \( \gamma_{nt}(T) = n/2 \). If \( n = 4 \), then either \( T = P_4 \) or \( T = K_{1,3} \). In both cases, \( \gamma_{nt}(T) = 2 = n/2 \). If \( T = P_4 \) (respectively, \( T = K_{1,3} \)), then \( T \in \mathcal{T} \) with \( P_2 \) (respectively, \( K_1 \)) as the unique underlying tree and the tree \( T \) itself as the corresponding base tree. This establishes the base case. Let \( n \geq 6 \) and assume that if \( T' \) is a tree of order \( n' \) where \( 4 \leq n' < n \) satisfying \( \gamma_{nt}(T) = n'/2 \), then \( T' \in \mathcal{T} \). Let \( T \) be a tree of order \( n \) satisfying \( \gamma_{nt}(T) = n/2 \). Our aim is to show that \( T \in \mathcal{T} \). For this purpose, we introduce some additional notation.

For a subtree \( T' \) of the tree \( T \) that belongs to the family \( \mathcal{T} \), we adopt the following notation in our proof. Let \( T'_0 \) be an underlying tree of \( T' \) with corresponding base tree \( T'_1 \). For each vertex \( x \in V(T'_1) \), we let \( N_x = N_{T'}(x) \setminus V(T'_1) \), and so \( N_x \) consists of all neighbors of \( x \) in \( T' \) that do not belong to the base tree \( T'_1 \). Further, we let \( L_x \) consist of all leaves of \( T' \) at distance 2 from \( x \) that do not belong to the base tree \( T'_1 \). Necessarily, a vertex in \( N_x \) is a support vertex of \( T' \) that belongs to a \( P_2 \) appended to \( x \), while a vertex in \( L_x \) is a leaf of \( T' \) that belongs to a \( P_2 \) appended to \( x \).
Let $A$ be the vertex set of the underlying tree $T'_0$; that is, $A = V(T'_0)$. Let $B$ be the set of vertices in the base tree $T'_1$ that do not belong to the underlying tree $T'_0$; that is, $B = V(T'_1) \setminus V(T'_0)$. Further, let $B_1$ be the set of all central vertices of star-units of $T'_1$. Let $C = V(T'_2) \setminus V(T'_1)$ be the set of vertices of $T'$ that belong to a Type-1 or Type-2 appended $P_2$. We note that $(A, B, C)$ is a partition of the vertex set $V(T')$, where possibly $C = \emptyset$. Let $C_1$ (respectively, $C_2$) be the set of all leaves (respectively, support vertices) of $T'$ that do not belong to the base tree $T'_1$. We note that $(C_1, C_2)$ is a partition of the set $C$. Let

$$D' = A \cup B_1 \cup C_1.$$ 

Then the set $D'$ is a NTD-set of $T'$ (recall that $n' \geq 4$). Since $|D'| = n'/2 = \gamma_{nt}(T')$, the set $D'$ is therefore a $\gamma_{nt}(T')$-set.

For each vertex $x \in A$, we let $A_x$ be the set of neighbors of $x$ in $A$ that have degree 2 in $T'$ and belong to a $P_2$-unit in $T'_1$. Let $B_x$ be the set of all vertices in $B$ that are neighbors of vertices in $A_x$ and let $C_x$ be the set of all vertices of $C_2$ that are neighbors of vertices in $B_x$. Further, let $D_x$ be the set of all vertices of $C_1$ that are neighbors of vertices in $C_x$. We note that each vertex in $C_x$ is a support vertex of $T'$ that belongs to a Type-1 appended $P_2$, while each vertex in $D_x$ is a leaf of $T'$ that belongs to a Type-1 appended $P_2$. We note that $|A_x| = |B_x|$ and $|C_x| = |D_x|$, although possibly $A_x = \emptyset$ (in which case $B_x = \emptyset$). Further we let $A^1_x$ be the set of vertices in $A_x$ that are support vertices in $T'$ and we let $B^1_x$ be the set of leaf-neighbors of vertices in $A_x$. Possibly, $A^1_x = \emptyset$. We note that $|A^1_x| = |B^1_x|$. If $A^1_x \neq \emptyset$ and $A_x = A^1_x$, then $B_x$ is the set $B^1_x$ of leaves of $T'$ (and in this case $C_x = D_x = \emptyset$).

We now return to our proof of Theorem 5. If $T$ is a star, then $\gamma_{nt}(T) = 2 < n/2$, a contradiction. If $T$ is a double star, then the two vertices that are not leaves form a NTD-set, implying that $\gamma_{nt}(T) = 2 < n/2$, a contradiction. Therefore, $\text{diam}(T) \geq 4$. Let $P$ be a longest path in $T$ and suppose that $P$ is an $(r, u)$-path. Necessarily, $r$ and $u$ are leaves in $T$. We now root the tree $T$ at the vertex $r$. Let $v$ be the parent of $u$, and let $w$ be the parent of $v$ in the rooted tree $T$. Among all such paths $P$, we may assume that $P$ is chosen so that $d_T(v)$ is minimum. Thus if $P'$ is an arbitrary longest path in $T$ and $P'$ is an $(r', u')$-path with $v'$ the neighbor of $u'$ on $P'$, then $d_T(v') \geq d_T(v)$.

We proceed further with the following claim.

**Claim A** If $d_T(v) = 2$, then $T \in T$.

**Proof of Claim A.** Suppose that $d_T(v) = 2$. Let $T' = T - \{u, v\}$ have order $n'$, and so $n' = n - 2 \geq 4$. Since $n$ is even, so too is $n'$. By Theorem 2, $\gamma_{nt}(T') \leq n'/2$. Let $D^*$ be a $\gamma_{nt}(T')$-set. If $v \in D^*$, let $D = D^* \cup \{v\}$. If $v \notin D^*$, let $D = D^* \cup \{u\}$. In both cases, the set $D$ is a NTD-set of $T$, and so

$$n/2 = \gamma_{nt}(T) \leq |D| = |D^*| + 1 \leq \frac{n'}{2} + 1 = \frac{n}{2}. $$

Hence we must have equality throughout the above inequality chain. In particular, this implies that $\gamma_{nt}(T') = n'/2$. Applying the inductive hypothesis to the tree $T'$, we have
$T' \in T$. Adopting our earlier notation, let $D' = A \cup B_1 \cup C_1$ and recall that $D'$ is a $\gamma_{\text{nt}}(T')$-set. We now consider the parent, $w$, of the vertex $v$ in the rooted tree $T$. If $w \in A$, then $T \in T$ with $T[A \cup \{v\}]$ as an underlying tree of $T$ and $T[A \cup B \cup \{u, v\}]$ as the corresponding base tree. If $w \in B$ and $w$ is not a blocked vertex, then $T \in T$ with $T'_w$ as an underlying tree of $T$ and $T'_w$ as the corresponding base tree. Therefore, we may assume that either $w \in B$ and $w$ is a blocked vertex or $w \in C$, for otherwise $T \in T$ as desired. We proceed further by considering the following three cases.

**Case 1.** $w \in B$ and $w$ is a blocked vertex. Thus, $w$ is a blocked vertex contained in a star-unit of $T'_w$. Let $x$ be the vertex of $A$ that belongs to the star-unit containing $w$ and let $y$ be the central vertex of the star-unit. If $w \in B_1$ (and so, $w = y$), then the set 

$$(D' \setminus (A_x \cup \{x\})) \cup (B_x \cup \{v\})$$

is a NTD-set of $T$ of size $|D'| + |B_1| - |A_1| = n'/2 = n/2 - 1$, implying that $\gamma_{\text{nt}}(T) < n/2$, a contradiction. Hence, $w \notin B_1$ and $w$ is therefore a leaf-neighbor of $y$ in the star-unit containing $y$. Recall that $a_y$ and $b_y$ denote the two leaf-neighbors of $y$ in the star-unit that do not belong to $A$. Since $w$ is a blocked vertex, by convention we have $w = b_y$. We note that at least one Type-2 $P_2$ is appended to $a_y$ in order for $b_y$ to be a blocked vertex. If $|A| \geq 2$, then the set 

$$(D' \setminus (A_x \cup D_x \cup \{x\})) \cup (B_x \cup C_x \cup \{u\})$$

is a NTD-set of $T$ of size $|D'| + (|B_x| - |A_x|) + (|C_x| - |D_x|) = |D'|$, a contradiction. Hence, $A$ consists only of the vertex $x$. Thus, $T \in T$ with $T[\{v, w\}]$ as an underlying tree of $T$ and $T'[u, v, w, x, y, a_y]$ as the corresponding base tree.

**Case 2.** $w \in C$ and $w$ belongs to a Type-1 appended $P_2$. Suppose firstly that $w$ is a leaf of $T'$, and so $w \in C_1$. Let $x$ be the neighbor of $w$ that belongs to $C_2$, let $y$ be the neighbor of $x$ that belongs to $B$ and let $z$ be the neighbor of $y$ that belongs to $A$. If the vertex $z$ has a neighbor in $A$ that is not a support vertex of degree 2 in $T'$, then the set 

$$(D' \setminus (A_x \cup \{w, z\})) \cup (B_z \cup \{u, x\})$$

is a NTD-set of $T$ of size $|D'| + |B_z| - |A_z| = |D'|$, a contradiction. Hence, $A = \{z\}$ or every neighbor of $z$ that belongs to $A$ is a support vertex of degree 2 in $T'$. In this case, $A = A_z \cup \{z\}$, $C_2 = N_y$ and $C_1 = L_y$. Thus, $T \in T$ with $T[N_y \cup \{y\}]$ as an underlying tree of $T$ and $T'[C \cup \{y, z\}]$ as the corresponding base tree.

Suppose secondly that $w$ is a support vertex of $T'$, and so $w \in C_2$. Let $x$ be the leaf-neighbor of $w$ in $T'$. Let $y$ be the neighbor of $w$ that belongs to $B$ and let $z$ be the neighbor of $y$ that belongs to $A$. If the vertex $z$ has a neighbor in $A$ that is not a support vertex of degree 2 in $T'$, then the set 

$$(D' \setminus (A_z \cup \{x, z\})) \cup (B_z \cup \{v, w\})$$

is a NTD-set of $T$ of size $|D'| + |B_z| - |A_z| = |D'|$, a contradiction. Hence, $A = \{z\}$ or every neighbor of $z$ that belongs to $A$ is a support vertex of degree 2 in $T'$. In this case,
\[ A = A_1 \cup \{ z \}, \quad C_2 = N_y \quad \text{and} \quad C_1 = L_y. \] Thus, \( T \in \mathcal{T} \) with \( T[N_y \cup \{ v, y \}] \) as an underlying tree of \( T \) and \( T[C \cup \{ u, v, y, z \}] \) as the corresponding base tree.

**Case 3.** \( w \in C \) and \( w \) belongs to a Type-2 appended \( P_2 \). Suppose firstly that \( w \) is a leaf of \( T' \), and so \( w \in C_1 \). Let \( x \) be the neighbor of \( w \) that belongs to \( C_2 \). Let \( a_y \) be the neighbor of \( x \) that belongs to \( B \) and let \( y \) be the central vertex of the star-unit that contains \( a_y \). We note that \( b_y \) is a leaf in \( T' \). Let \( z \) be the neighbor of \( y \) that belongs to \( A \). If the vertex \( z \) has a neighbor in \( A \) that is not a support vertex of degree 2 in \( T' \), then the set

\[
(D' \setminus (A_z^1 \cup \{ w, y, z \})) \cup (B_z^1 \cup \{ u, x, b_y \})
\]

is a NTD-set of \( T \) of size \( |D'| \), a contradiction. Hence, \( A = \{ z \} \) or every neighbor of \( z \) that belongs to \( A \) is a support vertex of degree 2 in \( T' \). In this case, \( A = A_1^1 \cup \{ z \} \) and \( C_2 = N_{a_y} \). Thus, \( T \in \mathcal{T} \) with \( T[C_2 \cup \{ a_y \}] \) as an underlying tree of \( T \) and \( T[C \cup \{ y, a_y, b_y, z \}] \) as the corresponding base tree.

Suppose secondly that \( w \) is a support vertex of \( T' \), and so \( w \in C_2 \). Let \( x \) be the leaf-neighbor of \( w \) in \( T' \). Let \( a_y \) be the neighbor of \( w \) that belongs to \( B \) and let \( y \) be the central vertex of the star-unit that contains \( a_y \). Let \( z \) be the neighbor of \( y \) that belongs to \( A \). If the vertex \( z \) has a neighbor in \( A \) that is not a support vertex of degree 2 in \( T' \), then the set

\[
(D' \setminus (A_z^1 \cup \{ x, y, z \})) \cup (B_z^1 \cup \{ v, w, b_y \})
\]

is a NTD-set of \( T \) of size \( |D'| \), a contradiction. Hence, \( A = \{ z \} \) or every neighbor of \( z \) that belongs to \( A \) is a support vertex of degree 2 in \( T' \). In this case, \( A = A_1^1 \cup \{ z \} \) and \( C_2 = N_{a_y} \). Thus, \( T \in \mathcal{T} \) with \( T[C_2 \cup \{ v, a_y \}] \) as an underlying tree of \( T \) and \( T[C \cup \{ u, v, y, a_y, b_y, z \}] \) as the corresponding base tree. In all three cases above, we have that \( T \in \mathcal{T} \). This completes the proof of Claim A. \( \Box \)

By Claim A, we may assume that \( d_T(v) \geq 3 \), for otherwise \( T \in \mathcal{T} \) as desired. By our choice of the path \( P \), every child of \( w \) that is not a leaf has degree at least as large as \( d_T(v) \). Let \( x \) be the parent of \( w \) in \( T \). Since \( \text{diam}(T) \geq 4 \), we note that \( x \neq r \), and so \( d_T(x) \geq 2 \). Let \( w \) have \( \ell \geq 0 \) leaf neighbors and \( k \geq 1 \) children that are support vertices. Let \( W \) be the set consisting of the vertex \( w \) and its \( k \) children that are support vertices. Then, \( |W| = k + 1 \) and, as observed earlier, every vertex in \( W \setminus \{ w \} \) has degree at least \( d_T(v) \geq 3 \). Let the subtree, \( T_w \), of \( T \) rooted at \( w \) have order \( n_w \), and so \( n_w \geq 3k + \ell + 1 \). Let \( T' = T - V(T_w) \) be the tree obtained from \( T \) by deleting the vertices in the subtree \( T_w \) of \( T \) rooted at \( w \). Let \( T' \) have order \( n' \). Then, \( n' \geq 2 \) and \( n' = n - n_w \).

**Claim B** If \( n' = 2 \), then \( T \in \mathcal{T} \).

**Proof of Claim B.** Suppose that \( n' = 2 \). Then, \( n \geq 3k + \ell + 3 \) and the set \( W \cup \{ x \} \) is a NTD-set of \( T' \), and so \( n/2 = \gamma_{\text{int}}(T) \leq |W| + 1 = k + 2 \leq k + (k + \ell + 3)/2 \leq n/2 \). Hence we must have equality throughout this inequality chain, implying that \( k = 1, \ell = 0, n_w = 4 \) and \( n = 6 \). Thus, \( T \in \mathcal{T} \) with \( T[\{ w, x \}] \) as the underlying tree of \( T \) and \( T \) itself as the corresponding base tree. \( \Box \)
By Claim B, we may assume that \( n' \geq 3 \), for otherwise \( T \in \mathcal{T} \) as desired. Applying Theorem 2 and Theorem 3 to the tree \( T' \), we have that \( \gamma_{int}(T') \leq (n' + 1)/2 \), with equality if and only if \( T' \) is a subdivided star.

**Claim C** \( T' \in \mathcal{T}, d_T(w) = 2, \) and \( d_T(v) = 3 \).

**Proof of Claim C.** We show firstly that \( \gamma_{int}(T') \leq n'/2 \). Suppose, to the contrary, that \( \gamma_{int}(T') = (n' + 1)/2 \) and \( T' \) is a subdivided star. Let \( y \) be the link vertex of \( T' \), and let \( Y_1 \) and \( Y_2 \) be the set of vertices at distance 1 and 2, respectively, from \( y \) in \( T' \). Select an arbitrary vertex \( y_2 \in Y_2 \) and let \( y_1 \) be the common neighbor of \( y \) and \( y_2 \), and so \( yy_1y_2 \) is a path in \( T' \). Renaming vertices if necessary, we may assume that \( x \in \{y, y_1, y_2\} \). If \( x = y \), let \( Y = Y_2 \). If \( x = y_1 \), let \( Y = (Y_2 \setminus \{y_2\}) \cup \{x\} \). If \( x = y_2 \), let \( Y = (Y_1 \setminus \{y_1\}) \cup \{y\} \). In all three cases, \( |Y| = (n' - 1)/2 \) and the set \( W \cup Y \) is a NTD-set of \( T \). Recall that \( n_w \geq 3k + \ell + 1 \) and \( n' = n - n_w \). Hence,

\[
\gamma_{int}(T) \leq |Y| + |W| = \frac{n' - 1}{2} + k + 1 \\
\leq \frac{n - 3k - \ell - 2}{2} + k + 1 \\
= \frac{n - k - \ell}{2} \\
\leq \frac{n - 1}{2},
\]

a contradiction. Therefore, \( \gamma_{int}(T') \leq n'/2 \). Every \( \gamma_{int}(T') \)-set can be extended to a NTD-set of \( T \) by adding to it the set \( W \). Hence,

\[
\frac{n}{2} = \gamma_{int}(T) \leq \gamma_{int}(T') + |W| \leq \frac{n'}{2} + k + 1 \leq \frac{n - k - \ell + 1}{2} \leq \frac{n}{2}.
\]

Consequently, we must have equality throughout this inequality chain, implying that \( k = 1 \), \( \ell = 0 \), \( d_T(w) = 2 \), \( d_T(v) = 3 \), and \( \gamma_{int}(T') = n'/2 \). Applying the inductive hypothesis to the tree \( T' \) of (even) order \( n' \geq 4 \), we deduce that \( T' \in \mathcal{T} \).

By Claim C, \( d_T(w) = 2 \) and \( d_T(v) = 3 \). Thus, \( N(w) = \{v, x\} \). Let \( u_1 \) and \( u_2 \) be the two children of \( v \) where \( u = u_1 \). By Claim C, \( T' \in \mathcal{T} \). Adopting our earlier notation, let \( T' \) have order \( n' \). In this case, \( n' = n - 4 \). Further, let \( D' = A \cup B_1 \cup C_1 \) and recall that \( D' \) is a \( \gamma_{int}(T') \)-set. We now consider the parent, \( x \), of the vertex \( w \) in the rooted tree \( T \). If \( x \in A \), then \( T \in \mathcal{T} \) with \( T[A \cup \{w\}] \) as an underlying tree of \( T \) and \( T[A \cup B \cup N[v]] \) as the corresponding base tree. Hence we may assume that \( x \in B \cup C \), for otherwise \( T \in \mathcal{T} \) as desired.

**Claim D** If \( x \in B \), then \( T \in \mathcal{T} \).

**Proof of Claim D.** Suppose that \( x \in B \). We consider two subclaims.
Claim D.1 If $x$ belongs to a $P_2$-unit in $T'_1$, then $T \in \mathcal{T}$.

Proof of Claim D.1 Suppose that $x$ belongs to a $P_2$-unit in $T'_1$. Let $y$ be the neighbor of $x$ that belongs to $A$. If the vertex $y$ has a neighbor in $A$ that is not a support vertex of degree 2 in $T'$, then the set

$$(D' \setminus (A_y \cup \{y\})) \cup (B_y \cup \{v, w\})$$

is a NTD-set of $T$ of size $|D'| + 1 + (|B_y| - |A_y|) = |D'| + 1 = n'/2 + 1 = n/2 - 1$, a contradiction. Hence, $A = \{y\}$ or every neighbor of $y$ that belongs to $A$ is a support vertex of degree 2 in $T'$. In this case, $A = A'_y \cup \{y\}$, $C_2 = N_y$ and $C_1 = L_x$. Thus, $T \in \mathcal{T}$ with $T[N_y \cup \{w, x\}]$ as an underlying tree of $T$ and $T[C \cup \{u_1, u_2, v, w, x, y\}]$ as the corresponding base tree. (2)

Claim D.2 If $x$ belongs to a star-unit in $T'_1$, then $T \in \mathcal{T}$.

Proof of Claim D.2. Suppose that $x$ belongs to a star-unit in $T'_1$. Let $z$ be the vertex of $A$ that belongs to the star-unit containing $x$ and let $y$ be the central vertex of the star-unit. If $x \in B_1$ (and so, $x = y$), then the set

$$(D' \setminus (A_z \cup \{z\})) \cup (B_z \cup \{v, w\})$$

is a NTD-set of $T$ of size $|D'| + 1 = n/2 - 1$, a contradiction. Hence, $x \notin B_1$ and $x$ is therefore a leaf-neighbor in its star-unit. Recall that $a_y$ and $b_y$ denote the two leaf-neighbors of $y$ in the star-unit that do not belong to $A$. By convention the vertex $b_y$ is a leaf in $T'$ and the vertex $a_y$ has $\ell \geq 0$ Type-2 $P_2$'s appended to it.

Suppose $x = a_y$. If the vertex $z$ has a neighbor in $A$ that is not a support vertex of degree 2 in $T'$, then the set

$$(D' \setminus (A_z \cup \{y, z\})) \cup (B_z \cup \{b_y, v, w\})$$

is a NTD-set of $T$ of size $|D'| + 1$, a contradiction. Hence, $A = \{z\}$ or every neighbor of $z$ that belongs to $A$ is a support vertex of degree 2 in $T'$. In this case $A = A'_2 \cup \{z\}$, $C_2 = N_{a_y}$ and $C_1 = L_{a_y}$. Thus, $T \in \mathcal{T}$ with $T[C \cup \{w, a_y\}]$ as an underlying tree of $T$ and $T[C \cup \{a_y, b_y, u_1, u_2, v, w, y, z\}]$ as the corresponding base tree.

Suppose that $x = b_y$. If $a_y$ is a leaf of $T'$, then renaming $a_y$ and $b_y$, we may assume that $x = a_y$. In this case, we have shown that $T \in \mathcal{T}$. Hence we may assume that $a_y$ has at least one Type-2 $P_2$ appended to it. If $|A| \geq 2$, then the set

$$(D' \setminus (A_z \cup D_z \cup \{z\})) \cup (B_z \cup C_z \cup \{v, w\})$$

is a NTD-set of $T$ of size $|D'| + 1$, a contradiction. Hence, $A = \{z\}$. Thus, $T \in \mathcal{T}$ with $T[\{b_y, w\}]$ as an underlying tree of $T$ and $T[\{a_y, b_y, u_1, u_2, v, w, y, z\}]$ as the corresponding base tree. This completes the proof of Claim D.2. (3)

Claim D now follows from Claim D.1 and Claim D.2. □

By Claim D, we may assume that $x \in C$, for otherwise $T \in \mathcal{T}$ as desired.
Claim E If $x$ belongs to a Type-1 appended $P_2$, then $T \in \mathcal{T}$.

Proof of Claim E. Suppose that $x$ belongs to a Type-1 appended $P_2$. Suppose firstly that $x$ is a leaf of $T'$, and so $x \in C_1$. Let $y_2$ be the neighbor of $x$ that belongs to $C_2$, let $y$ be the neighbor of $y_1$ that belongs to $B$ and let $z$ be the neighbor of $y$ that belongs to $A$. If $y$ has degree at least 3 in $T'$, then $|N_y| \geq 2$ and the set
\[(D' \setminus (A_z^1 \cup L_y \cup \{z\})) \cup \{v, w, y\}\]
is a NTD-set of $T$ of size $|D'|+|B_z^1|-|A_z^1|+(|N_y|-|L_y|)+3-2 = |D'|+1$, a contradiction. Hence, $y$ has degree 2 in $T'$. If $|A| \geq 2$, then the set
\[(D' \setminus (A_z \cup D_z \cup \{z\})) \cup \{v, w, y\}\]
is a NTD-set of $T$ of size $|D'|+|B_z|-|A_z|+(|C_z|-|D_z|)+3-2 = |D'|+1$, a contradiction. Hence, $A$ consists only of the vertex $z$. Therefore, $T$ is obtained from the path $u_1vwx_1y_1yz$ by adding a new vertex $u_2$ and the edge $u_2y$. Thus, $T \in \mathcal{T}$ with $T'[\{w, x\}]$ as an underlying tree of $T$ and $T[\{u_1, u_2, v, w, x, y_1\}]$ as the corresponding base tree.

Suppose secondly that $x$ is a support vertex of $T'$, and so $x \in C_2$. Let $x_1$ be the leaf-neighbor of $x$ in $T'$. Let $y$ be the neighbor of $x$ that belongs to $B$ and let $z$ be the neighbor of $y$ that belongs to $A$. If the vertex $z$ has a neighbor in $A$ that is not a support vertex of degree 2 in $T'$, then the set
\[(D' \setminus (A_z^1 \cup \{x_1, z\})) \cup \{v, w, x\}\]
is a NTD-set of $T$ of size $|D'|+1$, a contradiction. Hence, $A = \{z\}$ or every neighbor of $z$ that belongs to $A$ is a support vertex of degree 2 in $T'$. In this case, $A = A_z^1 \cup \{z\}$, $C_2 = N_y$ and $C_1 = L_y$. Thus, $T \in \mathcal{T}$ with $T[N_y \cup \{w, y\}]$ as an underlying tree of $T$ and $T'[\{u_1, u_2, v, w, y, z\}]$ as the corresponding base tree. □

By Claim E, we may assume that $x$ belongs to a Type-2 appended $P_2$, for otherwise $T \in \mathcal{T}$ as desired.

Claim F The vertex $x$ is a support vertex of $T'$.

Proof of Claim F. Suppose to the contrary that $x$ is a leaf of $T'$, and so $x \in C_1$. Let $x_1$ be the neighbor of $x$ that belongs to $C_2$. Let $a_y$ be the neighbor of $x_1$ that belongs to $B$ and let $y$ be the central vertex of the star-unit that contains $a_y$. We note that $b_y$ is a leaf in $T'$. Let $z$ be the neighbor of $y$ that belongs to $A$. Then the set
\[(D' \setminus (A_z^1 \cup \{x, z\})) \cup \{v, w, a_y\}\]
is a NTD-set of $T$ of size $|D'|+1$, a contradiction. □

We now return to the proof of Theorem [5] one last time. By Claim F, the vertex $x$ is a support vertex of $T'$, and so $x \in C_2$. Let $x_1$ be the leaf-neighbor of $x$ in $T'$. Let $a_y$ be
the neighbor of \(x\) that belongs to \(B\) and let \(y\) be the central vertex of the star-unit that contains \(a_y\). Let \(z\) be the neighbor of \(y\) that belongs to \(A\). If the vertex \(z\) has a neighbor in \(A\) that is not a support vertex of degree 2 in \(T'\), then the set

\[
(D' \setminus (A^1_z \cup \{x, y, z\})) \cup (B^1_z \cup \{v, w, b, y\})
\]

is a NTD-set of \(T\) of size \(|D'| + 1\), a contradiction. Hence, \(A = \{z\}\) or every neighbor of \(z\) that belongs to \(A\) is a support vertex of degree 2 in \(T'\). In this case, \(A = A^1_z \cup \{z\}\) and \(C_2 = N_{a_y}\). Thus, \(T \in \mathcal{T}\) with \(T[C_2 \cup \{w, a_y\}]\) as an underlying tree of \(T\) and \(T[C \cup \{u_1, u_2, v, w, y, a_y, b, z\}]\) as the corresponding base tree. This completes the proof of Theorem 5.

\[\square\]

5 Closing Remark

As a consequence of our tree characterization provided in Theorem 5, we remark that a complete solution to the Arumugam-Sivagnanam Problem 1 can be obtained as follows. Let \(G\) be a connected graph of (even) order \(n \geq 4\) satisfying \(\gamma_{nt}(G) = n/2\) and consider an arbitrary spanning tree \(T\) of \(G\). By Theorem 2, \(\gamma_{nt}(T) \leq n/2\). Every NTD-set of \(G\) implies that \(n/2 = \gamma_{nt}(G) \leq \gamma_{nt}(T) \leq n/2\). Consequently, \(\gamma_{nt}(T) = n/2\). Thus, by Theorem 5, \(T \in \mathcal{T}\). This is true for every spanning tree \(T\) of the graph \(G\). An exhaustive case analysis of the allowable edges that can be added to trees in the family \(\mathcal{T}\) without lowering their neighborhood total domination number produces the connected graphs \(G\) of order \(n \geq 4\) satisfying \(\gamma_{nt}(G) = n/2\). Since our detailed case analysis of the resulting such graphs exceeds the length of the current paper, we omit the details here which can be found in 9.

References


[9] M. A. Henning and K. Wash, Graphs with large neighborhood total domination number, manuscript.