Stratification and domination in graphs with minimum degree two

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Abstract

A graph $G$ is 2-stratified if its vertex set is partitioned into two classes (each of which is a stratum or a color class). We color the vertices in one color class red and the other color class blue. Let $F$ be a 2-stratified graph with one fixed blue vertex $v$ specified. We say that $F$ is rooted at $v$. The $F$-domination number of a graph $G$ is the minimum number of red vertices of $G$ in a red-blue coloring of the vertices of $G$ such that every blue vertex $v$ of $G$ belongs to a copy of $F$ rooted at $v$. We investigate the $F$-domination number when $F$ is a 2-stratified path $P_3$ on three vertices rooted at a blue vertex which is an end-vertex of the $P_3$ and is adjacent to a blue vertex with the remaining vertex colored red. We show that for a connected graph of order $n$ with minimum degree at least two this parameter is bounded above by $(n-1)/2$ with the exception of five graphs (one each of orders four, five and six and two of order eight). For $n \geq 9$, we characterize those graphs that achieve the upper bound of $(n-1)/2$.

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1. Introduction

In this paper we continue the study of stratification and domination in graphs started by Chartrand et al. [3] and studied further in [11]. A graph $G$ whose vertex set has been
partitioned into two sets \( V_1 \) and \( V_2 \) is called a \textit{2-stratified graph}. The sets \( V_1 \) and \( V_2 \) are called the \textit{strata} or sometimes the \textit{color classes} of \( G \). We ordinarily color the vertices of \( V_1 \) red and the vertices of \( V_2 \) blue.

In [13], Rashidi studied a number of problems involving stratified graphs; while distance in stratified graphs was investigated in [1,2,4].

A set \( S \subseteq V(G) \) of a graph \( G \) is a \textit{dominating set} if every vertex not in \( S \) is adjacent to a vertex in \( S \). The domination number of \( G \), denoted by \( \gamma(G) \), is the minimum cardinality of a dominating set. A dominating set of \( G \) of cardinality \( \gamma(G) \) is called a \( \gamma \)-set of \( G \). The concept of domination in graphs, with its many variations, is now well studied in graph theory. The book by Chartrand and Lesniak [5] includes a chapter on domination. For a more thorough study of domination in graphs, see Haynes et al. [8,9].

In [3] a new mathematical framework for studying domination is presented. It is shown that the domination number and many domination related parameters can be interpreted as restricted 2-stratifications or 2-colorings, with the red vertices forming the dominating set. This framework places the domination number in a new perspective and suggests many other parameters of a graph which are related in some way to the domination number.

More precisely, let \( F \) be a 2-stratified graph with one fixed blue vertex \( v \) specified. We say that \( F \) is \textit{rooted} at the blue vertex \( v \). An \( F \)-\textit{coloring} of a graph \( G \) is defined in [3] to be a red–blue coloring of the vertices of \( G \) such that every blue vertex \( v \) of \( G \) belongs to a copy of \( F \) (not necessarily induced in \( G \)) rooted at \( v \). The \textit{\( F \)-domination number} \( \gamma_F(G) \) of \( G \) is the minimum number of red vertices of \( G \) in an \( F \)-coloring of \( G \). In [3], an \( F \)-coloring of \( G \) that colors \( \gamma_F(G) \) vertices red is called a \( \gamma_F \)-\textit{coloring} of \( G \). The set of red vertices in a \( \gamma_F \)-coloring is called a \( \gamma_F \)-set. If \( G \) has order \( n \) and \( G \) has no copy of \( F \), then certainly \( \gamma_F(G) = n \).

For notation and graph theory terminology we, in general, follow [8]. Specifically, let \( G = (V, E) \) be a graph with vertex set \( V \) of order \( n \) and edge set \( E \). For a set \( S \subseteq V \), the subgraph induced by \( S \) is denoted by \( G[S] \). The minimum degree (resp., maximum degree) among the vertices of \( G \) is denoted by \( \delta(G) \) (resp., \( \Delta(G) \)).

A \textit{star} is the tree \( K_{1,n-1} \) of order \( n \geq 2 \). A \textit{subdivided star} is a star where each edge is subdivided exactly once. A \textit{cycle} on \( n \) vertices is denoted by \( C_n \) and a \textit{path} on \( n \) vertices by \( P_n \). A \textit{daisy} with \( k \geq 2 \) \textit{petals} is a connected graph that can be constructed from \( k \geq 2 \) disjoint cycles by identifying a set of \( k \) vertices, one from each cycle, into one vertex. In particular, if the \( k \) cycles have lengths \( n_1, n_2, \ldots, n_k \), we denote the daisy by \( D(n_1, n_2, \ldots, n_k) \). Further, if \( n_1 = n_2 = \cdots = n_k \), then we write \( D(n_1, n_2, \ldots, n_k) \) simply as \( D_k(n_1) \). The daisies \( D(3, 5) \), \( D(4, 4) \) and \( D_3(5) = D(5, 5, 5) \) are shown in Fig. 1.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{daisies.png}
\caption{The daisies \( D(3, 5) \), \( D(4, 4) \) and \( D_3(5) \).}
\end{figure}
For integers \( n_1 \geq n_2 \geq 3 \) and \( k \geq 0 \), we define a dumb-bell \( D_b(n_1, n_2, k) \) to be the graph of order \( n = n_1 + n_2 + k \) obtained from the cycles \( C_{n_1} \) and \( C_{n_2} \) by joining a vertex of \( C_{n_1} \) to a vertex of \( C_{n_2} \) and subdividing the resulting edge \( k \) times. The dumb-bells \( D_b(5, 4, 0) \) and \( D_b(5, 5, 1) \) are shown in Fig. 2.

2. Known results

If \( F \) is a \( K_2 \) rooted at a blue vertex \( v \) that is adjacent to a red vertex, then it is shown in [3] that \( \gamma_F(G) = \gamma(G) \). Thus domination can be interpreted as restricted 2-stratifications or 2-colorings, with the red vertices forming the dominating set. Clearly, this \( F \)-coloring is the only well-defined one for connected graphs \( F \) with order 2.

Let \( F \) be a 2-stratified \( P_3 \) rooted at a blue vertex \( v \). The five possible choices for the graph \( F \) are shown in Fig. 3. (The red vertices in Fig. 3 are darkened.)

The following result is established in [3].

**Theorem 1** (Chartrand and Haynes [3]). If \( G \) is a connected graph of order at least 3, then for \( i \in \{1, 2, 4, 5\} \), the parameter \( \gamma_{F_i}(G) \) is given by the following table:

\[
\begin{array}{ccccc}
   i & 1 & 2 & 4 & 5 \\
\gamma_{F_i}(G) = & \gamma_1(G) & \gamma(G) & \gamma_4(G) & \gamma_5(G) \\
\end{array}
\]

where \( \gamma_1(G) \) denotes the total domination number (see [8]), \( \gamma_4(G) \) denotes the restrained domination number (see [6,8]), and \( \gamma_5(G) \) denotes the 2-domination number (see [7,8]).

The parameter \( \gamma_{F_3}(G) \) (see Fig. 3) appears to be new and is investigated further in [11]. If \( T \) is a star \( K_{1,n-1} \) of order \( n \geq 3 \), then \( \gamma_{F_3}(T) = n \) since the central vertex of \( T \) must be colored red in any \( F_3 \)-coloring of \( T \). If \( T \) is a tree of diameter at least 3, then it is shown in [11] that \( \gamma_{F_3}(T) \leq 2n/3 \) and the trees achieving equality are characterized.

As pointed out in [3], \( F_3 \)-domination is not the same as the distance domination parameter called \( k \)-step domination introduced in [12]. A set \( S \subseteq V \) is a \( k \)-step dominating set if for every vertex \( u \in V - S \), there exists a path of length \( k \) from \( u \) to some vertex in \( S \).
The \textit{k-step domination number} is the minimum cardinality of any \textit{k-step} dominating set of \( G \). The difference in 2-step domination and \( F_3 \)-domination is that in \( F_3 \)-domination every blue vertex must have a blue–blue–red path (of length two) to some red vertex. Thus, every \( F_3 \)-dominating set is a 2-step dominating set, but not every 2-step dominating set is an \( F_3 \)-dominating set. If \( T \) is a star \( K_{1,n-1} \) of order \( n \geq 3 \), then \( \gamma_{F_3}(T) = n \) while the 2-step domination number of \( T \) equals 2 (the set consisting of the central vertex and any leaf of \( T \) is a 2-step dominating set of \( T \)). A survey of results on distance domination in graphs can be found in [8, Section 7.4]. For a more comprehensive survey, the reader is referred to [10].

3. Main results

In this paper we continue the study of the \( F_3 \)-domination number of a graph. We have two immediate aims: firstly to establish an upper bound on the \( F_3 \)-domination number of a connected graph with minimum degree at least two in terms of the order of the graph and to characterize those graphs achieving equality in this bound. Secondly, to characterize connected graphs of sufficiently large order with maximum possible \( F_3 \)-domination number.

We will refer to a graph \( G \) as an \( F_3 \)-\textit{minimal graph} if \( G \) is edge-minimal with respect to satisfying the following three conditions: (i) \( \delta(G) \geq 2 \), (ii) \( G \) is connected, and (iii) \( \gamma_{F_3}(G) \geq (n-1)/2 \), where \( n \) is the order of \( G \). To achieve our aims, we characterize \( F_3 \)-minimal graphs. To do this, we define four families of graphs.

Let \( A_1(4) = D_b(5, 4, 0) \) and \( A_1(5) = D_b(5, 5, 1) \) be the two dumb-bells shown in Fig. 2. For \( k \geq 2 \), let \( A_k(4) \) be the graph obtained from a daisy \( D_k(5) \) by adding a 4-cycle and joining the central vertex of the daisy to a vertex of the added 4-cycle. The graph \( A_2(4) \) is shown in Fig. 4(a). For \( k \geq 2 \), let \( A_k(5) \) be the graph obtained from a daisy \( D_k(5) \) by adding a 5-cycle and then adding a new vertex and joining it to the central vertex of the daisy and to a vertex of the added 5-cycle. The graph \( A_2(5) \) is shown in Fig. 4(b). Let \( \mathcal{A} = \{ A_k(4) \mid k \geq 1 \} \cup \{ A_k(5) \mid k \geq 1 \} \).

Let \( \mathcal{B} = \{ B_1, B_2, B_3, B_4, B_5 \} \) where \( B_1, B_2, B_3, B_4 \) and \( B_5 \) are the five graphs shown in Fig. 5. We call each graph in the family \( \mathcal{B} \) a \textit{bad graph}.

Next we define a subfamily \( \mathcal{C} \) of cycles and a subfamily \( \mathcal{D} \) of daisies by

\[ \mathcal{C} = \{ C_3, C_4, C_5, C_7, C_8, C_{11} \} \]

and

\[ \mathcal{D} = \{ D_k(5) \mid k \geq 2 \} \cup \{ D(3, 5), D(4, 4) \} \].

![Fig. 4. The graphs \( A_2(4) \) and \( A_2(5) \) in the family \( \mathcal{A} \).](attachment:image.png)
The following result, a proof of which is given in Section 5, characterizes $F_3$-minimal graphs.

**Theorem 2.** A graph $G$ is an $F_3$-minimal graph if and only if $G \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$.

Let $H_1$ (resp., $H_2$) be the graph obtained from $C_8$ (resp., $C_{11}$) by adding an edge joining two vertices at distance four apart on the cycle. The graphs $H_1$ and $H_2$ are shown in Fig. 6.

As a consequence of Theorem 2 we have our first main result, a proof of which is given in Section 6.

**Theorem 3.** If $G$ is a connected graph of order $n$ with $\delta(G) \geq 2$, then $\gamma_{F_3}(G) \leq (n-1)/2$ unless $G \in \{B_2, C_4, C_8, H_1\}$, in which case $\gamma_{F_3}(G) = n/2$, or $G = C_5$, in which case $\gamma_{F_3}(G) = (n+1)/2$.

Our second main result provides a characterization of connected graphs with minimum degree at least two and order at least nine that have maximum possible $F_3$-domination number. A proof of Theorem 4 is given in Section 7.

**Theorem 4.** If $G$ is a connected graph of order $n \geq 9$ with $\delta(G) \geq 2$, then $\gamma_{F_3}(G) \leq (n-1)/2$ with equality if and only if $G \in \mathcal{A} \cup (\mathcal{D} - \{D(3, 5), D(4, 4)\})$ or $G \in \{B_4, B_5, C_{11}, H_2\}$.

### 4. Preliminary results

Our aim in this section is to establish some preliminary results that we will need later when proving our main results. We begin with the following observation, a proof of which is presented in Section 4.1.

**Observation 5.** Let $G$ be a connected graph with $\delta(G) \geq 2$ and let $F$ be obtained from $G$ by subdividing any edge four times. Then, $\gamma_{F_3}(F) \leq \gamma_{F_3}(G) + 2$.

Next we establish the value of $\gamma_{F_3}(C_n)$ for a cycle $C_n$. A proof of Proposition 6 is presented in Section 4.2.
Proposition 6. For \( n \geq 3 \), \( \gamma_{F_3}(C_n) = \lceil (n - 1)/3 \rceil + \lceil n/3 \rceil - \lfloor n/3 \rfloor \).

Equivalently, Proposition 6 states that \( \gamma_{F_3}(C_n) = \lceil n/3 \rceil + \lceil n/3 \rceil - \lfloor n/3 \rfloor \) if \( n \equiv 2 \pmod{3} \) and \( \gamma_{F_3}(C_n) = \lceil n/3 \rceil - 1 \) otherwise. For an example of a \( \gamma_{F_3} \)-coloring of an \( n \)-cycle \( C_n \): \( v_1, v_2, \ldots, v_n, v_1 \), let \( R = \{ v_1 \mid i \equiv 1 \pmod{3} \} \), and so \( |R| = \lfloor n/3 \rfloor \). If \( n \equiv 2 \pmod{3} \), then coloring the vertices of \( R \cup \{ v_n \} \) red and coloring all other vertices blue produces an \( F_3 \)-coloring of \( C_n \). If \( n \not\equiv 2 \pmod{3} \), then coloring the vertices of \( R \) red and coloring all other vertices blue produces an \( F_3 \)-coloring of \( C_n \). As an immediate consequence of Proposition 6 we can characterize the \( F_3 \)-minimal graphs that are cycles.

Corollary 7. A cycle \( G \) is an \( F_3 \)-minimal graph if and only if \( G \in \mathcal{C} \).

Next we characterize the \( F_3 \)-minimal graphs that are daisies. Proofs of Propositions 8 and 9 are presented in Sections 4.3 and 4.4, respectively.

Proposition 8. If \( G \) is a daisy of order \( n \) with two petals, then \( \gamma_{F_3}(G) \leq (n - 1)/2 \). Furthermore, \( \gamma_{F_3}(G) = (n - 1)/2 \) if and only if \( G = D(3, 5) \) or \( G = D(4, 4) \) or \( G = D(5, 5) \).

Proposition 9. If \( G \) is a daisy of order \( n \), then \( \gamma_{F_3}(G) \leq (n - 1)/2 \). Furthermore, \( \gamma_{F_3}(G) = (n - 1)/2 \) if and only if \( G = D(3, 5) \), \( G = D(4, 4) \) or \( G = D_k(5) \) for some \( k \geq 2 \).

As an immediate consequence of Proposition 9 we can characterize the \( F_3 \)-minimal graphs that are daisies.

Corollary 10. A daisy \( G \) is an \( F_3 \)-minimal graph if and only if \( G \in \mathcal{D} \).

Next we characterize the \( F_3 \)-minimal graphs that are dumb-bells. We begin with the following result, a proof of which is presented in Section 4.5.

Proposition 11. If \( G \) is a dumb-bell of order \( n \), then \( \gamma_{F_3}(G) \leq (n - 1)/2 \) with equality if and only if \( G \in \{ A_1(4), A_1(5) \} \).

Corollary 12. A dumb-bell \( G \) is an \( F_3 \)-minimal graph if and only if \( G \in \{ A_1(4), A_1(5) \} \).

The following two observations about graphs in the families \( \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \) will prove to be useful.

Observation 13. Let \( G \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \) have order \( n \). Then, \( G \) is a connected graph with \( \delta(G) = 2 \), and

\[
\gamma_{F_3}(G) = \begin{cases} 
\frac{n + 1}{2} & \text{if } G = C_5, \\
\frac{n}{2} & \text{if } G \in \{ B_2, C_4, C_8 \}, \\
\frac{n - 1}{2} & \text{otherwise}.
\end{cases}
\]

Corollary 14. Each graph in \( \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \) is an \( F_3 \)-minimal graph.
Observation 15. Let $G \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$. Then for any vertex $v$ of $G$, there is a minimum $F_3$-coloring in which $v$ is colored blue and in which every blue vertex is adjacent to a red vertex. Further for any vertex $v$ of $G$, except for the central vertex of a daisy $D_k(5)$, there is a minimum $F_3$-coloring of $G$ in which $v$ is colored red.

We close our preliminary results with a characterization of $F_3$-minimal graphs of small order. A proof of Lemma 16 is presented in Section 4.6.

Lemma 16. If $G$ is an $F_3$-minimal graph of order $n$, $3 \leq n \leq 6$, then $G \in \{B_1, B_2, C_3, C_4, C_5\}$.

4.1. Proof of Observation 5

Let $uv$ be the edge of $G$ that is subdivided four times to obtain the graph $F$, and let $u, u_1, u_2, u_3, u_4, v$ be the resulting path in $F$. Any minimum $F_3$-coloring of $G$ can be extended to an $F_3$-coloring of $F$ as follows. If both $u$ and $v$ are colored red, then color $u_1$ and $u_4$ red and $u_2$ and $u_3$ blue. If exactly one of $u$ and $v$, say $u$, is colored red, then color $u_3$ and $u_4$ red and $u_1$ and $u_2$ blue. Suppose both $u$ and $v$ are colored blue. If each of $u$ and $v$ has a neighbor colored red, then color $u_2$ and $u_3$ red and $u_1$ and $u_4$ blue. If exactly one of $u$ and $v$, say $u$, has a neighbor colored red, then color $u_2$ red, color $u_1, u_3$ and $u_4$ blue, and recolor $v$ red (note that in any $F_3$-coloring of a graph $H$ that colors a vertex $w$ and all its neighbors blue, we can recolor $w$ red and leave all other vertices unchanged to produce a new $F_3$-coloring of $H$). If neither $u$ nor $v$ has a neighbor colored red, then color $u_1$ and $u_4$ red and $u_2$ and $u_3$ blue. In all cases, we produce an $F_3$-coloring of $F$ that colors exactly two more vertices red than does the original $F_3$-coloring of $G$. It follows that $\gamma_{F_3}(F) \leq \gamma_{F_3}(G) + 2$.

4.2. Proof of Proposition 6

We proceed by induction on the order $n$ of a cycle $C_n$. The result is straightforward to verify for $n \in \{3, 4, 5\}$. Assume then that $n \geq 6$ and that the result of the proposition is true for all cycles on fewer than $n$ vertices. Consider a cycle $C_5: v_1, v_2, \ldots, v_n, v_1$. Let $\mathcal{C}$ be a $\gamma_{F_3}$-coloring of $C$. Since every blue vertex is rooted at a copy of $F_3$, every blue vertex on the cycle is adjacent to a red vertex and a blue vertex. Renaming if necessary, we may assume that $\mathcal{C}$ colors $v_2$ and $v_3$ blue, and therefore colors $v_1$ and $v_4$ red. Let $C'$ be the cycle obtained from $C$ by deleting the vertices $v_1, v_2$ and $v_3$ and adding the edge $v_4v_n$, i.e., $C' = (C - \{v_1, v_2, v_3\}) \cup \{v_4v_n\}$. Then $C'$ is a cycle of order $n - 3 \geq 3$ and the restriction of $\mathcal{C}$ to $C'$ is an $F_3$-coloring of $C'$ that colors $\gamma_{F_3}(C) - 1$ vertices red. Hence, $\gamma_{F_3}(C') \leq \gamma_{F_3}(C) - 1$. On the other hand, any $\gamma_{F_3}$-coloring of $C'$ can be extended to a $\gamma_{F_3}$-coloring of $C$ by colorig exactly one of $v_1, v_2$ and $v_3$ red: If $\mathcal{C}'$ colors $v_n$ and $v_4$ blue, then color $v_2$ red and $v_1$ and $v_3$ blue. If $\mathcal{C}'$ colors $v_n$ red and $v_4$ blue or if $\mathcal{C}'$ colors $v_n$ and $v_4$ red, then color $v_1$ and $v_2$ blue and $v_3$ red. If $\mathcal{C}'$ colors $v_n$ blue and $v_4$ red, then color $v_1$ red and $v_2$ and $v_3$ blue. Thus, $\gamma_{F_3}(C) \leq \gamma_{F_3}(C') + 1$. Consequently, $\gamma_{F_3}(C) = \gamma_{F_3}(C') + 1$. Since $C \cong C_n$ and $C' \cong C_{n-3}$, the result now follows by applying the inductive hypothesis to the cycle $C'$. 
4.3. Proof of Proposition 8

Let $G = D(n_1 + 1, n_2 + 1)$, and so $n = n_1 + n_2 + 1$. Let $v$ denote the vertex of degree 4 in $G$ and let $F_1$ and $F_2$ denote the two cycles passing through $v$, where $F_i \cong C_{n_i+1}$ for $i = 1, 2$. Let $F_1$ be the cycle $v, v_1, v_2, \ldots, v_{n_1}, v$ and let $F_2$ be the cycle $v, u_1, u_2, \ldots, u_{n_2}, v$. We consider four possibilities.

Case 1. $n_i \equiv 2 \pmod{3}$ for $i = 1$ or $i = 2$.

We may assume $n_1 \equiv 2 \pmod{3}$. Let $R_1 = \{v_i \mid i \equiv 0 \pmod{3}\}$, and so $|R_1| = (n_1 - 2)/3$. Let $\mathcal{C}_2$ be a $\gamma_{F_3}$-coloring of $F_2$ that colors $v$ red. By Proposition 6, if $n_2 \not\equiv 1 \pmod{3}$, then $\mathcal{C}_2$ colors at most $(n_2 + 3)/3$ vertices red, while if $n_2 \equiv 1 \pmod{3}$, then $\mathcal{C}_2$ colors $(n_2 + 5)/3$ vertices red. We can extend $\mathcal{C}_2$ to an $F_3$-coloring $\mathcal{C}$ of $G$ by coloring the vertices in $R_1$ red and all remaining uncolored vertices of $F_1$ blue. If $n_2 \not\equiv 1 \pmod{3}$, then $\mathcal{C}$ colors at most $(n_1 - 2)/3 + (n_2 + 3)/3 = n/3 < (n - 1)/2$ vertices red. If $n_2 \equiv 1 \pmod{3}$, then $n \geq 7$ and $\mathcal{C}$ colors $(n_1 - 2)/3 + (n_2 + 5)/3 = (n + 2)/3 \leq (n - 1)/2$ vertices red with strict inequality if $n > 7$. Hence, $\gamma_{F_3}(G) < (n - 1)/2$ unless $G = D(3, 5)$, in which case $\gamma_{F_3}(G) = 3 = (n - 1)/2$.

Case 2. $n_i \equiv 0 \pmod{3}$ for $i = 1, 2$.

Then, $n \geq 7$. Let $R_1 = \{v_i \mid i \equiv 1 \pmod{3}\}$, and so $|R_1| = n_1/3$. Let $\mathcal{C}_2$ be a $\gamma_{F_3}$-coloring of $F_2$ that colors $v$ red. By Proposition 6, $\mathcal{C}_2$ colors $(n_2 + 3)/3$ vertices red. We can extend $\mathcal{C}_2$ to an $F_3$-coloring $\mathcal{C}$ of $G$ by coloring the vertices in $R_1$ red and all remaining uncolored vertices of $F_1$ blue. Then, $\mathcal{C}$ colors $(n_1 + n_2 + 3)/3 = (n + 2)/3 \leq (n - 1)/2$ vertices red with strict inequality if $n > 7$. Hence, $\gamma_{F_3}(G) < (n - 1)/2$ unless $G = D(4, 4)$, in which case $\gamma_{F_3}(G) = 3 = (n - 1)/2$.

Case 3. $n_1 \equiv 0 \pmod{3}$ and $n_2 \equiv 1 \pmod{3}$.

Then, $n \geq 8$. Let $R_1 = \{v_i \mid i \equiv 2 \pmod{3}\}$, and so $|R_1| = n_1/3$. Let $R_2 = \{u_i \mid i \equiv 1 \pmod{3}\}$, and so $|R_2| = (n_2 + 2)/3$. Then coloring the vertices in $R_1 \cup R_2$ red and all remaining uncolored vertices blue produces an $F_3$-coloring of $G$ that colors $(n_1 + n_2 + 2)/3 = (n + 1)/3 < (n - 1)/2$ vertices red.

Case 4. $n_1 \equiv 1 \pmod{3}$ for $i = 1, 2$.

Then, $n \geq 9$. Let $R_1 = \{v_i \mid i \equiv 1 \pmod{3}\}$, and so $|R_1| = (n_1 + 2)/3$. Let $R_2 = \{u_i \mid i \equiv 2 \pmod{3}\} \cup \{u_{n_2 - 1}\}$, and so $|R_2| = (n_2 + 2)/3$. Then coloring the vertices in $R_1 \cup R_2$ red and all remaining uncolored vertices blue produces an $F_3$-coloring of $G$ that colors $(n_1 + n_2 + 4)/3 = (n + 3)/3 \leq (n - 1)/2$ vertices red with strict inequality if $n > 9$. Hence, $\gamma_{F_3}(G) < (n - 1)/2$ unless $G = D(5, 5)$, in which case $\gamma_{F_3}(G) = 4 = (n - 1)/2$.

4.4. Proof of Proposition 9

We proceed by induction on the order $n \geq 5$ of a daisy $G$ to show that $\gamma_{F_i}(G) \leq (n - 1)/2$. If $n = 5$, then $G = D(3, 3)$ and $\gamma_{F_i}(G) = 1 < (n - 1)/2$, while if $n = 6$, then $G = D(3, 4)$ and $\gamma_{F_i}(G) = 2 < (n - 1)/2$. This establishes the base cases. Assume, then, that $n \geq 7$ and that if $G'$ is a daisy of order $n' < n$, then $\gamma_{F_i}(G') \leq (n' - 1)/2$. Let $G$ be a daisy of order $n$ with $k \geq 2$ petals. If $k = 2$, then the result follows from Proposition 8. Hence we may assume $k \geq 3$. Let $v$ denote the vertex of degree $2k$ in $G$, and let $F_1, F_2, \ldots, F_k$ denote the $k$ cycles passing through $v$, where $F_i \cong C_{n_i+1}$ for $i = 1, 2, \ldots, k$. Thus, $n = 1 + \sum_{i=1}^{k} n_i$. Let $F_1$ be the cycle $v, v_1, v_2, \ldots, v_{n_1}, v$. \[ \text{\ldots} \]
Let $G' = D(n_2, \ldots, n_k)$. Then, $G'$ is a daisy of order $n' = n - n_1$. Applying the inductive hypothesis to $G'$, $\gamma_{F_3}(G') \leq (n' - 1)/2 = (n - n_1 - 1)/2$. Let $\mathcal{C}'$ be a $\gamma_{F_3}$-coloring of $G'$. Note that if $\mathcal{C}'$ colors $v$ blue, then $v$ must be adjacent to at least one vertex colored red under $\mathcal{C}'$. We extend $\mathcal{C}'$ to an $F_3$-coloring of $G$ as follows. If $n_1 \equiv 2 \pmod{3}$, let $R = \{v_i \mid i \equiv 0 \pmod{3}\}$, and so $|R| = (n_1 - 2)/3$. If $n_1 \equiv 0 \pmod{3}$ and $v$ is colored blue in $\mathcal{C}'$, let $R = \{v_i \mid i \equiv 2 \pmod{3}\}$, and so $|R| = n_1/3$. In all other cases, let $R = \{v_i \mid i \equiv 1 \pmod{3}\}$, and so $|R| = [n_1/3]$. Then, $\mathcal{C}'$ can be extended to an $F_3$-coloring $\mathcal{C}$ of $G$ by coloring the vertices in $R$ red and all remaining uncolored vertices of $F_1$ blue. If $n_1 \equiv 2 \pmod{3}$, then $\mathcal{C}'$ colors at most $|R| + (n' - 1)/2 = (n_1 - 2)/3 + (n - n_1 - 1)/2 < (n - 1)/2$ vertices red. If $n_1 \equiv 0 \pmod{3}$, then $\mathcal{C}'$ colors at most $(n_1 + 2)/3 + (n - n_1 - 1)/2 = (3n - n_1 + 1)/6 \leq (n - 1)/2$ vertices red with strict inequality if $n_1 > 4$. Hence in all cases, $\mathcal{C}$ colors strictly less than $(n - 1)/2$ vertices red unless $n_1 = 4$ (and so, $F_1 = C_5$) and $\gamma_{F_3}(G') = (n' - 1)/2$. An identical argument shows that if $n_1 \neq 4$ for some $i$, $1 \leq i \leq k$, then there is an $F_3$-coloring of $G$ that colors strictly less than $(n - 1)/2$ vertices red. Thus we have shown that $\gamma_{F_3}(G) < (n - 1)/2$ unless $G = D_k(5)$, in which case $\gamma_{F_3}(G) \leq (n - 1)/2$.

We show next that $\gamma_{F_3}(G) = (n - 1)/2$ if and only if $G = D(3, 5)$, $G = D(4, 4)$ or $G = D_k(5)$ for some $k \geq 2$. By Proposition 8, the result is proven if $G$ is a daisy with two petals. Hence we may assume $G$ has at least three petals. If $\gamma_{F_3}(G) = (n - 1)/2$, then we have shown that $G = D_k(5)$. Conversely, suppose $G = D_k(5)$. Then $G$ has order $n = 4k + 1$ and any $F_3$-coloring of $G$ colors at least two vertices from each $F_i$, $1 \leq i \leq k$, red, and so $\gamma_{F_3}(G) \geq 2k = (n - 1)/2$. On the other hand, if $v$ denotes the vertex of degree $2k$ in $G$ and if $v, v_1, v_2, v_3$ denote a $5$-cycle in $G$, then coloring all vertices in $(N[v] - \{v_1, v_4\}) \cup \{v_2, v_3\}$ blue and all remaining vertices red, produces an $F_3$-coloring of $G$ that colors exactly $2k = (n - 1)/2$ vertices red, and so $\gamma_{F_3}(G) \leq (n - 1)/2$. Consequently, $\gamma_{F_3}(G) = (n - 1)/2$.

4.5. Proof of Proposition 11

We proceed by induction on the order $n \geq 6$. If $n = 6$, then $G = D_6(3, 3, 0)$ and $\gamma_{F_3}(G) = 2 = (n - 2)/2$. Let $n \geq 7$, and assume that the result is true for all dumb-bells of order $n'$, where $n' < n$. Let $G = D_b(n_1, n_2, k)$ be a dumb-bell of order $n = n_1 + n_2 + k$. Suppose $G$ contains a path on six vertices each internal vertex of which has degree $2$ in $G$ and whose end-vertices, say $u$ and $v$, are not adjacent. Let $G'$ be the graph obtained from $G$ by removing the four internal vertices of this path and adding the edge $uv$. Then, $G'$ is a dumb-bell of order $n' = n - 4$. By Observation 5, $\gamma_{F_3}(G) \leq \gamma_{F_3}(G') + 2$. Applying the inductive hypothesis to $G'$, $\gamma_{F_3}(G') \leq (n' - 1)/2$. If $\gamma_{F_3}(G') < (n' - 1)/2$, then $\gamma_{F_3}(G) < (n - 1)/2$. On the other hand, if $\gamma_{F_3}(G') = (n' - 1)/2$, then by the inductive hypothesis, $G' \in \{A_1(4), A_1(5)\}$. Now $G$ is obtained from $G'$ by subdividing the edge $uv$ of $G'$ four times. Irrespective of whether the edge $uv$ is a cycle edge or a bridge of $G'$, it is straightforward to check that $\gamma_{F_3}(G) \leq (n - 3)/2$. Hence we may assume that $G$ contains no path on six vertices each internal vertex of which has degree $2$ in $G$ and whose end-vertices are not adjacent, for otherwise $\gamma_{F_3}(G) < (n - 1)/2$. With this assumption, $3 \leq n_1, n_2 \leq 6$ and $0 \leq k \leq 3$. It is now a simple exercise to check that $\gamma_{F_3}(G) \leq (n - 1)/2$ with equality if and only if $(n_1, n_2, k) \in \{(5, 4, 0), (5, 5, 1)\}$.
4.6. Proof of Lemma 16

Let \( G = (V, E) \). Let \( u \) be a vertex of maximum degree in \( G \). If \( n \in \{3, 4\} \), then \( G = C_n \).
Suppose \( n = 5 \). If \( \Delta(G) = 4 \), then coloring \( u \) red and coloring all other vertices blue produces
an \( F_3 \)-coloring of \( G \), and so \( \gamma_{F_3}(G) = 1 < (n - 1)/2 \), a contradiction. If \( \Delta(G) = 3 \), then it
follows from Observation 13 that \( G = B_1 \). If \( \Delta(G) = 2 \), then \( G = C_5 \).

Suppose \( n = 6 \). If \( \Delta(G) = 2 \), then \( \gamma_{F_3}(G) \leq 2 < (n - 1)/2 \), a contradiction. If \( \Delta(G) = 4 \),
let \( V - N[u] = \{v, w\} \). Then, \( u \) and \( v \) have at least two common neighbors. Coloring \( v \) and any
neighbor of \( u \) red and coloring all other vertices blue produces an \( F_3 \)-coloring of \( G \), and so
\( \gamma_{F_3}(G) \leq 2 < (n - 1)/2 \), a contradiction. If \( \Delta(G) = 5 \), then coloring \( u \) red and coloring all other vertices blue produces an \( F_3 \)-coloring of \( G \), and so \( \gamma_{F_3}(G) = 1 < (n - 1)/2 \), a contradiction. Hence, \( \Delta(G) = 3 \). Let \( V - N[u] = \{v, w\} \). If \( v \) and \( w \) have a common
neighbor \( x \), let \( y \in N(u) - \{x\} \). Coloring \( v \) and \( y \) red and coloring all other vertices blue
produces an \( F_3 \)-coloring of \( G \), and so \( \gamma_{F_3}(G) \leq 2 < (n - 1)/2 \), a contradiction. Hence, \( v \)
and \( w \) have no common neighbor, whence \( G = B_2 \).

5. Proof of Theorem 2

The sufficiency follows from Corollary 14. To prove the necessary, we proceed by induction on the order \( n \geq 3 \) of an \( F_3 \)-minimal graph. By Lemma 16, the result is true for \( n \leq 6 \). Let \( n \geq 7 \), and assume that the result is true for all \( F_3 \)-minimal graphs of order less
than \( n \). Let \( G = (V, E) \) be an \( F_3 \)-minimal graph of order \( n \). If \( e \) is an edge of \( G \), then
\( \gamma_{F_3}(G - e) \geq \gamma_{F_3}(G) \). Hence, by the minimality of \( G \), we have the following observation.

**Observation 17.** If \( e \in E \), then either \( e \) is a bridge of \( G \) or \( \delta(G - e) = 1 \).

Since the \( F_3 \)-domination number of a graph cannot decrease if edges are removed, the next result is a consequence of the inductive hypothesis.

**Observation 18.** If \( G' \) is a connected subgraph of \( G \) of order \( n' < n \) with \( \delta(G') \geq 2 \), then
either \( G' \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \) or \( \gamma_{F_3}(G') < (n' - 1)/2 \).

The following observation will prove useful.

**Observation 19.** Let \( G' \) be a graph and let \( v \) be a vertex of \( G' \) all of whose neighbors have
degree at most 2 in \( G' \). Then in any \( F_3 \)-coloring of \( G' \), at least one vertex in \( N[v] \) is colored red.

**Proof.** If every vertex in \( N[v] \) is colored blue in some \( F_3 \)-coloring of \( G' \), then there must
be a red vertex at distance 2 from \( v \). But then the neighbor of \( v \) that is adjacent to such a
red vertex is not rooted at a copy of \( F_3 \), a contradiction. \( \Box \)

We now return to the proof of Theorem 2. If \( G = C_n \), then, by Corollary 7, \( G \in \mathcal{C} \). If
\( G \) is a daisy, then by Corollary 10, \( G \in \mathcal{D} \). If \( G \) is a dumb-bell, then, by Corollary 12,
\( G \in \{A_1(4), A_1(5)\} \). So we may assume that \( G \) is neither a cycle, nor a daisy, nor a dumb-
bell. Hence, \( G \) contains at least two vertices of degree at least 3. Let \( S = \{ v \in V | \deg v \geq 3 \} \). Then, \(|S| \geq 2\) and each vertex of \( V - S \) has degree 2 in \( G \).

For each \( v \in S \), we define the 2-graph of \( v \) to be the component of \( G - (S - \{ v \}) \) that contains \( v \). So each vertex of the 2-graph of \( v \) has degree 2 in \( G \), except for \( v \). Furthermore, the 2-graph of \( v \) consists of edge-disjoint cycles through \( v \), which we call 2-graph cycles, and paths emanating from \( v \), which we call 2-graph paths.

Using the inductive hypothesis, we shall prove the following lemma, a proof of which is given in Section 5.1.

**Lemma 20.** If \( S \) is not an independent set, then \( G = A_k(4) \) for some \( k \geq 2 \).

By Lemma 20, we may assume that \( S \) is an independent set, for otherwise \( G \in \mathcal{A} \). Let \( u \) and \( v \) be two vertices of \( S \) that are joined by a path \( u, u_1, \ldots, u_m, v \) every internal vertex of which has degree 2 in \( G \). By assumption, \( d(u, v) \geq 2 \), whence \( m \geq 1 \). If \( m \) is large, then the following result, a proof of which is presented in Section 5.2, shows that \( G = B_4 \).

**Lemma 21.** If \( m \geq 4 \), then \( G = B_4 \).

By Lemma 21, we may assume that every 2-graph path has length at most 3. In particular, \( m \leq 3 \). Let \( P \) denote the path \( u_1, \ldots, u_m \). Let \( H = G - V(P) \). Then, \( H \) has order \( n' = n - m \) and \( \delta(H) \geq 2 \). Possibly, \( H \) is disconnected in which case \( H \) has two components, one containing \( u \) and the other \( v \). Since \( S \) is an independent set, we observe that each neighbor of a vertex of \( S \) has degree 2 in \( G \). In particular each neighbor of \( u \) and \( v \) in \( H \) has degree 2. Thus any \( F_3 \)-coloring of \( H \) that colors \( u \) (resp., \( v \)) blue must color at least one neighbor of \( u \) (resp., \( v \)) red. A proof of the following lemma is given in Section 5.3.

**Lemma 22.** If \( H \) is disconnected, then \( G = A_k(5) \) for some \( k \geq 2 \).

By Lemma 22, we may assume that removing the vertices in \( V - S \) from any 2-graph path in \( G \) produces a connected graph, for otherwise \( G \in \mathcal{A} \). In particular, \( H \) is connected.

In what follows, for each vertex \( u \in S \), let \( G_u = G - N[u] \). A proof of Lemma 23 is given in Sections 5.4, 5.5 and 5.6.

**Lemma 23.** If every 2-graph path has length exactly 1, then

(a) There is no 2-graph cycle in \( G \).
(b) If \( u, v \in S \), then \( N(u) \not \subseteq N(v) \).
(c) \( \delta(G_u) = 1 \) for every \( u \in S \).

A proof of Lemma 24 is given in Section 5.7.

**Lemma 24.** At least one 2-graph path has length 2 or 3.

By Lemma 24, we may assume that \( m \in \{2, 3\} \). By Observation 18, \( H \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \) or \( \gamma_{F_3}(H) < (n' - 1)/2 \). Let \( \mathcal{C} \) be a minimum \( F_3 \)-coloring of \( H \). A proof of the following two lemmas are given in Sections 5.8 and 5.9.
Lemma 25. If $m = 3$, then $G = B_5$.

Lemma 26. If $m = 2$, then $G = B_3$.

This completes the proof of Theorem 2.

5.1. Proof of Lemma 20

Let $e = uv$ be an edge, where $u, v \in S$. By Observation 17, $e$ must be a bridge of $G$.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be the two components of $G - e$ where $u \in V_1$. For $i = 1, 2$, let $|V_i| = n_i$. Each $G_i$ satisfies $\delta(G_i) \geq 2$ and is connected. Hence by Observation 18, $G_i \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$ or $\gamma_{F_3}(G_i) < (n_i - 1)/2$ for $i = 1, 2$. If $\gamma_{F_3}(G_i) < (n_i - 1)/2$ for $i = 1, 2$, then $\gamma_{F_3}(G) \leq \gamma_{F_3}(G_1) + \gamma_{F_3}(G_2) < (n - 1)/2$, a contradiction. Hence at least one of $G_1$ and $G_2$, say $G_1$, must belong to $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$.

Claim 1. $G_1 \neq C_5$.

Proof. Suppose $G_1 = C_5$. By assumption, $G$ is not a dumb-bell, and so $G_2$ is not a cycle. Thus if $\gamma_{F_3}(G_2) \geq n_2/2$, then, by Observation 13, $G_2 = B_2$. But then $G$ is not an $F_3$-minimal graph (either we contradict Observation 17 or $\gamma_{F_3}(G) < (n - 1)/2$), a contradiction. Thus, $\gamma_{F_3}(G_2) < (n_2 - 1)/2$.

Suppose $\gamma_{F_3}(G_2) = (n_2 - 1)/2$ (and still $G_2$ is not a cycle). Then, $G_2 \in (\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} - \{B_2\}$.

Suppose $G_2 = D_k(5)$ for some $k \geq 2$ and $v$ is the central vertex of $G_2$. Then a minimum $F_3$-coloring of $G_2$ can be extended to an $F_3$-coloring of $G$ by coloring the two neighbors of $u$ in $G_1$ red and the three remaining vertices of $G_1$ blue. Thus, $\gamma_{F_3}(G) \leq (n_2 - 1)/2 + 2 = (n - 2)/2$, a contradiction. If $v$ is not the central vertex of a daisy $D_k(5)$, then by Observation 15, there is a minimum $F_3$-coloring of $G$ in which $v$ is colored red. Such an $F_3$-coloring of $G_2$ can be extended to an $F_3$-coloring of $G$ by coloring $u$ and its two neighbors in $G_1$ blue and coloring the remaining two vertices of $G_1$ red. Thus, $\gamma_{F_3}(G) \leq (n_2 - 1)/2 + 2 = (n - 2)/2$, a contradiction. Hence, $\gamma_{F_3}(G_2) \leq (n_2 - 2)/2$.

If $\gamma_{F_3}(G_2) \leq (n_2 - 3)/2$, then $\gamma_{F_3}(G) \leq \gamma_{F_3}(G_1) + \gamma_{F_3}(G_2) \leq 3 + (n_2 - 3)/2 = (n - 3)/2$, a contradiction. Hence, $\gamma_{F_3}(G_2) = (n_2 - 2)/2$. If there exists a minimum $F_3$-coloring of $G_2$ in which $v$ or a neighbor of $v$ is colored red, then such an $F_3$-coloring of $G_2$ can be extended to an $F_3$-coloring of $G$ by coloring exactly two vertices of $G_1$ red, and so $\gamma_{F_3}(G) \leq (n_2 - 2)/2 + 2 = (n - 3)/2$, a contradiction. On the other hand, suppose that every minimum $F_3$-coloring of $G_2$ colors $v$ and all its neighbors blue. Then at least one neighbor $w$ of $v$ in $G_2$ must have degree at least 3, and so $w \in S$. By Observation 17, the edge $vw$ must be a bridge of $G$. Let $H_1$ and $H_2$ be the two components of $G - vw$ where $v \in V(H_1)$. For $i = 1, 2$, let $H_i$ have order $n_i'$. Each $H_i$ satisfies $\delta(H_i) \geq 2$ and is connected. Hence by Observation 18, $H_i \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$ or $\gamma_{F_3}(H_i) < (n_i - 1)/2$ for $i = 1, 2$. Since $H_1 \notin \{B_2, C_4, C_5, C_8\}$, $\gamma_{F_3}(H_1) \leq (n_1' - 1)/2$. If $H_2 \in \{B_2, C_4, C_5, C_8\}$, then there would exist a minimum $F_3$-coloring of $G_2$ in which $w$ is colored red, contrary to our earlier assumption that there is no such minimum $F_3$-coloring of $G_2$. Hence, $\gamma_{F_3}(H_2) \leq (n_2' - 1)/2$. Thus, $\gamma_{F_3}(G) \leq \gamma_{F_3}(H_1) + \gamma_{F_3}(H_2) < (n - 1)/2$, a contradiction. □
By Claim 1, $G_1 \neq C_5$. Similarly, $G_2 \neq C_5$.

**Claim 2.** $G_1 \notin \{B_2, C_8\}$.

**Proof.** Suppose $G_1 \in \{B_2, C_8\}$. If $G_2 \in \{B_2, C_4, C_8\}$, then $G$ is not an $F_3$-minimal graph (either we contradict Observation 17 or $\gamma_{F_3}(G) < (n - 1)/2$, a contradiction. If $\gamma_{F_3}(G_2) \leq (n_2 - 2)/2$, then $\gamma_{F_3}(G) \leq n_1/2 + (n_2 - 2)/2 < (n - 1)/2$, a contradiction. Hence, $\gamma_{F_3}(G_2) = (n_2 - 1)/2$, and so $G_2 \notin (\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \\mathcal{D}) - \{B_2, C_4, C_5, C_8\}$. By Observation 15, there is a minimum $F_3$-coloring $\mathcal{C}_1$ of $G_2$ in which $v$ is colored blue and is adjacent to a vertex colored red.

Suppose $G_1 = B_2$. If $u$ is a vertex of degree 2 in $G_1$, then at least one of the two edges incident with $u$ in $G_1$ joins two vertices of $S$ but is not a bridge of $G$, contradicting Observation 17. Hence the vertex $u$ must be a vertex of degree 3 in $G_1$. The $F_3$-coloring $\mathcal{C}_1$ of $G_2$ can be extended to an $F_3$-coloring of $G$ as follows: color one neighbor of $u$ on the 4-cycle in $G_1$ red, color the neighbor of $u$ in $G_1$ that does not belong to the 4-cycle red, and color the remaining four vertices of $G_1$ blue. Thus, $\gamma_{F_3}(G) \leq \gamma_{F_3}(G_2) + 2 = (n_2 - 1)/2 + 2 = (n - 3)/2$, a contradiction.

Suppose $G_1 = C_8$. The $F_3$-coloring $\mathcal{C}_1$ of $G_2$ can be extended to an $F_3$-coloring of $G$ by coloring the two neighbors of $u$ in $G_1$ red, coloring the vertex at maximum distance 4 from $u$ in $G_1$ red, and coloring the remaining five vertices of $G_1$ (including $u$) blue. Thus, $\gamma_{F_3}(G) \leq (n_2 - 1)/2 + 3 = (n - 3)/2$, a contradiction. □

By Claim 2, $G_1 \notin \{B_2, C_8\}$. Similarly, $G_2 \notin \{B_2, C_8\}$. If neither $G_1$ nor $G_2$ is a 4-cycle, then for $i = 1, 2$, $\gamma_{F_3}(G_i) \leq (n_i - 1)/2$, and so $\gamma_{F_3}(G) \leq \gamma_{F_3}(G_1) + \gamma_{F_3}(G_2) < (n - 1)/2$, a contradiction. Hence at least one of $G_1$ and $G_2$, say $G_1$, is a 4-cycle.

If $G_2 = C_4$, then $\gamma_{F_3}(G) = (n - 2)/2$, contradicting the fact that $G$ is an $F_3$-minimal graph. If $\gamma_{F_3}(G_2) \leq (n_2 - 2)/2$, then $\gamma_{F_3}(G) \leq n_1/2 + (n_2 - 2)/2 < (n - 1)/2$, a contradiction. Hence, $\gamma_{F_3}(G_2) = (n_2 - 1)/2$, and so $G_2 \notin (\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}) - \{B_2, C_4, C_5, C_8\}$. If $v$ is not the central vertex of a daisy $D_k(5)$, then by Observation 15, there is a minimum $F_3$-coloring of $G_2$ in which $v$ is colored red. Such an $F_3$-coloring of $G_2$ can be extended to an $F_3$-coloring of $G$ by coloring the vertex in $G_1$ at distance 2 from $u$ with the color red and coloring the remaining three vertices of $G_1$ blue. Thus, $\gamma_{F_3}(G) \leq (n_2 - 1)/2 + 1 = (n - 3)/2$, a contradiction. Thus, $v$ must be the central vertex of a daisy $D_k(5)$ for some $k \geq 2$, whence $G = A_k(4)$.

5.2. **Proof of Lemma 21**

Let $G'$ be the graph obtained from $G$ by removing the vertices $u_1, u_2, u_3, u_4$, and either adding the edge $uu_5$ if $m \geq 5$ or adding the edge $uv$ if $m = 4$. Then, $G'$ is a connected graph of order $n' = n - 4$ with $\delta(G') \geq 2$. By Observation 5, $\gamma_{F_3}(G) \leq \gamma_{F_3}(G') + 2$. If $\gamma_{F_3}(G') < (n' - 1)/2$, then $\gamma_{F_3}(G) < (n - 1)/2$, a contradiction. Hence, $\gamma_{F_3}(G') \geq (n' - 1)/2$. Since $G$ is an $F_3$-minimal graph, it follows that $G'$ is an $F_3$-minimal graph. Since $G$ is neither a dumbbell nor a dumbbell, $G'$ is not a cycle or a dumbbell. Further the degree of each vertex of $S$ is unchanged in $G$ and $G'$, and so $G'$ has at least two vertices of degree at least 3. Hence applying the inductive hypothesis to $G'$, $G' \in \mathcal{A} \cup \{B_1, B_2, \ldots, B_3\}$. A straightforward
check confirms that if $G' \neq B_1$, then $\gamma_{F_3}(G) < (n - 1)/2$. Therefore, $G' = B_1$, whence $G = B_4$.

5.3. Proof of Lemma 22

Let $H_1$ and $H_2$ be the two components of $H$, where $u \in V(H_i)$. For $i = 1, 2$, let $H_i$ have order $n_i$. Each $H_i$ is a connected graph with $\delta(H_i) \geq 2$. By Observation 18, $H_i \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$ or $\gamma_{F_3}(H_i) < (n_i - 1)/2$.

**Claim 3.** $\gamma_{F_3}(H_i) \geq n_i/2$ for $i = 1$ or $i = 2$.

**Proof.** Suppose $\gamma_{F_3}(H_i) \leq (n_i - 1)/2$ for $i = 1, 2$. Let $\mathcal{C}_{12}$ be a minimum $F_3$-coloring of $H_1 \cup H_2$. Then the restriction of $\mathcal{C}_{12}$ to $V(H_i)$ is a minimum $F_3$-coloring of $H_i$ for $i = 1, 2$, and so $\mathcal{C}_{12}$ colors at most $(n_1 + n_2)/2 - 1$ vertices of $H$ red.

Suppose $m = 3$. Then, $n_1 + n_2 = n - 3$. If at least one of $u$ and $v$, say $u$, is colored red in $\mathcal{C}_{12}$, then $\mathcal{C}_{12}$ can be extended to an $F_3$-coloring of $G$ by coloring the vertex $u_1$ red and the vertices $u_1$ and $u_2$ blue. On the other hand, if both $u$ and $v$ are colored blue in $\mathcal{C}_{12}$, then $\mathcal{C}_{12}$ can be extended to an $F_3$-coloring of $G$ by coloring the vertex $u_2$ red and the vertices $u_1$ and $u_3$ blue. Hence, $\gamma_{F_3}(G) \leq (n - 3)/2$, a contradiction.

Suppose $m = 2$. Then, $n_1 + n_2 = n - 2$. If $u$ and $v$ are colored with the same color in $\mathcal{C}_{12}$, then $\mathcal{C}_{12}$ can be extended to an $F_3$-coloring of $G$ by coloring both $u_1$ and $u_2$ blue, whence $\gamma_{F_3}(G) \leq (n - 4)/2$, a contradiction. If $u$ and $v$ are colored with different colors in $\mathcal{C}_{12}$, say $u$ is colored red and $v$ blue, then $\mathcal{C}_{12}$ can be extended to an $F_3$-coloring of $G$ by coloring $u_1$ red and $u_2$ blue, and so $\gamma_{F_3}(G) \leq (n - 2)/2$, a contradiction. Hence, $m = 1$ and $n_1 + n_2 = n - 1$.

If $u$ or $v$ is colored blue in $\mathcal{C}_{12}$, then $\mathcal{C}_{12}$ can be extended to an $F_3$-coloring of $G$ by coloring $u_1$ blue, and $\gamma_{F_3}(G) \leq (n - 3)/2$, a contradiction. Hence in every minimum $F_3$-coloring of $H$, the vertices $u$ and $v$ are colored red. There is therefore no minimum $F_3$-coloring of $H_1$ that colors $u$ blue, and so it follows from Observation 15 that $\gamma_{F_3}(H_1) \leq (n_1 - 2)/2$. Similarly, $\gamma_{F_3}(H_2) \leq (n_2 - 2)/2$. Thus, $\mathcal{C}_{12}$ colors at most $(n_1 + n_2 - 4)/2 = (n - 5)/2$ vertices of $H$ red. The coloring $\mathcal{C}_{12}$ can be extended to an $F_3$-coloring of $G$ by coloring $u_1$ red, and so $\gamma_{F_3}(G) \leq (n - 3)/2$, a contradiction. \(\Box\)

By Claim 3 and Observation 13, we may assume that $H_1 \in \{B_2, C_4, C_5, C_8\}$. We consider each possibility in turn.

**Claim 4.** If $H_1 = C_5$, then $G = A_k(5)$ for some $k \geq 2$.

**Proof.** Since $G$ is not a dumb-bell, $H_2$ is not a cycle. If $H_2 = B_2$, then $v$ must be one of the two vertices of degree 3 in $B_2$ and it is easy to check that for each value of $m \in \{1, 2, 3\}$, $\gamma_{F_3}(G) < (n - 1)/2$, a contradiction. Hence, by Observation 18, $H_2 \in \mathcal{A} \cup (\mathcal{B} - \{B_2\}) \cup \mathcal{D}$ or $\gamma_{F_3}(H_2) < (n_2 - 1)/2$. In particular, $\gamma_{F_3}(H_2) \leq (n_2 - 1)/2$. Let $\mathcal{C}_2$ be a minimum $F_3$-coloring of $H_2$.

Suppose $m = 3$. If $v$ is colored red in the coloring $\mathcal{C}_2$, then $\mathcal{C}_2$ can be extended to an $F_3$-coloring of $G$ by coloring $u_1$ and the two vertices in $H_1$ not adjacent to $u$ with the color
red and coloring all remaining uncolored vertices of $G$ blue. On the other hand, if $v$ is colored blue in $\mathcal{C}_2$, then $\mathcal{C}_2$ can be extended to an $F_3$-coloring of $G$ by coloring $u_2$ and the two neighbors of $u$ in $H_1$ with the color red and coloring all remaining uncolored vertices of $G$ blue. In both cases we color at most $(n-3)/2$ vertices red, and so $\gamma_{F_3}(G) \leq (n-3)/2$, a contradiction.

Suppose $m = 2$. If $v$ is colored red in $\mathcal{C}_2$, then $\mathcal{C}_2$ can be extended to an $F_3$-coloring of $G$ by coloring $u$ and its two neighbors in $H_1$ with the color red and coloring all remaining uncolored vertices of $G$ blue. On the other hand, if $v$ is colored blue in $\mathcal{C}_2$, then $\mathcal{C}_2$ can be extended to an $F_3$-coloring of $G$ by coloring $u_1$ and the two vertices in $H_1$ not adjacent to $u$ with the color red and coloring all remaining uncolored vertices of $G$ blue. In both cases we color at most $(n-2)/2$ vertices red, and so $\gamma_{F_3}(G) \leq (n-2)/2$, a contradiction. Hence, $m = 1$.

If $v$ is colored red in $\mathcal{C}_2$, then $\mathcal{C}_2$ can be extended to an $F_3$-coloring of $G$ by coloring the two neighbors of $u$ in $H_1$ with the color red and coloring all remaining uncolored vertices of $G$ blue, whence $\gamma_{F_3}(G) \leq (n-3)/2$, a contradiction. Hence, by Observation 15, either $H_2 = D_k(5)$ for some $k \geq 2$ with $v$ the central vertex of this daisy, or $\gamma_{F_3}(H_2) \leq (n_2 - 2)/2$ and $v$ is colored blue in $\mathcal{C}_2$. In the latter case, $\mathcal{C}_2$ can be extended to an $F_3$-coloring of $G$ by coloring $u$ and its two neighbors in $H_1$ with the color red and coloring all remaining uncolored vertices of $G$ blue, whence $\gamma_{F_3}(G) \leq (n-2)/2$, a contradiction. Hence, $H_2 = D_k(5)$ for some $k \geq 2$ with $v$ the central vertex of this daisy. Thus, $G = A_k(5)$ for some $k \geq 2$. □

By Claim 4, we may assume that neither $H_1$ nor $H_2$ is a 5-cycle, for otherwise the desired result follows. Hence, $\gamma_{F_3}(H_2) \leq n_2/2$.

**Claim 5.** $H_1 \neq B_2$.

**Proof.** Suppose $H_1 = B_2$. Let $\mathcal{C}_2$ be a minimum $F_3$-coloring of $H_2$. Suppose $m = 3$. If $v$ is colored red in the coloring $\mathcal{C}_2$, then $\mathcal{C}_2$ can be extended to an $F_3$-coloring of $G$ by coloring $u_1$ and the two vertices in $H_1$ not adjacent to $u$ with the color red and coloring all remaining uncolored vertices of $G$ blue. On the other hand, if $v$ is colored blue in $\mathcal{C}_2$, then $\mathcal{C}_2$ can be extended to an $F_3$-coloring of $G$ by coloring $u_2$ and two neighbors of $u$ in $H_1$ that lie on a common 5-cycle with the color red and coloring all remaining uncolored vertices of $G$ blue. In both cases we color at most $(n-3)/2$ vertices red, and so $\gamma_{F_3}(G) \leq (n-3)/2$, a contradiction.

Suppose $m = 2$. If $v$ is colored red in $\mathcal{C}_2$, then $\mathcal{C}_2$ can be extended to an $F_3$-coloring of $G$ by coloring $u$ and two neighbors of $u$ in $H_1$ that lie on a common 5-cycle with the color red and coloring all remaining uncolored vertices of $G$ blue. On the other hand, if $v$ is colored blue in $\mathcal{C}_2$, then $\mathcal{C}_2$ can be extended to an $F_3$-coloring of $G$ by coloring $u_1$ and the two vertices in $H_1$ not adjacent to $u$ with the color red and coloring all remaining uncolored vertices of $G$ blue. In both cases we color at most $(n-2)/2$ vertices red, and so $\gamma_{F_3}(G) \leq (n-2)/2$, a contradiction.

Suppose $m = 1$. If $v$ is colored red in $\mathcal{C}_2$, then $\mathcal{C}_2$ can be extended to an $F_3$-coloring of $G$ by coloring two neighbors of $u$ in $H_1$ that lie on a common 5-cycle with the color red and coloring all remaining uncolored vertices of $G$ blue, and so $\gamma_{F_3}(G) \leq (n-3)/2$,
a contradiction. Hence we may assume that every minimum $F_3$-coloring of $H_2$ colors $v$ blue, for otherwise we reach a contradiction. Thus by Observations 13 and 15, $\gamma_{F_3}(H_2) \leq (n_2 - 1)/2$. Since $v$ is colored blue in $\mathcal{C}_2$, the coloring $\mathcal{C}_2$ can be extended to an $F_3$-coloring of $G$ by coloring $u_1$ and the two vertices in $H_1$ not adjacent to $u$ with the color red and coloring all remaining uncolored vertices of $G$ blue, whence $\gamma_{F_3}(G) \leq (n - 2)/2$, a contradiction. □

By Claim 5, $H_1 \neq B_2$. Similarly, $H_2 \neq B_2$.

Claim 6. $H_1 \neq C_4$.

Proof. Suppose $H_1 = C_4$. Let $H_1$ be the 4-cycle $u, w, x, y, u$. Since $G$ is not a dumb-bell, $H_2$ is not a cycle. Hence, $\gamma_{F_3}(H_2) \leq (n_2 - 1)/2$. Let $\mathcal{C}_2$ be a minimum $F_3$-coloring of $H_2$.

Suppose $m = 3$. If $v$ is colored red in $\mathcal{C}_2$, then $\mathcal{C}_2$ can be extended to an $F_3$-coloring of $G$ by coloring $u_1$ and $x$ red and coloring all remaining uncolored vertices of $G$ blue. On the other hand, if $v$ is colored blue in $\mathcal{C}_2$, then $\mathcal{C}_2$ can be extended to an $F_3$-coloring of $G$ by coloring the vertices $u_2$, $x$ and $y$ red, and coloring all remaining uncolored vertices of $G$ blue. Hence, $\gamma_{F_3}(G) \leq (n - 2)/2$, a contradiction.

Suppose $m = 2$. If $v$ is colored red in $\mathcal{C}_2$, then $\mathcal{C}_2$ can be extended to an $F_3$-coloring of $G$ by coloring $u$ and $w$ red and coloring all remaining uncolored vertices of $G$ blue. On the other hand, if $v$ is colored blue in $\mathcal{C}_2$, then $\mathcal{C}_2$ can be extended to an $F_3$-coloring of $G$ by coloring the vertices $u_1$ and $x$ red, and coloring all remaining uncolored vertices of $G$ blue. Hence, $\gamma_{F_3}(G) \leq (n - 3)/2$, a contradiction.

Suppose $m = 1$. If $v$ is colored red in $\mathcal{C}_2$, then $\mathcal{C}_2$ can be extended to an $F_3$-coloring of $G$ by coloring $x$ and $y$ red and coloring all remaining uncolored vertices of $G$ blue. On the other hand, if $v$ is colored blue in $\mathcal{C}_2$, then $\mathcal{C}_2$ can be extended to an $F_3$-coloring of $G$ by coloring the vertices $u$ and $w$ red, and coloring all remaining uncolored vertices of $G$ blue. Hence, $\gamma_{F_3}(G) \leq (n - 2)/2$, a contradiction. □

We now return to the proof of Lemma 22. By Claim 6, $H_1 \neq C_4$. Similarly, $H_2 \neq C_4$. Hence, $H_1 = C_8$. Let $H_1$ be the 8-cycle $u = w_1, w_2, \ldots, w_8, u$. Since $G$ is not a dumb-bell, $H_2$ is not a cycle. Hence, $\gamma_{F_3}(H_2) \leq (n_2 - 1)/2$. Let $\mathcal{C}_2$ be a minimum $F_3$-coloring of $H_2$.

Suppose $m = 3$. If $v$ is colored red in $\mathcal{C}_2$, then $\mathcal{C}_2$ can be extended to an $F_3$-coloring of $G$ by coloring the vertices in the set $\{u_1, w_3, w_4, w_7\}$ red and coloring all remaining uncolored vertices of $G$ blue. On the other hand, if $v$ is colored blue in $\mathcal{C}_2$, then $\mathcal{C}_2$ can be extended to an $F_3$-coloring of $G$ by coloring the vertices in the set $\{u_2, w_2, w_5, w_8\}$ red, and coloring all remaining uncolored vertices of $G$ blue. Hence, $\gamma_{F_3}(G) \leq (n - 4)/2$, a contradiction.

Suppose $m = 2$. If $v$ is colored red in $\mathcal{C}_2$, then $\mathcal{C}_2$ can be extended to an $F_3$-coloring of $G$ by coloring the vertices in the set $\{w_1, w_2, w_5, w_8\}$ red and coloring all remaining uncolored vertices of $G$ blue. On the other hand, if $v$ is colored blue in $\mathcal{C}_2$, then $\mathcal{C}_2$ can be extended to an $F_3$-coloring of $G$ by coloring the vertices in the set $\{u_1, w_3, w_4, w_7\}$ red, and coloring all remaining uncolored vertices of $G$ blue. Hence, $\gamma_{F_3}(G) \leq (n - 3)/2$, a contradiction.

Suppose $m = 1$. If $v$ is colored red in $\mathcal{C}_2$, then $\mathcal{C}_2$ can be extended to an $F_3$-coloring of $G$ by coloring the vertices in the set $\{w_2, w_5, w_8\}$ red and coloring all remaining uncolored vertices of $G$ blue. On the other hand, if $v$ is colored blue in $\mathcal{C}_2$, then $\mathcal{C}_2$ can be extended to an $F_3$-coloring of $G$ by coloring the vertices in the set $\{u_1, w_3, w_4, w_7\}$ red, and coloring all remaining uncolored vertices of $G$ blue. Hence, $\gamma_{F_3}(G) \leq (n - 2)/2$, a contradiction. □
an $F_3$-coloring of $G$ by coloring the vertices in the set \{w_1, w_2, w_5, w_8\} red, and coloring all remaining uncolored vertices of $G$ blue. Hence, $\gamma_{F_3}(G) \leq (n - 2)/2$, a contradiction. This completes the proof of Lemma 22.

5.4. Proof of Lemma 23(a)

Suppose that there is a 2-graph cycle in $G$. Since $|S| \geq 2$, each vertex of $S$ that has a 2-graph cycle also has a 2-graph path. Hence we may assume that the vertex $u$ (defined earlier) has a 2-graph cycle $C_u$ of order $n_1 + 1$. Let $H_u = G - (V(C_u) - \{u\})$ have order $n_2$. Then, $H_u$ is a connected graph of order $n_2 = n - n_1$. If $\deg_G(u) = 3$, then the graph $H = G - u_1$ (defined earlier) is disconnected, contrary to assumption. Hence, $\deg_G(u) \geq 4$, and so $\delta(H_u) \geq 2$. By Observation 18, $H_u \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$ or $\gamma_{F_3}(H_u) < (n_2 - 1)/2$.

Since $v$ is a vertex of degree at least 3 in $H_u$, the graph $H_u$ is not a cycle. Further by our earlier assumptions (that every 2-graph path has length exactly 1; that the set $S$ is an independent set with $|S| \geq 2$; that removing the vertices not in $S$ of any 2-graph path from $G$ produces a connected graph), it follows that $H_u \notin \mathcal{A} \cup \mathcal{B} - \{B_1\} \cup \mathcal{C} \cup \mathcal{D} - D(4, 4)$. Hence by Observation 13, either $H_u \in \{B_1, D(4, 4)\}$ or $\gamma_{F_3}(H_u) < (n_2 - 2)/2$.

Let $\mathcal{E}^\ast$ be a minimum $F_3$-coloring of $H_u$. If $n_1 \neq 4$ (i.e., if $C_u$ is not a 5-cycle), then irrespective of whether $u$ is colored red or blue in $\mathcal{E}^\ast$, the coloring $\mathcal{E}^\ast$ can be extended to an $F_3$-coloring of $G$ by coloring at most $(n_1 - 1)/2$ additional vertices in $C_u$ red, and so $\gamma_{F_3}(G) \leq (n - 2)/2$, a contradiction. If $n_1 = 4$, then $\mathcal{E}^\ast$ can be extended to an $F_3$-coloring of $G$ by coloring $n_1/2$ additional vertices red. Thus, if $\gamma_{F_3}(H_u) < (n_2 - 2)/2$, then $\gamma_{F_3}(G) \leq (n - 2)/2$, a contradiction. Hence, $n_1 = 4$ and $H_u \in \{B_1, D(4, 4)\}$. But once again, $\gamma_{F_3}(G) \leq (n - 3)/2$, a contradiction.

5.5. Proof of Lemma 23(b)

By Lemma 23(a), $G$ is a bipartite graph with partite sets $S$ and $V - S$. Every vertex in $V - S$ has degree exactly 2, while every vertex in $S$ has degree at least 3.

Suppose that $N(u) \subseteq N(v)$ for some pair of vertices $u$, $v \in S$. If $|S| = 2$ (and still $n \geq 7$), then coloring $u$ and one neighbor of $u$ red and coloring all remaining vertices blue produces an $F_3$-coloring of $G$, and so $\gamma_{F_3}(G) = 2 \leq (n - 3)/2$, a contradiction. Hence, $|S| \geq 3$, and so at least one neighbor of $v$ is not a neighbor of $u$.

Suppose $\deg_G(v) = \deg_G(u) + 1$. Let $G' = G - \{v\} - u$ have order $n'$. Then, $n' \leq n - 6$ and $G'$ is an induced subgraph of $G$ with $\delta(G') \geq 2$. Since $G'$ is a bipartite graph, $G'$ has no 5-cycles, and so, by the inductive hypothesis, $\gamma_{F_3}(G') \leq n'/2 \leq (n - 6)/2$. Any minimum $F_3$-coloring of $G'$ can be extended to an $F_3$-coloring of $G$ by coloring $u$ and one neighbor of $u$ red and coloring all remaining uncolored vertices blue. Thus, $\gamma_{F_3}(G) \leq 2 + \gamma_{F_3}(G') \leq (n - 2)/2$, a contradiction.

On the other hand, suppose $\deg_G(v) \geq \deg_G(u) + 2$. Let $G^* = G - \{u\}$ have order $n^*$. Then, $n^* \leq n - 4$ and $G^*$ is an induced subgraph of $G$ with $\delta(G^*) \geq 2$. Since $G^*$ is a bipartite graph, $G^*$ has no 5-cycles, and so, by the inductive hypothesis, $\gamma_{F_3}(G^*) \leq n^*/2 \leq (n - 4)/2$. Any minimum $F_3$-coloring of $G^*$ that colors $v$ red can be extended to an $F_3$-coloring of $G$ by coloring one neighbor of $u$ red (and coloring all remaining uncolored vertices blue), while any minimum $F_3$-coloring of $G^*$ that colors $v$ blue can be extended to an $F_3$-coloring
of $G$ by coloring the vertex $u$ red. Hence, $\gamma_{F_3}(G) \leq 1 + \gamma_{F_3}(G^e) \leq (n-2)/2$, a contradiction.

We deduce, therefore, that for any pair of vertices $u, v \in S, N(u) \subseteq N(v)$.

5.6. Proof of Lemma 23(c)

Suppose $\delta(G_u) \geq 2$ for some vertex $u \in S$. Then, $G_u$ is an induced subgraph (possible disconnected) of $G$. Since $G_u$ is a bipartite graph, $G_u$ is $C_5$-free. Hence by the inductive hypothesis, each component of $G_u$ has $F_3$-domination number at most one-half its order, and so $\gamma_{F_3}(G_u) \leq |V(G_u)|/2 \leq (n-4)/2$.

Let $S_u = \{w \in S \mid d_G(u, w) = 2\}$. By Lemma 23(b), $|S_u| \geq 2$. Let $C_u$ be a minimum $F_3$-coloring of $G_u$. Suppose $C_u$ colors a vertex $x$ in $S_u$ red. Let $x'$ be a common neighbor of $u$ and $x$. Then, $C_u$ can be extended to an $F_3$-coloring of $G$ by coloring one vertex in $N(u) - \{x'\}$ red (and coloring all other vertices in $N[u]$ blue). On the other hand, if $C_u$ colors no vertex in $S_u$ red, then it follows by Observation 19 that $C_u$ can be extended to an $F_3$-coloring of $G$ by coloring the vertex $u$ red (and coloring all remaining uncolored vertices blue). Hence, $\gamma_{F_3}(G) \leq 1 + \gamma_{F_3}(G_u) \leq (n-2)/2$, a contradiction.

5.7. Proof of Lemma 24

Suppose that every 2-graph path has length exactly 1. Let $u \in S$. By Lemma 23, $\delta(G_u) = 1$ and $G$ is a bipartite graph with partite sets $S$ and $V-S$. We may assume that $u$ has degree 1 in $G_u$. Thus, $u$ and $v$ have at least two common neighbors. Let $c$ be one such common neighbor of $u$ and $v$. Let $v'$ be the neighbor of $v$ in $G_u$, and let $N(v') = \{v, w\}$. Then, $w \in S - \{u, v\}$ and $w$ has degree at least 2 in $G_v$. Since $\delta(G_v) = 1$ by Lemma 23(c), the vertex $u$ must have degree 1 in $G_v$. Let $u'$ be the neighbor of $u$ in $G_v$, and let $N(u') = \{u, z\}$. Then, $z \in S - \{u, v\}$ and $z$ has degree at least 2 in $G_v$ (possibly, $z = w$). Let $G' = G - N[u] - N[v]$ have order $n'$. Then, $n' \leq n - 6$ and $G'$ is an induced subgraph of $G$. In particular, $G'$ has no 5-cycles.

Let $C'$ be a minimum $F_3$-coloring of $G'$. We now consider two possibilities.

Suppose $z \neq w$. Then, $\delta(G') \geq 2$. It follows from the inductive hypothesis that $\gamma_{F_3}(G') \leq n'/2 \leq (n-6)/2$. If $C'$ colors both $w$ and $z$ red, then $C'$ can be extended to an $F_3$-coloring of $G$ by coloring the vertex $c$ red and coloring all remaining uncolored vertices blue. If $C'$ colors both $w$ and $z$ blue, then it follows from Observation 19 that $C'$ can be extended to an $F_3$-coloring of $G$ by coloring the vertices $c$ and $u$ red and coloring all remaining uncolored vertices blue. Suppose, finally, that $C'$ colors exactly one of $w$ and $z$ red. By symmetry, we may assume that $w$ is colored red. Then $C'$ can be extended to an $F_3$-coloring of $G$ by coloring the vertices $c$ and $u$ red and coloring all remaining uncolored vertices blue. Thus, $\gamma_{F_3}(G) \leq 2 + \gamma_{F_3}(G') \leq (n-2)/2$.

Suppose, on the other hand, that $z = w$. If $\deg_G(w) = 3$, then $\delta(G_w) \geq 2$, which contradicts Lemma 23(c). Hence, $\deg_G(w) \geq 4$. Then, $\delta(G') \geq 2$. It follows from the inductive hypothesis that $\gamma_{F_3}(G') \leq n'/2 \leq (n-6)/2$. If $C'$ colors $w$ red, then $C'$ can be extended to an $F_3$-coloring of $G$ by coloring the vertex $c$ red and coloring all remaining uncolored vertices blue. If $C'$ colors $w$ blue, then it follows from Observation 19 that $C'$ can be extended to an $F_3$-coloring of $G$ by coloring the vertices $c$ and $u$ red and coloring all remaining uncolored vertices blue. Thus, $\gamma_{F_3}(G) \leq 2 + \gamma_{F_3}(G') \leq (n-2)/2$. 
5.8. Proof of Lemma 25

Suppose \( \gamma_{F_3}(H) \leq (n' - 1)/2 = (n - 4)/2 \). If at least one of \( u \) and \( v \), say \( u \), is colored red in \( \mathcal{C}_H \), then \( \mathcal{C}_H \) can be extended to an \( F_3 \)-coloring of \( G \) by coloring the vertex \( u_1 \) red and the vertices \( u_1 \) and \( u_2 \) blue. On the other hand, if both \( u \) and \( v \) are colored blue in \( \mathcal{C}_H \), then \( \mathcal{C}_H \) can be extended to an \( F_3 \)-coloring of \( G \) by coloring the vertex \( u_2 \) red and the vertices \( u_1 \) and \( u_2 \) blue. Hence, \( \gamma_{F_3}(G) \leq (n - 2)/2 \), a contradiction. Thus, \( \gamma_{F_3}(H) \geq n'/2 \), whence \( H \in \{B_2, C_4, C_5, C_8\} \). If \( H \in \{B_2, C_4, C_5\} \), then it is easily checked that \( \gamma_{F_3}(G) < (n - 1)/2 \), a contradiction. If \( H = C_8 \) and if \( u \) and \( v \) are at distance 2 or 3 apart in \( H \), then \( \gamma_{F_3}(G) < (n - 1)/2 \), a contradiction. Thus, \( H = C_8 \) and the vertices \( u \) and \( v \) are at distance 4 apart in \( H \), and so \( G = B_5 \).

5.9. Proof of Lemma 26

Note that \( n' = n - 2 \geq 5 \). If \( H \in \{B_2, C_4, C_5, C_8\} \), then it is easily checked that \( \gamma_{F_3}(G) < (n - 1)/2 \), a contradiction. Hence, \( \gamma_{F_3}(H) \leq (n' - 1)/2 = (n - 3)/2 \). If \( u \) and \( v \) are both colored with the same color in \( \mathcal{C}_H \), then \( \mathcal{C}_H \) can be extended to an \( F_3 \)-coloring of \( G \) by coloring both \( u_1 \) and \( u_2 \) blue, and so \( \gamma_{F_3}(G) < (n - 1)/2 \), a contradiction. Hence every minimum \( F_3 \)-coloring of \( H \) colors \( u \) and \( v \) with different colors. We may assume that \( u \) is colored red in \( \mathcal{C}_H \). If \( \gamma_{F_3}(H) \leq (n - 4)/2 \), then \( \mathcal{C}_H \) can be extended to an \( F_3 \)-coloring of \( G \) by coloring \( u_1 \) red and \( u_2 \) blue, and so \( \gamma_{F_3}(G) \leq (n - 2)/2 \), a contradiction. Hence, \( \gamma_{F_3}(H) = (n' - 1)/2 \). Since the set \( S \) is independent and since \( u \) and \( v \) must receive different colors in every minimum \( F_3 \)-coloring of \( H \), it therefore follows that \( H \in \{B_1, B_2, B_3, A_1(5), D(4, 4), D_2(5)\} \). If \( H \neq B_1 \), then it is easily checked that \( \gamma_{F_3}(G) < (n - 1)/2 \), a contradiction. Hence, \( H = B_1 \), and so \( G = B_3 \).

6. Proof of Theorem 3

The proof of Theorem 3 follows readily from Theorem 2. Since the \( F_3 \)-domination number of a graph cannot decrease if edges are removed, it follows from Theorem 2 and Observation 13 that the \( F_3 \)-domination number of \( G \) is at most \( (n + 1)/2 \). Further suppose \( \gamma_{F_3}(G) \geq n/2 \). Then by removing edges of \( G \), if necessary, we produce an \( F_3 \)-minimal graph \( G' \). By Theorem 2 and Observation 13, \( G' \in \{B_2, C_4, C_5, C_8\} \). In all cases it can be readily checked that \( G = G' \) unless possibly if \( G' = C_8 \) in which case \( G \in \{C_8, H_1\} \).

7. Proof of Theorem 4

The proof of Theorem 4 follows readily from Theorem 2. Since the \( F_3 \)-domination number of a graph cannot decrease if edges are removed, and since \( n \geq 9 \), it follows from Theorem 2 and Observation 13 that the \( F_3 \)-domination number of \( G \) is at most \( (n - 1)/2 \). Further suppose \( \gamma_{F_3}(G) = (n - 1)/2 \). Then by removing edges of \( G \), if necessary, we produce an \( F_3 \)-minimal graph \( G' \). By Theorem 2 and Observation 13, \( G' \in \mathcal{A} \cup (\varnothing - \{D(3, 5), D(4, 4)\}) \).
or $G' \in \{B_4, B_5, C_{11}\}$. In all cases it is straightforward (though tedious) to check that $G=G'$ unless possibly if $G' = C_{11}$ in which case $G \in \{C_{11}, H_2\}$.

References