1. INTRODUCTION

An $\ell$-arrangement $\mathcal{A}$ is a finite set of linear hyperplanes in $\mathbb{C}^\ell$. Its complement $C = C(\mathcal{A})$ is the open $2\ell$-manifold $\mathbb{C}^\ell \setminus \mathcal{A}$. If the defining linear forms have real coefficients we say $\mathcal{A}$ is complexified. In this case the same equations define an arrangement $\mathcal{A}_R$ in $\mathbb{R}^\ell$ which we call the real part of $\mathcal{A}$. If $C$ is an aspherical space we say $\mathcal{A}$ is a $K(\pi, 1)$ arrangement.

We are interested in the problem of determining necessary and/or sufficient conditions for $C$ to be aspherical. With roots in geometric topology and singularity theory, this “$K(\pi, 1)$ problem” has developed into a major focus of research in the topological theory of arrangements. For the sake of completeness we will begin with a survey of the origins of this problem and a compendium of the known results. Refer to [20] or [16] for background material; see also [9].

Consider for example the arrangement $\mathcal{A}_c = \{H_{ij} | 0 \leq i < j \leq \ell\}$, where $H_{ij}$ is defined by $x_i - x_j = 0$. The complement $C(\mathcal{A}_c)$ is the configuration space of $\ell + 1$ distinct labelled points in the plane. Its fundamental group is the pure braid group on $\ell + 1$ strands; $\mathcal{A}_c$ is usually called the braid arrangement. The projection $\mathbb{C}^{\ell + 1} \to \mathbb{C}^\ell$ restricts to a fibration $C(\mathcal{A}_c) \to C(\mathcal{A}_c)_{\ell - 1}$ with fiber a plane with $\ell$ punctures. Since $C(\mathcal{A}_c) \cong \mathbb{C}^* \times \mathbb{C}$ it follows that $\mathcal{A}_c$ is a $K(\pi, 1)$ arrangement [5].

This classical result has been generalized in two directions. First of all, $\mathcal{A}_c$ is a complexified arrangement whose real part consists of the “mirrors” for the reflections in the canonical representation of the symmetric group $G = S_{\ell + 1}$ on $\mathbb{R}^{\ell + 1}$. Projecting $\mathbb{R}^{\ell + 1}$ to $x_0 + \cdots + x_\ell = 0$ along the kernel $x_0 = x_1 = \cdots = x_\ell$, one obtains an irreducible representation of $S_{\ell + 1}$ on $\mathbb{R}^\ell$, whose image is the Weyl group of type $A_\ell$. Complexifying, we obtain a projection of $C(\mathcal{A}_c)$ onto the complement $C$ of a complexified arrangement whose real part consists of the reflecting hyperplanes of the Weyl group. We have $C(\mathcal{A}_c) \cong \mathbb{C} \times \mathbb{C}$, so $C$ is also aspherical.

The Weyl group in this example can be replaced with any finite irreducible real linear group $G$ generated by reflections to obtain the associated (complexified) Coxeter arrangement $\mathcal{A}_G$. The fundamental group of $C(\mathcal{A}_G)$ is the “generalized pure braid group” associated with $G$. In [2] Brieskorn introduced these arrangements and proved that most of them are $K(\pi, 1)$. All Coxeter arrangements have the property that their real parts induce on the unit sphere $S^{\ell - 1}$ in $\mathbb{R}^\ell$, a cellular subdivision which is simplicial. A simplicial arrangement is any real arrangement with this property. In [4] Deligne extended Brieskorn’s work by proving the following theorem: if $\mathcal{A}$ is the complexification of a real simplicial arrangement, then $C(\mathcal{A})$ is aspherical. It is this remarkable result which has inspired much of the interest in the $K(\pi, 1)$ problem.

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Part of the appeal of the theory of arrangements comes from the interplay between topology and combinatorics which permeates the subject. The second generalization of the braid arrangement arises from combinatorial considerations. The fundamental combinatorial invariant of an arrangement $\mathcal{A}$ is its intersection lattice $L(\mathcal{A}) = \{X \mid X = \bigcap \mathcal{B} \text{ for some } \mathcal{B} \subseteq \mathcal{A}\}$. With the partial ordering defined by reverse inclusion, this set of subspaces becomes a geometric lattice. For example, the intersection lattice of $\mathcal{A}$ is isomorphic to the lattice of partitions of the set $\{0, \ldots, \ell\}$, ordered by refinement. The characteristic polynomial of $L = L(\mathcal{A})$ is defined by $\chi_L(t) = \sum_{X \in L} \mu(X) t^{r(X)}$, where $r(X) = \ell - \dim(X)$ denotes the rank of $X$, $r(L) = r(\bigcap \mathcal{A})$, and $\mu$ is the one variable Möbius function of the lattice $L$ [20]. This polynomial generalizes the chromatic polynomial of a graph and has many interesting combinatorial properties. It also has topological significance: the coefficients of the characteristic polynomial of $L(\mathcal{A})$ agree with the Betti numbers of $C(\mathcal{A})$ in reverse order (up to an alternating sign) [17].

In many cases of interest the complex roots of the characteristic polynomial of $L(\mathcal{A})$ turn out to be positive integers. In fact this property holds for $\mathcal{A}$, and any Coxeter arrangement $\mathcal{A}_G$; in this case the integer roots have algebraic interpretations in terms of the representation theory and the invariant theory of $G$ [20]. In [31] Stanley introduced a class of lattices with this factorization property and gave a combinatorial interpretation of the integer roots. These lattices are called supersolvable. The partition lattice $L(\mathcal{A}_G)$ is supersolvable, though not every Coxeter arrangement has supersolvable intersection lattice. If $\mathcal{A}$ is an arrangement with supersolvable lattice the roots of $\chi_L(t)$ have topological significance. In fact $L(\mathcal{A})$ is supersolvable precisely when $C(\mathcal{A})$ sits atop a tower of linear fibrations over hyperplane complements having punctured planes as fibers, with $C^*$ at the very bottom [34]. Thus supersolvable arrangements give a natural generalization of the braid arrangement. Arrangements with this fibration property (called fiber-type arrangements) had been studied from the topological point of view in [8]. In that paper it was shown that the roots of the characteristic polynomial are the numbers of punctures in successive fibers. From the sequence of fibrations with aspherical fibers we conclude that any supersolvable arrangement is a $K(\pi, 1)$ arrangement.

We now proceed to outline the major unsolved problems and collect the known results related to $K(\pi, 1)$ arrangements. Complexified simplicial arrangements and supersolvable arrangements form the only two general classes of arrangements known to be $K(\pi, 1)$. In both cases the hypotheses depend only on the intersection lattice. According to [17] the intersection lattice also determines the cohomology algebra of $C(\mathcal{A})$. This raises a natural question: does the asphericity of $C(\mathcal{A})$ in general depend only on $L(\mathcal{A})$? This question remains unresolved.

It follows from Deligne's theorem that all Coxeter arrangements are $K(\pi, 1)$. A unitary reflection arrangement $\mathcal{A}_G$ is the set of fixed hyperplanes of reflections in a finite unitary group generated by reflections [20]. It is natural to ask whether all unitary reflection arrangements are $K(\pi, 1)$, and this question remains open. There is an infinite family of unitary reflection groups, the monomial groups $G(p, q, r)$, which includes the Weyl groups of types $B \ (p = 2, q = 1)$ and $D \ (p = q = 2)$. If $q < p$ the associated arrangements are all supersolvable [18]. For arbitrary $q$ the associated reflection arrangements are shown to be $K(\pi, 1)$ in [15]. The symmetry groups of complex polytopes (called Shephard groups in [20]) are generated by reflections, and the associated reflection arrangements are shown to be $K(\pi, 1)$ in [19]. There remain six unitary reflection arrangements for which the $K(\pi, 1)$ problem is open. The smallest of these is the arrangement associated with the double cover of Klein's simple group of order 168.
Terao [32] introduced the study of free arrangements. Consider the space of rational 1-forms on $C'$ which have logarithmic singularities along $\bigcup \mathcal{A}$. These form a module over the polynomial ring $S = C[x_1, \ldots, x_r]$. If this is a free module, then $\mathcal{A}$ is said to be free. This is a specialization and refinement of the Saito's notion of free divisor, which is defined in terms of meromorphic forms. For arrangements one usually works with the dual module of polynomial vector fields on $C'$ which are tangent along $\bigcup \mathcal{A}$ [20]. This module has a purely algebraic description as the module $D(\mathcal{A})$ of derivations $\theta \in \text{Der}(S)$ with the property that for each $H \in \mathcal{A}$, the defining linear form $\phi_H$ divides $\theta(\phi_H)$. For example, the Euler derivation $\sum x_i \partial/\partial x_i$ will have this property for any arrangement. The arrangement $\mathcal{A}$ is free if $D(\mathcal{A})$ is a free $S$-module. The geometric meaning of this condition remains a mystery.

The class of free arrangements includes all Coxeter arrangements, unitary reflection arrangements (see Chapter 6 of [20] for references), and supersolvable arrangements [14]. In addition, the characteristic polynomial of a free arrangement has positive integer roots [33]. The roots are equal to the degrees of the coefficients appearing in any set of homogeneous derivations which form a free basis for $D(\mathcal{A})$. The natural conjecture, due to Saito [20], is that all free arrangements are $K(\pi, 1)$. This is the most prominent open problem in the field. It is known that the converse is false; the counterexample is the complexification of a three-dimensional real simplicial arrangement of 13 planes [9].

Since Deligne's proof 20 years ago, there has been only modest progress on the $K(\pi, 1)$ problem for more general arrangements. Hendriks [12] proves that if $\mathcal{A}$ is a locally finite complexified arrangement and all codimension two flats have multiplicity at least three (that is, each $X \in L(\mathcal{A})$ of rank two is contained in at least three hyperplanes), then $C(\mathcal{A})$ is aspherical. No finite arrangements have this property. The result applies also to any subspace of $C$ lying over an open convex subset of $\mathbb{R}^C$.

Randell [27] has shown that a 3-arrangement $\mathcal{A}$ is $K(\pi, 1)$ if and only if $H^*(C) \cong H^*(\pi_1(C))$ as $\mathbb{Z}$-modules and there exists a $K(\pi_1(C), 1)$ with the homotopy type of a finite complex. It is not known whether the second hypothesis can be eliminated. To date this result has not yielded any new examples. More recently, Paris has given a simplified proof of Deligne's theorem [21], and Cordovil has generalized Deligne's theorem to arbitrary (not necessarily realizable) oriented matroids [3].

Not all arrangements are $K(\pi, 1)$, and there has been some success in the search for necessary conditions. A classical result due to Hattori asserts that general position arrangements are not $K(\pi, 1)$, except in trivial cases [11]. An $\mathcal{A}$-arrangement $\mathcal{A}$ is general position if $\text{codim}(\bigcap \mathcal{B}) = |\mathcal{B}|$ for $|\mathcal{B}| \leq \mathcal{A}$. It is known that $K(\pi, 1)$ arrangements are necessarily formal [9, 23]; loosely speaking this means the arrangement is generic among arrangements with the same structure through codimension two. If $\mathcal{A}$ is $K(\pi, 1)$ then the localization $A_X := \{H \in \mathcal{A} | H \ni X\}$ is also $K(\pi, 1)$ for each $X \in L(\mathcal{A})$. There are also some folk theorems for affine 2-arrangements. (We will see below that the theory for central 3-arrangements reduces to that of affine 2-arrangements.) If an affine 2-arrangement contains one line in general position with respect to the other lines, the arrangement cannot be a $K(\pi, 1)$. And if the real part of such an arrangement agrees with that of some non-$K(\pi, 1)$ arrangement throughout some convex open subset of the plane, it cannot be a $K(\pi, 1)$ (see e.g. [9, 29]). Finally, for arbitrary arrangements the homotopy type of the complement remains unchanged throughout a lattice isotopy, that is, a one-parameter family of arrangements with constant intersection lattice [26]. It is often the case that the last result for affine 2-arrangements applies only after such an isotopy is carried out.

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1 Added in proof: A counter-example to Saito's conjecture was recently discovered by P. Edelman and V. Reiner.
In this paper we develop a new approach to this problem, and apply it to derive a weight test for a complexified 3-arrangement \( \mathcal{A} \) to be \( K(\pi, 1) \). This test provides some new examples, and specializes to Hendriks' result cited above. Recently, the weight test has been used by Paris [22] to generalize the theorem that supersolvable (complexified 3-) arrangements are \( K(\pi, 1) \) to the class of complexified factored 3-arrangements, as defined in [20, Section 3.3].

The key to all our results is a simple model for the homotopy type of the universal cover of \( C(\mathcal{A}) \). This is a 2-complex \( T \) whose structure is encoded in the bounded complex \( \Gamma \) of the real part \( \mathcal{A} \) relative to a fixed hyperplane at infinity. The construction of \( T \) follows the standard procedure of Bass–Serre theory, as generalized to 2-complexes of groups in [30]. This procedure is applied to a modified version \( S \) of the Salvetti complex, a 2-complex which carries the homotopy type of \( C(\mathcal{A}) \). The covering group \( \pi \) acts on \( T \) with orbit complex \( \Gamma \).

The face groups are trivial, edge groups are infinite cyclic, and vertex groups are direct products of free groups. The construction of \( T \) is carried out in Section 2.

One can easily express the cellular chain complex of \( T \) in terms of permutation modules over \( \pi \) (Theorem 2.5). This can be used to derive negative results by producing explicit non-zero 2-cycles in \( T \). As a simple application we give another proof of Theorem 3.1 of [9] (Example 2.7). Radloff has used this technique to establish some new results along these lines [23].

We proceed in Section 3 to develop a combinatorial test for \( \mathcal{A} \) to be \( K(\pi, 1) \). This is based on a straightforward generalization of the condition of [30] for the total space of a "triangle group" to be aspherical. One assigns weights to the corners of the 2-cells of \( T \) so that, at each vertex, the sum of weights along any cycle of faces is at least two. Using the \( \pi \)-action it suffices to assign weights at the level of the orbit complex \( \Gamma \). The simple nature of the vertex groups makes it easy to determine the possible images in \( \Gamma \) of cycles of faces at vertices of \( T \) (Lemmas 3.2 and 3.6). These images are called full circuits in \( \Gamma \) relative to \( \mathcal{L} \). A system of weights on \( \mathcal{L} \) is \( \mathcal{L} \)-admissible if the sum of weights around any full circuit in \( \Gamma \) is at least 2. Then by Gersten's weight test [10] we conclude that \( T \) is aspherical provided \( \Gamma \) supports an \( \mathcal{L} \)-admissible system of weights such that the sum of weights around any \( d \)-gon is at most \( d - 2 \), for each \( d \) (Theorem 3.7).

In Corollary 3.10 we apply this result to a "balanced distribution" of weights to obtain a generalization of Hendriks' theorem [12]. In Theorem 3.12 we state L. Paris' recent result that factored complexified 3-arrangements are \( K(\pi, 1) \). Neither of these general results utilizes the full strength of the weight test. In Example 3.13 we construct an infinite family of (non-simplicial, non-supersolvable, non-factored) free arrangements, and prove they are all \( K(\pi, 1) \) using unbalanced systems of weights.

Our development depends on properties of the Salvetti complex \( S_0 \) and the fundamental group \( \pi \) of a complexified affine arrangement in \( \mathbb{C}^2 \). The construction of \( S_0 \) is described in Section 2. Properties of \( \pi \) are established as needed. A detailed exposition of the Salvetti complex of a general complexified arrangement may be found in [20]. In addition, both the complex \( S_0 \) and our modified version \( S \), as well as the group \( \pi \), are described and illustrated in [6].

2. THE UNIVERSAL COVER COMPLEX

Let \( \mathcal{A} \) be a complexified 3-arrangement. Thus \( \mathcal{A} \) consists of hyperplanes in \( \mathbb{C}^3 \) which have real defining forms. Fix \( H \in \mathcal{A} \) and let \( \mathcal{L} = \mathcal{L}_H \) denote the associated affine line arrangement in \( \mathbb{C}^2 \); \( \mathcal{L} \) is obtained by taking \( H \) to infinity in the projective plane and dehomogenizing the remaining hyperplanes in \( \mathcal{A} \). Note \( \mathcal{L} \) is also a complexified arrange-
ment. Let $C = C(\mathcal{A}) = C^2 \setminus \cup \mathcal{A}$ and $B = B(\mathcal{L}) = C^2 \setminus \cup \mathcal{L}$ denote the respective complements. Then $C$ is the total space of a trivial $C^*$-bundle over $B$, hence $C \cong C^* \times B$ [20].

Let $\tilde{C}$ denote universal cover. Then $\tilde{C} \cong C \times \tilde{B} \cong \tilde{B}$. We set $\pi = \pi_1(B)$. Then $\pi_1(C) \cong \mathbb{Z} \times \pi$. Note that the homotopy equivalence $\tilde{C} \to \tilde{B}$ is equivariant with respect to the action of $\pi_1(C)$, with $\mathbb{Z} \times \{1\}$ acting trivially on $\tilde{B}$. Finally, $C$ is aspherical if and only if $\pi_2(B) = 0$, since $B$ has the homotopy type of a 2-complex.

The purpose of this section is to construct a 2-complex $T$ which is homotopy equivalent to $\tilde{B}$ and whose structure can be understood in terms of the real part of $\mathcal{L}$. Our first task is to describe an explicit 2-complex $S$ with the homotopy type of $B$. This complex is obtained from the Salvetti complex $S_0$ of $\mathcal{L}$ by "blowing up" some vertices and edges. First we describe $S_0$.

Let $\mathcal{L}_R$ denote the real part of $\mathcal{A}$. Thus $\mathcal{L}_R$ is a collection of lines in the plane, determining a subdivision of the plane with vertices, edges (including rays), and (possibly unbounded) faces. These will be referred to as the vertices, edges, and faces of $\mathcal{L}_R$. The multiplicity of a vertex is the number of lines of $\mathcal{L}_R$ containing that vertex. The Salvetti complex $S_0$ of $\mathcal{L}$ is the regular 2-complex defined as follows. The vertices of $S_0$ are in one-to-one correspondence with the faces of $\mathcal{L}_R$. Vertices corresponding to adjacent faces (i.e., those sharing an edge of $\mathcal{L}_R$) are connected by a pair of edges in $S_0$. Each vertex $v$ of $\mathcal{L}_R$ of multiplicity $m$ gives rise to $2m$ 2-cells of $S_0$. In order to describe the attaching maps we orient the edges of $S_0$ so that the pair of edges connecting each pair of adjacent vertices receive opposite orientations. Then for each face $f$ of $\mathcal{L}_R$ incident with $v$, there is a 2-cell of $S_0$ whose boundary consists of the two minimal oriented edge-paths in $S_0$ proceeding from $f$ to the face opposite to $f$ at $v$. This complex was introduced by Salvetti in [28]. The following result is the case $\epsilon = 2$ of the main theorem of that paper.

**Theorem 2.1.** There is an embedding $S_0 \to B$ whose image is a strong deformation retract of $B$.

For our purposes we need to construct a modified version $S$ of the complex $S_0$. Let $\Gamma$ denote the planar 2-complex consisting of the bounded cells of $\mathcal{L}_R$. For each vertex $v$ of $\Gamma$ let $\mathcal{L}_v$ denote the set of lines in $\mathcal{L}$ containing $v$. Let $S_v$ be the Salvetti complex of $\mathcal{L}_v$, with a maximal tree collapsed to a point. Thus $S_v$ has a unique vertex. Let $C(\mathcal{L}_v)$ denote the complement of the central 2-arrangement $\mathcal{L}_v$.

If $e$ is an edge of $\Gamma$ incident with $v$, let $\alpha(e, v)$ be the corresponding loop in $S_v$. If $\epsilon \in \mathcal{L}$ is the affine span of $e$, then $\alpha(e, v)$ maps to generator of the infinite cyclic group $\pi_1(C^2 \setminus \epsilon)$ under the inclusion $C(\mathcal{L}_v) \subset C^2 \setminus \epsilon$.

To construct the complex $S$ start with $\Gamma$ and attach to each vertex $v$ a copy of $S_v$. Then to each edge $e = vw$ of $\Gamma$ attach an annulus $A(e)$ with boundary components identified with $\alpha(e, v)$ and $\alpha(e, w)$, such that $e$ forms a spanning arc of annulus. The resulting complex $S$ was used in [6], where it is denoted $\tilde{S}$. The reader will find in that reference a complete proof of the following assertion.

**Theorem 2.2.** $S$ is homotopy equivalent to $B$.

**Sketch of proof.** One recovers the Salvetti complex $S_0$ of $\mathcal{L}$ from $S$ by blowing up the vertices of the $S_v$ back to maximal trees, and then collapsing the faces of $\Gamma$ to vertices and the annuli $A(e)$ to loops.

Let $F_r$ denote the free group of rank $r$. 

\[\]
LEMMA 2.3. (i) \( S_v \) is an aspherical complex.
(ii) \( \pi_1(S_v) \cong F_1 \times F_{m-1} \), where \( m = |\mathcal{L}_v| \).
(iii) \( \pi_1(S_v) \to \pi_1(S) \) is injective.
(iv) \( \alpha(e, v) \) generates an infinite cyclic subgroup of \( S_v \).

Proof: Both (i) and (ii) hold because \( S_v \) is homotopy equivalent to the complement \( C(\mathcal{L}_v) \) of the central 2-arrangement \( \mathcal{L}_v \), and \( C(\mathcal{L}_v) \) is the total space of a trivial \( C^\ast \)-bundle over a plane with \( m - 1 \) punctures. The third assertion is proved using the “Brieskorn trick” [2, Lemma 3]: the composite \( S_v \to S \to B \subset C(\mathcal{L}_v) \) is a homotopy equivalence. The last item follows from our previous observation concerning \( \alpha(e, v) \). \( \square \)

Let \( \pi_v \) denote the image of \( \pi_1(S_v) \) in \( \pi_1 \). Thus \( \pi_v \cong F_1 \times F_{m-1} \), where \( m = m(v) = |\mathcal{L}_v| \) is the multiplicity of \( v \). Note that the images of \( \alpha(e, v) \) and \( \alpha(e, w) \) coincide in \( \pi_1 \); let \( \pi_v \) denote the infinite cyclic subgroup of \( \pi_1 \) generated by this element. If \( f \) is a face of \( \Gamma \) let \( \pi_f \) denote the trivial subgroup. Let \( \Gamma^{(i)} \) denote the \( i \)-skeleton of \( \Gamma \), for \( i = 0, 1, 2 \). We are now prepared to state and prove the main result of this section.

THEOREM 2.4. There exists a 2-dimensional \( CW \) complex \( T \) and a cellular action of \( \pi \) on \( T \) such that
(i) \( T \) is equivariantly homotopy equivalent to \( \tilde{B} \), and
(ii) the orbit complex \( T/\pi \) is isomorphic to \( \Gamma \).
Furthermore, if \( q : T \to \Gamma \) denotes the orbit map, then
(iii) \( q \) has a section, and
(iv) the isotropy subgroup corresponding to the cell \( \phi \) of \( T \) is conjugate to \( \pi_v \), where \( \phi = q(\phi) \).

Proof. Let \( p: \tilde{S} \to S \) denote the universal covering projection. By Lemma 2.1 \( \tilde{S} \cong \tilde{B} \). We may assume \( \tilde{S} \) is a \( CW \) complex and \( p \) is a cellular map. Let \( v \) be a vertex of \( \Gamma \), and consider \( p^{-1}(S_v) \). By Lemma 2.3(iii), this \( \pi \)-invariant subcomplex of \( \tilde{S} \) is a disjoint union of copies of \( \tilde{S}_v \). The components of \( \tilde{S}_v \) correspond to right cosets of \( \pi_v \) in \( \pi_1 \). Furthermore, by 2.2(i), \( \tilde{S}_v \) is contractible. Similarly, for each edge \( e \) of \( \Gamma \), \( p^{-1}(A(e)) \) is a \( \pi \)-invariant subcomplex of \( \tilde{S} \) consisting of a disjoint union of copies of \( \tilde{A}(e) \), one for each right coset of \( \pi_v \) in \( \pi_1 \). Note that \( \tilde{A}(e) \cong R \times [0, 1] \cong [0, 1] \). The desired complex \( T \) is obtained from \( \tilde{S} \) by simultaneously collapsing the components of \( \bigcup_{e \in \Gamma} p^{-1}(S_v) \) to points and the components of \( \bigcup_{e \in \Gamma} p^{-1}(A(e)) \) to arcs. These points and arcs become the vertices and edges of \( T \).

The covering group action of \( \pi \) on \( \tilde{S} \) induces an action of \( \pi \) on the quotient space \( T \). Since collapsing the \( S_v \) and \( A(e) \) in \( S \) (to points and arcs respectively) yields \( \Gamma \), the projection \( p \) induces a map \( q: T \to \Gamma \), which coincides with the orbit map of the \( \pi \)-action.

According to [1, Theorem 4.5.7], \( \Gamma \) is contractible. Thus \( \Gamma \) is a simply-connected subcomplex of \( S \), so we may construct a partial section \( \Gamma \to \tilde{S} \) of the covering map \( p \). Fix such a partial section to play the role of base point in \( \tilde{S} \). A vertex \( \tilde{v} \) of \( T \) is fixed by \( g \in \pi \) if and only if the corresponding component of \( p^{-1}(S_v) \) is invariant under \( g \), where \( v = q(\tilde{v}) \). This is the case precisely when \( g \in \gamma \pi \gamma^{-1} \), where \( \gamma \in \pi \) carries \( \tilde{v} \) back to the base region. The remainder of assertion (iv) holds by a similar argument. The partial section of \( p \) followed by the quotient map \( \tilde{S} \to T \) is a section of \( q \). \( \square \)

The construction of \( T \) in the preceding proof mirrors the construction of the tree associated with a graph of groups in Bass-Serre theory. The idea of mimicking this procedure with a 2-complex of groups is due to Gersten and Stallings [30]. It is quite
remarkable that this construct manifests itself naturally in our context. The first indication of this connection between line arrangements and graphs of groups is Randell's description of the homotopy type of $B_2$ [25], which motivated the modification $S$ of Salvetti's complex.

The statement of Theorem 2.4 is intended to both assert the existence of $T$ and describe its structure. Indeed, the conditions (ii) and (iv) determine $T$ uniquely. The picture is clarified further by the following corollary.

**Corollary 2.5.** The cellular chain complex of $\bar{S}$ is equivariantly chain homotopy equivalent to the complex of right $\mathbb{Z}[\pi]$-modules

$$C_i = C_i(\mathcal{L}) = \bigoplus_{\phi \in \Gamma^m} \mathbb{Z}[\pi/\pi_{\phi}],$$

with boundary maps $\partial : C_i \rightarrow C_{i-1}$, $i = 1, 2$ defined by extending the boundary maps in $\Gamma$ to $\mathbb{Z}[\pi]$-module homomorphisms.

**Proof.** By fixing a section of $q$ we may identify each $q^{-1}(\phi)$ with the set $\pi/\pi_{\phi}$ of right cosets. This gives an isomorphism $C_i(T) \cong C_i(\mathcal{L})$. The description of $\partial$ follows from an analysis of the quotient map $\bar{S} \rightarrow T$. 

**Corollary 2.6.** $\mathcal{A}$ is a $K(\pi, 1)$ arrangement if and only if the kernel of $\partial : C_2(\mathcal{L}) \rightarrow C_1(\mathcal{L})$ is trivial.

This last result yields an algebraic method of showing explicitly that a given arrangement is not $K(\pi, 1)$. This method is used in [23] to prove new results of this type. We demonstrate the technique with a simple example [9, Theorem 3.1].

**Example 2.7.** Suppose $\mathcal{L}$ has a triangular face $f$ each of whose vertices is contained in exactly two lines of $\mathcal{L}$. Then $\mathcal{A}$ is not $K(\pi, 1)$. To see this, let $e_i$, $i = 1, 2, 3$ denote the edges of $f$. Let $\alpha_i = \alpha(e_i)$. Since the vertices of $f$ are double points, we have $\alpha_i \alpha_j = \alpha_j \alpha_i$ (see [19] or

![Fig. 1](image-url)
3. A WEIGHT TEST FOR $K(n, 1)$ ARRANGEMENTS

Let $K$ be an arbitrary regular CW complex of dimension two. The 0-, 1-, and 2-cells of $K$ will be referred to as vertices, edges, and faces. The degree $d(f)$ of a face $f$ is the number of vertices (or, equivalently, the number of edges) in $f$. Define a corner of $K$ to be a flag $v < f$, with $v \in K^{(0)}$ and $f \in K^{(2)}$. The link graph $\Lambda_v$ of $K$ at $v$ is the graph whose vertices are the edges of $K$ incident with $v$ and whose edges are the corners of $K$ at $v$. The vertices of an edge $v < f$ of $\Lambda_v$ are the edges in $f$ incident with $v$. In our situation $\Lambda_v$ will have no loops or multiple edges. A circuit at $v$ is a closed walk in the link graph; circuits may have repeated edges or vertices. A circuit is reduced if no edge of $\Lambda_v$ (i.e. corner at $v$) appears twice in succession.

A system of weights on $K$ is a function from the set of corners of $K$ to the non-negative reals. A weight system is admissible if for every vertex of $K$, the sum of weights around any reduced circuit at $v$ is at least 2. A weight system is aspherical if, for any face $f$ of $K$, the sum of weights at the corners of $f$ is at most $d(f) - 2$.

Our main tool in this section will be the following result, proved by S. Gersten in [10].

**Theorem 3.1.** Suppose $K$ supports an admissible, aspherical system of weights. Then $K$ is aspherical.

**Sketch of proof.** One first proves that every map $S^2 \to K$ is homotopic to a sum of spherical diagrams. A spherical diagram is a map $S^2 \to K$ which is cellular with respect to some regular cell decomposition of $S^2$, with each cell mapping to a cell of the same dimension. This definition is more restrictive than the usual one, but suffices for our purposes. A spherical diagram is reducible if it contains a fold, that is, pair of adjacent 2-cells mapped to the same cell of $K$ with opposite orientations. Such a pair of faces can be eliminated up to homotopy. It is then shown that, under the hypotheses of the theorem, every spherical diagram over $K$ is reducible. In this case $K$ is said to be diagrammatically reducible, and is necessarily aspherical. To show $K$ is diagrammatically reducible one pulls back the system of weights on $K$ via a given spherical diagram $\Delta$ to a system of weights on a subdivision of $S^2$. This system is necessarily aspherical, but cannot be admissible by a simple euler characteristic argument. It follows that some circuit at some vertex of $S^2$ maps to a non-reduced circuit in $K$, yielding a fold in $\Delta$. 

In our situation, the 2-complex $K = T$ is simply-connected, so $\pi_2(T) \cong H_2(T)$. The spherical diagrams over $T$ correspond to spherical cellular homology classes which generate $H_2(T)$. The argument above gives an inductive proof that each of these classes is null-homologous.

The proof of Theorem 3.1. can be applied to more general, multi-valued systems of weights. This is the approach taken in [13] for instance. In the original preprint version of this paper we formulated such a generalized weight test for arrangements. However, technical problems involving composition of circuits make it virtually impossible at this point to produce new examples using this generalized weight test.
We now proceed to apply 3.1 to the universal cover complex $T$. This method is used in [30] to study properties of certain group presentations.

The crucial step in our application is the determination of the possible images under $q: T \to \Gamma$ of reduced circuits in $T$. This requires some analysis of the vertex groups $\pi_v$. The first lemma establishes the connection between circuits in $T$ and relations in $\pi_v$. This lemma is adapted from [30].

**Lemma 3.2.** Suppose $v \in T^{(0)}$ with $q(v) = v$. Then each circuit $\xi = (\varepsilon_0, \ldots, \varepsilon_n)$ in $T$ at $v$ gives rise to a relation $w_1 \ldots w_n = 1$ in $\pi_v$ satisfying $w_i \in \pi_\varepsilon$, for each $i$, where $e_i = q(\varepsilon_i)$. Conversely, each circuit $\zeta = (e_0, \ldots, e_n)$ in $\Gamma$ at $v$ and relation $w_1 \ldots w_n = 1$ with $w_i \in \pi_{e_i}$ yields a $\pi_v$-orbit of circuits $\xi = (\varepsilon_0, \ldots, \varepsilon_n)$ in $T$ at $v$ with $e_i = q(\varepsilon_i)$.

**Proof.** Fix a section of $q: T \to \Gamma$ and identify $\Gamma$ with its image. Suppose $\varepsilon_i \in \varepsilon$ and $\varepsilon_j \in \varepsilon'$ are corners of $T$ incident along an edge $\varepsilon$. Then there exist unique $\tau, \tau' \in \pi$ such that $f' \cdot \tau' \in \Gamma$ and $f'' \cdot \tau'' \in \Gamma$. Then $\varepsilon \cdot \tau = \varepsilon' \cdot \tau' \in \Gamma$ is fixed by $\tau^{-1} \tau'$, so $\tau^{-1} \tau' \in \pi_\varepsilon$. Thus $\tau' = \tau w$ for some $w \in \pi_\varepsilon$. Now apply this observation to the corners which occur in the circuit $\xi$. We write $v \in f_i$ for the corner connecting $\varepsilon_{i-1}$ and $\varepsilon_i$, and choose $\tau_i \in \pi$ such that $f_i \cdot \tau_i \in \Gamma$. Then for each $i = 1, \ldots, n$ there is an element $w_i \in \pi_\varepsilon$, such that $\tau_{i+1} = \tau_i w_i$. Since $\xi$ is a circuit, we have $\varepsilon_n = \varepsilon_0$, so $f_n$ and $f_1$ are adjacent along $\varepsilon_n$. Thus there exists $w_n \in \pi_\varepsilon$, such that $\tau_1 = \tau_n w_n$. Picturing all this together, we have $w_i \in \pi_{e_i}$, satisfying $\tau_1 = \tau_n w_n \ldots w_1$, or $w_1 \ldots w_n = 1$.

Clearly this process may be reversed to construct $\zeta$ from the image circuit $\zeta$ and the $w_i$. The fact that $w_1 \ldots w_n = 1$ will guarantee that $\zeta$ is a circuit. The only ambiguity arises from the choice or $\tau_1$, which is subject only to the condition that $v \cdot \tau_1 = \varepsilon$. So any two choices of $\tau_1$ will differ by an element of $\pi_v$.

**Remark 3.3.** The $w_i$ in the preceding result need not be non-trivial. However, an analysis of the proof will show that $\zeta$ is reduced if and only if there is no index $i$ such that $e_i = e_i$ (up to cyclic permutation) and $w_i = 1$.

Next we require a more detailed description of $\pi_v$. Define the **carrier** $\ell(e) \in L$ of an edge $e \in \pi$ to be the affine span of $e$. Recall $\varepsilon(e, v)$ is the loop in $S$ corresponding to $e$ and $L_v = \{ \ell \in L | v \in \ell \}$. Let $[x_1, \ldots, x_m]$ denote the set of relations $x_1 \ldots x_m = a_2 \ldots a_m x_1 = \ldots = x_m x_1 \ldots x_{m-1}$.

**Lemma 3.4.** Let $v$ be a vertex of $\Gamma$ of multiplicity $m$. Let $(e_1, \ldots, e_m)$ be a path in $\Lambda_v$ satisfying

$L_v = \{ \ell(e_i) | 1 \leq i \leq m \}$.

Then

(i) $\pi_v$ is given by the presentation $\langle x_1, \ldots, x_m | [x_1, \ldots, x_m] \rangle$, where $x_i = \varepsilon(e_i, v)$.

(ii) For any edge $e$ incident with $v$, $\varepsilon(e, v)$ is conjugate in $\pi_v$ to $x_i$ if and only if $\ell(e) = \ell(e_i)$.

**Proof.** Assertion (i) and half of (ii) derive from the Randell presentation of $\pi_v$ [24]; see also [6, Lemma 2.4]. The “only if” part of (ii) can be proved by abelianizing. The details are left to the reader.

**Lemma 3.5.** Let $m = |L_v|$ and let $e_j$, $1 \leq j \leq m - 1$, be any $m - 1$ edges incident with $v$ such that the carriers $\ell(e_j)$ are distinct. Then $\{\varepsilon(e_j) | 1 \leq j \leq m - 1\}$ generates a free subgroup of rank $m - 1$ in $\pi_v$. 


Proof. Let \( a_1, \ldots, a_n \) be generators of \( \pi \) as in Lemma 3.4. Let \( \alpha = \alpha(e_j, v) \). By 3.4 we may relabel the edges \( e_j \) so that \( \alpha \) is conjugate to \( a_j \). Let \( \zeta = a_1 \ldots a_n \). Note that \( \zeta \) is central and \( \pi / \langle \zeta \rangle \) is free of rank \( m - 1 \), generated by the images of \( a_1, \ldots, a_{m-1} \). Furthermore, the images of \( e_1, \ldots, e_{m-1} \) also generate this free group, since these elements generate a free subgroup by Schreier's Theorem and map to generators of the abelianization \( \mathbb{Z}^{m-1} \). It follows that \( e_1, \ldots, e_{m-1} \) generate a free subgroup of rank \( m - 1 \) in \( \pi_v \).

Lemma 3.6. Suppose \( w_1 \ldots w_n = 1 \) is a relation in \( \pi_v \), with each \( w_j \in \pi_{e_j} \), for some edge \( e_j \). Assume \( j \neq j+1 \) for \( 1 \leq j \leq n - 1 \) and that some \( w_j \neq 1 \). Then for each \( \ell \in \mathcal{L}_e \) there are indices \( j_1 \neq j_2 \) such that \( \ell'(e_j) = \ell' = \ell(e_j) \).

Proof. Note that \( w_j \) is a power of \( \alpha(e_j, v) \), non-trivial for some \( j \). It follows that each \( \ell \in \mathcal{L}_e \) occurs as some \( \ell'(e_j) \) with \( w_j \neq 1 \), by Lemma 3.5. Let \( \alpha \) denote the generator in the presentation of \( \pi_v \) corresponding to \( \ell' \). The stronger assertion of the lemma follows from the fact that the abelianization of \( \pi_v \) is free abelian with the image of \( \alpha \) as one generator. Thus we have that the exponent sum on \( \alpha \) in the word \( w_1 \ldots w_n \) must equal zero.

The last result tells us what the shape of \( q(\xi) \) can be, for reduced circuits \( \xi \) in \( T \). Let us define a circuit \( \zeta = (e_0, \ldots, e_n) \) at \( v = q(\xi) \) to be full relative to \( \mathcal{L} \) if for each \( \ell \in \mathcal{L}_e \), there exist indices \( j_1 \neq j_2 \) satisfying \( \ell'(e_{j_1}) = \ell' = \ell'(e_{j_2}) \). So, if \( \zeta \) is a reduced circuit at \( v \) in \( T \), then \( q(\zeta) \) is a full circuit at \( v = q(\xi) \) in \( \Gamma \).

We now define a system of weights on \( \Gamma \) to be \( \mathcal{L} \)-admissible if, for each vertex \( v \) of \( \Gamma \), the sum of weights around any full circuit at \( v \) is at least 2. Here finally is the main result of this section.

Theorem 3.7. Suppose \( \Gamma \) admits a system of weights which is \( \mathcal{L} \)-admissible and aspherical. Then \( \mathcal{A} \) is a \( K(\pi, 1) \) arrangement.

Proof. Given such a weight system on \( \Gamma \), define a weight system on \( T \) by assigning to the corner \( v \in \mathcal{F} \) the weight at the corner \( q(v) \in \mathcal{F} \) of \( \Gamma \). This system is admissible by Lemma 3.6, and is aspherical because \( q \) maps \( d \)-gons to \( d \)-gons. Thus \( T \) is aspherical by Theorem 3.1. From this it follows that the original central 3-arrangement \( \mathcal{A} \) is \( K(\pi, 1) \).

In order to apply Theorem 3.7 we will need a description of the minimal full circuits relative to \( \mathcal{L} \). Note that, for \( v \in \Gamma^{(0)} \) of multiplicity \( m \), the link graph \( \Lambda_v \) is either a cycle of length \( 2m \) or a disjoint union of paths of length at most \( 2m \). Only in rare cases will \( \Lambda_v \) be connected; see [36, Figure 3.11] for an example. For the purposes of describing full circuits in \( \Lambda_v \), we may assume without loss that \( \Lambda_v \) is connected. Let \( e_1, \ldots, e_k \) denote the vertices of \( \Lambda_v \), with \( e_i \) adjacent to \( e_{i+1} \) for \( 1 \leq i \leq k - 1 \), and \( e_k \) possibly adjacent to \( e_1 \) (only if \( k = 2m \)).

We list below the four types of minimal full circuits in \( \Lambda_v \):

(i) If \( \Lambda_v \) is a cycle, then \( \zeta = (e_1, \ldots, e_{2m}, e_1) \) is a full circuit.

(ii) If \( k > m \), then \( \zeta = (e_j, \ldots, e_{m+j-1}, e_{m+j+1}, \ldots, e_j) \) is a full circuit for \( 1 \leq j \leq k - m - 1 \).

(iii) \( \zeta = (e_j, \ldots, e_{m+j-1}, e_j, \ldots, e_{m+j}, \ldots, e_j) \) is a full circuit for \( 1 \leq j \leq k - m \).

(iv) \( \zeta = (e_j, e_{m+j}, e_{m+j+1}, e_{j+1}, e_{j+1}) \) is a full circuit for \( 1 \leq j \leq k - m \).

Here the ellipsis always stands for the minimal path connecting the indicated vertices. The reader may check that these circuits do indeed encounter each line of \( \mathcal{L} \) at
least twice. A circuit \( \xi \) is a sum of circuits \( \xi_1 \) and \( \xi_2 \) if \( \xi \) is obtained by concatenating \( \xi_1 \) and \( \xi_2 \). If \( \xi = \xi_1 + (\xi_2 + (\xi_3 + (\ldots + (\xi_{n-1} + \xi_n))) \) we say \( \xi \) contains \( \xi_i \) for \( 1 \leq i \leq n \).

**Proposition 3.8.** If \( v \in \Gamma^{(0)} \), then any full circuit in \( A_v \) contains a full circuit of one of the types (i)–(iv) listed above.

**Proof:** Suppose \( \xi \) is a full circuit at \( v \). If \( A = A_v \) is a cycle and \( \xi \) traverses all the edges of \( A \), then \( \xi \) contains a circuit of type (i). If \( A \) is a path, or if \( \xi \) does not involve all the edges of \( A \), we have two cases. If \( \xi \) contains two distinct vertices \( e_j \) and \( e_{n+j} \) having the same carrier \( l \in L \), then \( \xi \) contains a circuit of type (ii). Otherwise \( \xi \) involves exactly \( m = m(v) \) vertices, and each is visited at least twice. Without loss these vertices are \( e_1, \ldots, e_m \). The vertices \( e_1 \) and \( e_m \) each occur twice. Up to cyclic permutation these occurrences are seen in one of two sequences: \( e_1, \ldots, e_m, \ldots, e_1, \ldots \), or \( e_1, \ldots, e_1, \ldots, e_m, \ldots, e_m, \ldots \). In the first case \( \xi \) will contain the circuit of type (iii). In the second \( \xi \) will contain a circuit of type (iv).

By 3.8 a weight system will be \( L \)-admissible if and only if the weight sums around circuits of types (i)–(iv) are at least two.

**Corollary 3.9.** Suppose \( \xi \) is a full circuit in \( A_v \). If the components of \( A_v \) are paths of length \( m = m(v) \geq 3 \), then \( \xi \) has length at least \( 2(m + 1) \). In any case \( \xi \) has length at least \( 2m(v) \).

**Proof.** If the components of \( A_v \) are all paths of length \( m(v) \) vertices, then any full circuit \( \xi \) contains a circuit of type (iii) or (iv). In either case the interior edges are each traversed at least twice while the extreme edges are crossed four times. Thus the number of edges in \( \xi \) is at least \( 2(m - 3) + 4(2) = 2(m + 1) \). For the general case note that each circuit of type (i)–(iv) has length at least \( 2m(v) \).

One way of defining an \( L \)-admissible weight system is to assign to each corner of \( \Gamma \) at \( v \) the weight \( 2/c \), where \( c \) is the length of the shortest full circuit at \( v \). This type of "balanced distribution" leads to the following theorem. Let \( m(v) \) denote the multiplicity of \( v \) and \( d(v) \) the number of vertices in the largest component of \( A_v \). Usually \( d(v) \) will be the degree of \( v \) in the graph \( \Gamma^{(1)} \).

**Corollary 3.10.** Let \( \Omega \) be the weight system on \( \Gamma \) given by

\[
\Omega(v < f) = \begin{cases} 
0 & \text{if } d(v) < m(v) \\
\frac{1}{m(v) + 1} & \text{if } d(v) = m(v) \geq 3 \\
\frac{1}{m(v)} & \text{otherwise}. 
\end{cases}
\]

Suppose \( \Sigma_v \Omega(v < f) \leq d(f) - 2 \) for every \( f \in \Gamma^{(2)} \). Then \( \mathcal{A} \) is a \( K(\pi, 1) \) arrangement.

**Proof.** We need only show that \( \Omega \) is \( L \)-admissible, that is, the sum of weights around any full circuit at \( v \) is at least two, for each \( v \in \Gamma^{(0)} \). But this is clear from 3.9 if \( d(v) \geq m(v) \), and if \( d(v) < m(v) \) then there are no full circuits at \( v \).
Remark 3.11. If \( m(v) \geq 3 \) for all vertices \( v \) then the condition of Corollary 3.10 is automatically satisfied, since \( d(f) \geq 3 \) for all faces \( f \). No finite arrangements have this property. However, the methods of this paper can be applied to the complement of an infinite, locally finite arrangement, or the portion of such an arrangement over an open convex subset of \( \mathbb{R}^2 \). Thus 3.10 can be considered as a generalization of Hendrik's theorem [12] on hyperplane arrangements of large type, at least in dimension 2.

Corollary 3.10 can also be used to prove some finite 3-arrangements are \( K(\pi, 1) \). As an exercise the reader may apply this test to the simplicial arrangement with defining equation \( xyz(x + y + z)(x + y - z)(x - y + z)(x - y - z) \), which is illustrated as arrangement \( \mathcal{A} \) in [20, Figure 2.8].

A more general application of Theorem 3.7 and Proposition 3.8 was recently discovered by Paris [22]. An arrangement \( \mathcal{A} \) is said to be factored if the Orlik–Solomon algebra \( \mathcal{A} \), or equivalently, the cohomology algebra \( H^*(C(\mathcal{A})) \), is isomorphic as a \( \mathbb{Z} \)-module to an internal tensor product of submodules determined by a partition of \( \mathcal{A} \) [35, 20, 7]. A partition of \( \mathcal{A} \) yielding such an isomorphism is called a "nice partition" in [35, 20] and a "proper root coloring" in [7]. A partition of \( \mathcal{A} \) induces a factorization if and only if, for each vertex \( X \in L(\mathcal{A}) \), \( \mathcal{A} \) meets precisely \( r(X) \) blocks of the partition, and meets one of those blocks in a singleton [7]. For more information refer to [20, Section 3.3] or [22]. The characteristic polynomial of a factored arrangement has positive integer roots equal to the sizes of blocks in the associated partition. Every supersolvable arrangement is factored, but there are free arrangements which are not factored. It is not known if every factored arrangement (over \( \mathbb{C} \)) is free.

**Theorem 3.12** (Paris). If \( \mathcal{A} \) is a complexified factored 3-arrangement, then \( \mathcal{A} \) is \( K(\pi, 1) \).

The proof of this theorem is an easy exercise; one uses a weight system with all weights equal to 0 or 1/2.

Neither Theorem 3.12 nor the balanced distribution scheme of 3.10 utilizes the full strength of Theorem 3.7. We close this section with another new example which demonstrates the full power of the weight test.

**Example 3.13.** A free \( K(\pi, 1) \) arrangement which is neither supersolvable nor simplicial. Let \( \mathcal{A} \) be the arrangement with defining equation 
\[
(x + y)(x - y)(x - z)(y + z)(y - z)(x + 2z)(x - 2z)(y + 2z)(y - 2z).z.
\]
Then \( \mathcal{A} \) is free with exponents 1, 5, and 5, and is not known to be \( K(\pi, 1) \) by previous methods. The bounded complex \( \Gamma \) of \( \mathcal{A} \) relative to \( \{z = 0\} \) is illustrated below.

We define a system of weights on \( \Gamma \) as follows:
\[
p = r = u = v = \frac{1}{2}; \quad q = s = t = \frac{1}{4}
\]
with the remaining weights determined by symmetry. This weight system is clearly aspherical. Using Proposition 3.8, the reader may show that it is also \( \mathcal{L} \)-admissible. Therefore \( \mathcal{A} \) is a \( K(\pi, 1) \) arrangement.

Terao pointed out to us that the same method may be applied to an infinite family of free, non-simplicial, non-supersolvable arrangements. Let \( \mathcal{A}_p \) be the arrangement with defining equation 
\[
(x \pm y)(x \pm z)(y \pm x)(y \pm z).
\]
The deletion of the hyperplane \( x + y = 0 \) results in a supersolvable arrangement with exponents \((1, p, p + 1)\), so, by the addition-removal theorem [20, Theorem 4.51], \( \mathcal{A}_p \) is free with exponents \((1, p + 1, p + 1)\). The weight system defined for \( A_2 \) above may be propagated to produce an aspherical \( \mathcal{P} \)-admissible weight system for \( \mathcal{A}_p \) for any \( p \geq 2 \). Note that if the number of "horizontal lines" \( ky + z = 0 \) is not the same as the number of "vertical lines" \( kx + z = 0 \), the arrangement will be neither free nor \( K(\pi, 1) \).

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