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RAINBOW CONNECTION NUMBER OF SOME WHEEL-RELATED GRAPHS

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ABSTRACT: Let $f : E(G) \to \{1, 2, ..., k\}$ be an edge coloring of G, not necessarily proper. A path P in G is called a rainbow path if its edges have distinct colors. A graph G is said to be rainbow-connected, if every two distinct vertices of G is connected by a rainbow path. In this case, we say that f is a rainbow k-coloring of G. The smallest k such that G has a rainbow k-coloring is called the rainbow connection number of G, denoted by $rc(G)$.

This study gave the rainbow connection number of lotus inside a circle, helms and sunflower graphs.

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Key Words: rainbow connection number, rainbow path, sunflower, helms

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1. INTRODUCTION

The concept rainbow coloring was introduced by Chartrand et al. [1]. Since then the concept is studied by many authors.

Some authors studied the rainbow connection number of graphs belonging to some special classes, for example: Sy et al. [16] determined the rainbow connection number of fans and suns; As cited by Salman et al. [15], the rainbow connection numbers of (flower graphs) origami graphs, pizza graphs, n-crossed prism graphs, pencil graphs and rocket graphs and stellar graphs were given by Nabila et al. [23], Resty et al. [24], Simamora et al. [25] and Susilwati et al. [27]; Shulhany et al. [13] introduced a new class of graphs, namely stellar graphs, and determined the rainbow connection number of these graphs; Sy et al. [12] determined the rainbow connection number of Gear graphs, Book graphs and cycle-chain graph; Das et al. [5] also studied interval graph, asteroidal triple-free graphs, circular graphs, threshold graphs or chain graphs and bridgeless chordal graphs; Schiermeyer [3] studied the rainbow connection number of complete graphs; Ekstein et al. [11] studied the rainbow connection number of 2-connected graphs; Basavaraju et al. [21] and Li et al. [17] studied the rainbow connection number connected (bridgeless) graph; Li et al. [17] studied the rainbow connection number of line graphs of triangle free graphs and 2-connected triangle-free graphs according to their ear decomposition; Li et al. [22] gave an upperbound for the rainbow connection number of 3-connected graphs; And, Chartrand et al. [1] gave the rainbow connection number of cycles, wheels and complete k-partite graphs.

Others studied the rainbow connection number of graphs resulting from graph operations, for example: Gologranc et al. [10] studied the rainbow connection number on direct, strong and lexicographic product of graphs, and gave the (strong) rainbow connection number of the direct, strong, and lexicographic product; Li et al. [17] studied the rainbow connection number of product graphs including Cartesian product, composition and Join of graphs; Fitriani et al. [14] determined the rainbow connection number of the amalgamation of some graphs; Basavaraju et al. [4] and Basavaraju et al. [9] also studied the rainbow connection number of graphs resulting from Cartesian product of two graphs, the lexicographic product product of two graphs, the strong product of two graphs and the operations of taking the power of a graph according to the radius of graphs.

Bounds for the rainbow connection number were also given by several authors, for example: Krivelevich et al. [20] and Das et al. [5] gave some upperbounds for the rainbow connection number of a graph in terms of the minimum degree; Das et al. [5] gave some bounds for their rainbow connection number in terms of k-connectivity, connected domination number, diameter and radius; Schiermeyer [3] gave some upperbounds for the rainbow connection number in terms of the degree and in terms of the rainbow connection number of a contacted subgraph; And, Scheirmeyer [19] gave some upperbounds for the rainbow connection number of a graph in terms of its order.

Characterizations of the rainbow connection were also given, for example: Caro et al. [18] and Kemnitz et al. [2] gave some conditions that a graph must satisfy for it to have a rainbow connection number equal to 2; Kemnitz et al. [2] also characterized graphs G with rainbow connection number equal to 2; Chartrand et al. [1] characterize trees in terms of the rainbow connection number and order.

The following classes of graphs are found [30]. The *graph lotus inside circle*, denoted by LC_n, is the graph of order $2n+1$ obtained by joining each vertex u_i of the star $K_{1,n} = (\{u\}, \varnothing) + (\{u_1, u_2, \ldots, u_n\}, \varnothing)$ to vertices w_i and $w_{i+1(mod\, n)}$ of the cycle $C_n = [w_1, w_2, \ldots, w_n]$. The *Helm* H_n of order $2n+1$ is the graph obtained from $W_n = (\{u\}, \varnothing) + [w_1, w_2, \ldots, w_n]$ by attaching pendant edges $v_i w_i$ for every $i = 1, 2, \ldots, n$. The *sunflower* graph SF_n of order $2n+1$ obtained by adding a vertex w_i joined by edges to vertices v_i and $v_{i+1(mod\, n)}$ of the $W_n = (\{v\}, \varnothing) + [v_1, v_1, \ldots, v_n]$ for all $i = 1, 2, \ldots, n$.

Hereafter, please refer to [29] for concepts that were used but were not discussed in this paper.

2. RAINBOW CONNECTION NUMBER OF LOTUS INSIDE A CIRCLE GRAPH

This section gives the rainbow connection number of lotus inside a circle graph. Theorem 2.1 is due to Chartrand et al. [1]. It states that the rainbow connection number of a graph G is less than or equal its size, but is greater than or equal its diameter.

Theorem 2.1. *Let G be a graph of size m. Then*

$$diam(G) \leq rc(G) \leq m.$$

The next theorem gives the rainbow connection number of lotus inside circle graph LC_n.

Theorem 2.2. *Let $C_n = [w_1, w_2, \ldots, w_n]$ be a cycle of order n, and $K_{1,n} = (\{u\}, \varnothing) + (\{u_1, u_2, \ldots, u_n\}, \varnothing)$ be a star of order $n+1$. Let LC_n be the graph lotus inside circle obtained by joining each vertex u_i to vertices w_i and $w_{i+1(mod\ n)}$. Then*

$$rc(LC_n) = \begin{cases} 4, & \text{if } n \geq 7 \\ 3, & \text{if } n = 4, 5, 6 \\ 2, & \text{if } n = 3 \end{cases}$$

Proof. For $n = 3$, define $f : E(LC_3) \to \{1, 2\}$ as follows

$$\begin{array}{cccccc} w_1w_2 \mapsto 2, & w_2w_3 \mapsto 1, & w_1w_3 \mapsto 2, & u_1w_1 \mapsto 1, & u_1w_2 \mapsto 1, & u_2w_2 \mapsto 2 \\ u_2w_3 \mapsto 1, & u_3w_3 \mapsto 2, & u_3w_1 \mapsto 1, & uu_1 \mapsto 2, & uu_2 \mapsto 1, & uu_3 \mapsto 1 \end{array}$$

Then f is a rainbow 2-coloring of LC_3. Hence, $rc(LC_3) \leq 2$. Since the diameter of LC_3 is equal to 2, by Theorem 2.1, $2 \leq rc(LC_3)$. Accordingly, $rc(LC_3) = 2$. For $n = 4$, define $f : E(LC_4) \to \{1, 2, 3\}$ as follows

$$\begin{array}{cccccc} w_1w_2 \mapsto 1, & w_2w_3 \mapsto 2, & w_3w_4 \mapsto 1, & w_1w_4 \mapsto 2, & u_1w_1 \mapsto 1, & u_1w_2 \mapsto 1 \\ u_2w_2 \mapsto 2, & u_2w_3 \mapsto 2, & u_3w_3 \mapsto 1, & u_3w_4 \mapsto 1, & u_4w_4 \mapsto 2, & u_4w_1 \mapsto 2 \\ uu_1 \mapsto 2, & uu_2 \mapsto 1, & uu_3 \mapsto 1, & uu_4 \mapsto 3. & & \end{array}$$

Then f is a rainbow 3-coloring of LC_4. Hence, $rc(LC_4) \leq 3$. Suppose that $rc(LC_4) < 3$. Note that the path of length 2 connecting w_1 and w_3 is (w_1, w_2, w_3) and (w_1, w_4, w_3). Hence, one of these paths must be rainbow, say the former is rainbow, that is, without loss of generality, $w_1w_2 \mapsto 1$ and $w_2w_3 \mapsto 2$. The path of length 2 connecting w_2 and w_4 is (w_2, w_3, w_4) and (w_2, w_1, w_4). Hence, one of these paths must be rainbow, say without loss of generality the later is rainbow, that is, $w_1w_4 \mapsto 2$.

The path of length 2 connecting w_2 and u_3 is (w_2, w_3, u_3). Hence, $u_3w_3 \mapsto 1$. Similarly, the path of length 2 connecting w_2 and u_4 is (w_2, w_1, u_4). Hence, $w_1u_4 \mapsto 2$.

The path of length 2 connecting w_1 and u_2 is (w_1, w_2, u_2). Hence, $u_2w_2 \mapsto 2$. Similarly, the path of length 2 connecting w_1 and u_3 is (w_1, w_4, u_3). Hence, $u_3w_4 \mapsto 1$.

The path of length 2 connecting w_3 and u_1 is (w_3, w_2, u_1). Hence, $u_1w_2 \mapsto 1$. Similarly, the path of length 2 connecting w_4 and u_1 is (w_4, w_1, u_1). Hence, $u_1w_1 \mapsto 1$.

If $w_3w_4 \mapsto 2$, then $u_2w_3 \mapsto 1$ and $u_4w_4 \mapsto 1$. Thus, either $uu_3 \mapsto 2$ or $uu_4 \mapsto 2$. If $uu_3 \mapsto 2$, then $uu_1 \mapsto 1$. If $uu_1 \mapsto 1$, then $uu_2 \mapsto 1$. If $uu_2 \mapsto 1$, then $uu_4 \mapsto 2$. This is a contradiction since there is no rainbow path connecting u and w_1. On the other hand, if $uu_4 \mapsto 2$, then $uu_2 \mapsto 1$. If $uu_2 \mapsto 1$, then $uu_3 \mapsto 2$. If $uu_3 \mapsto 2$, then $uu_1 \mapsto 1$. This is a contradiction since there is no rainbow path connecting u and w_1.

If $w_3w_4 \mapsto 1$, then $u_2w_3 \mapsto 2$ and $u_4w_4 \mapsto 2$. If $uu_3 \mapsto 1$, then $uu_1 \mapsto 2$ and $uu_4 \mapsto 1$. If $uu_4 \mapsto 1$, then $uu_2 \mapsto 2$. This is a contradiction since there is no rainbow path connecting u and w_3.

Accordingly, $rc(LC_4) = 3$.

For $n = 5$, define $f : E(LC_5) \to \{1, 2, 3\}$ as follows

$$\begin{array}{cccccc} w_1w_2 \mapsto 2, & w_2w_3 \mapsto 1, & w_3w_4 \mapsto 3, & w_4w_5 \mapsto 2, & w_1w_5 \mapsto 1, & u_1w_1 \mapsto 3 \\ u_1w_2 \mapsto 2, & u_2w_2 \mapsto 3, & u_2w_3 \mapsto 2, & u_3w_3 \mapsto 3, & u_3w_4 \mapsto 3, & u_4w_4 \mapsto 1 \\ u_4w_5 \mapsto 3, & u_5w_5 \mapsto 1, & u_5w_1 \mapsto 2, & uu_1 \mapsto 1, & uu_2 \mapsto 1, & uu_3 \mapsto 2 \\ uu_4 \mapsto 2, & uu_5 \mapsto 3. & & & & \end{array}$$

Then f is a rainbow coloring of LC_5. Hence, $rc(LC_5) \leq 3$. Since the diameter of LC_5 is equal to 3, by Theorem 2.1, $3 \leq rc(LC_5)$. Accordingly, $rc(LC_5) = 3$. For $n = 6$, define $f : E(LC_6) \to \{1, 2, 3\}$ as follows

$$\begin{array}{cccccc} w_1w_2 \mapsto 1, & w_2w_3 \mapsto 3, & w_3w_4 \mapsto 2, & w_4w_5 \mapsto 1, & w_5w_6 \mapsto 3, & w_6w_1 \mapsto 2 \\ u_1w_1 \mapsto 1, & u_1w_2 \mapsto 1, & u_2w_2 \mapsto 3, & u_2w_3 \mapsto 3, & u_3w_3 \mapsto 2, & u_3w_4 \mapsto 2 \\ u_4w_4 \mapsto 1, & u_4w_5 \mapsto 1, & u_5w_5 \mapsto 3, & u_5w_6 \mapsto 3, & u_6w_6 \mapsto 2, & u_6w_1 \mapsto 2 \\ uu_1 \mapsto 2, & uu_2 \mapsto 2, & uu_3 \mapsto 3, & uu_4 \mapsto 3, & uu_5 \mapsto 1, & uu_6 \mapsto 1 \end{array}$$

Then f is a rainbow coloring of LC_6. Hence, $rc(LC_6) \leq 3$. Since the diameter of LC_6 is equal to 3, by Theorem 2.1, $3 \leq rc(LC_6)$. Accordingly, $rc(LC_6) = 3$.

For $n = 7$, define $f : E(LC_7) \to \{1, 2, 3, 4\}$ as follows

$$\begin{array}{cccccc} w_1w_2 \mapsto 1, & w_2w_3 \mapsto 1, & w_3w_4 \mapsto 1, & w_4w_5 \mapsto 1, & w_5w_6 \mapsto 1, & w_6w_7 \mapsto 1 \\ w_7w_1 \mapsto 1, & u_1w_1 \mapsto 4, & u_1w_2 \mapsto 4, & u_2w_2 \mapsto 3, & u_2w_3 \mapsto 3, & u_3w_3 \mapsto 4 \\ u_3w_4 \mapsto 4, & u_4w_4 \mapsto 3, & u_4w_5 \mapsto 3, & u_5w_5 \mapsto 4, & u_5w_6 \mapsto 4, & u_6w_6 \mapsto 3 \\ u_6w_7 \mapsto 3, & u_7w_7 \mapsto 4, & u_7w_1 \mapsto 4, & uu_1 \mapsto 1, & uu_2 \mapsto 2, & uu_3 \mapsto 1 \\ uu_4 \mapsto 2, & uu_5 \mapsto 1, & uu_6 \mapsto 2, & uu_7 \mapsto 3. & & \end{array}$$

Then f is a rainbow coloring of LC_7. Hence, $rc(LC_7) \leq 4$. Since the diameter of LC_7 is equal to 3, by Theorem 2.1, $3 \leq rc(LC_7)$. Thus, $3 \leq rc(LC_7) \leq 4$.

Suppose that $rc(LC_7) \leq 3$, say $rc(LC_7) = 3$. Let $f : E(LC_7) \to \{1, 2, 3\}$ be a rainbow coloring of LC_7. Without loss of generality, assume that $w_1w_2 \mapsto 1$. Since f is a rainbow coloring, either $w_2w_3 \mapsto 2$ and $w_7w_1 \mapsto 3$, or $w_2w_3 \mapsto 3$ and $w_7w_1 \mapsto 2$. Without loss of generality, assume that $w_2w_3 \mapsto 2$ and $w_7w_1 \mapsto 3$. Since f is a rainbow coloring, we must have $w_3w_4 \mapsto 3$, $w_4w_5 \mapsto 1$ and $w_5w_6 \mapsto 2$. Consider vertices w_1 and w_5. Note that the only path of length 3 connecting the two is (w_5, w_6, w_7, w_1). Hence, $w_6w_7 \mapsto 1$. Now, consider vertices w_2 and w_6. Note that the only path of length 3 connecting the two is (w_2, w_1, w_7, w_6) and this path is not a rainbow path. This is a contradiction. Hence, we must have $rc(LC_7) = 4$.

For $n \geq 8$ and n is even, define $f_1 : E(LC_n) \to \{1, 2, 3, 4\}$ as follows

$$f_1(e) = \begin{cases} 1, & \text{if } e = w_iw_{i+1(mod\, n)} \ \forall i, \text{ or } e = uu_i \text{ with } i \equiv 1(mod\, 2) \\ 2, & \text{if } e = uu_i \text{ with } i \equiv 0(mod\, 2) \\ 3, & \text{if } e = u_iw_i \text{ or } e = u_iw \text{ with } i \equiv 1(mod\, 2) \\ 4, & \text{if } e = u_iw_i \text{ or } e = u_iw_{i+1(mod\, n)} \text{ with } i \equiv 0(mod\, 2) \end{cases}$$

Let $w, v \in V(LC_n)$ and consider the following cases:

Case 1. $deg_{LC_n}(w) = 4$

If $deg_{LC_n}(w) = 4$, then $w = w_i$ for some $i = 1, 2, \ldots, n$. Consider the following subcases:

Subcase 1. $deg_{LC_n}(v) = 4$

If $deg_{LC_n}(v) = 4$, then $v = w_j$ for some $j = 1, 2, \ldots, n$ with $j \neq i$. Note that if i and j have the same parity, then $(w_i, u_i, u, u_{j-1}, w_j,)$ is a rainbow path connecting w and v, and if i and j have the different parity, then $(w_i, u_i, u, u_j, w_j,)$ is a rainbow path connecting w and v.

Subcase 2. $deg_{LC_n}(v) = 3$

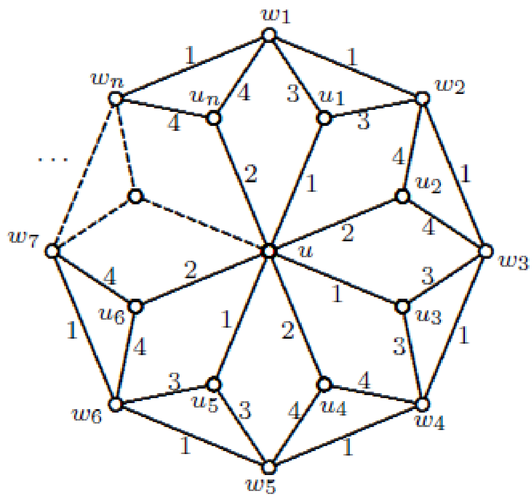

Figure 1: The function f_1 in LC_n

If $deg_{LC_n}(v) = 3$, then $v = u_j$ for some $j = 1, 2, \ldots, n$. If $i = j - 1$, then (u_i, w_j) is a rainbow path connecting u and v. If $i = j$, then (u_i, w_j) is a rainbow path connecting w and v. If i and j have the same parity and $j \neq j - 1, j$, then (w_i, u_{i-1}, u, u_j) is a rainbow path connecting w and v, and if i and j have the different parity and $j \neq j - 1, j$, then (w_i, u_i, u, u_j) is a rainbow path connecting w and v.

Subcase 3. $deg_{LC_n}(v) = n$

If $deg_{LC_n}(v) = n$, then $v = u$. Note that (w_i, u_i, u) is a rainbow path connecting w and v.

Case 2. $deg_{LC_n}(w) = 3$

If $deg_{LC_n}(w) = 3$, then $w = u_i$ for some $i = 1, 2, \ldots, n$. Consider the following subcases:

Subcase 1. $deg_{LC_n}(v) = 3$

If $deg_{LC_n}(v) = 3$, then $v = u_j$ for some $j = 1, 2, \ldots, n$. If i and j have the same parity, then $(u_i, u, u_{j+1}, w_{j+1}, u_j)$ is a rainbow path connecting w and v. If i and j have the different parity, then (u_i, u, u_j) is a rainbow path connecting w and v.

Subcase 2. $deg_{LC_n}(v) = n$

If $deg_{LC_n}(v) = n$, then $v = u$. Note that (u_i, u) is a rainbow path connecting w and v.

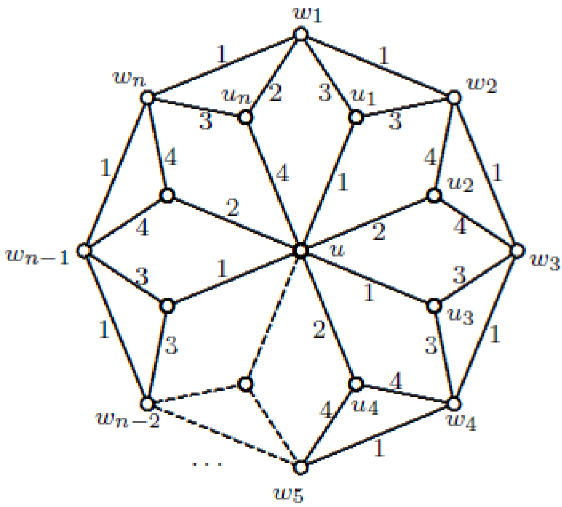

Figure 2: The function f_2 in LC_n

Hence, f is a rainbow 4-coloring of LC_n. Thus, $rc(LC_n) \leq 4$. Since the diameter of LC_n is equal to 4, by Theorem 2.1, $4 \leq rc(LC_n)$. Therefore, $rc(LC_n) = 4$ for $n \geq 8$ and n is even.

For $n \geq 9$ and n is odd, define $f_2 : E(LC_n) \to \{1, 2, 3, 4\}$ as follows

$$f_2(e) = \begin{cases} 1, & \text{if } e = w_i w_{i+1(mod\, n)}\ \forall i, \text{ or } e = uu_i \text{ with } i \equiv 1(mod\, 2) \\ & \text{with } i \neq n \\ 2, & \text{if } e = uu_i \text{ with } i \equiv 0(mod\, 2), \text{ or } e = u_n w_1 \\ 3, & \text{if } e = u_i w_i \text{ or } e = u_i w_{i+1(mod\, n)} \text{ with } i \equiv 1(mod\, 2) \text{ and } i \neq n, \\ & \text{or } e = u_n w_n \\ 4, & \text{if } e = u_i w_i \text{ or } e = u_i w_{i+1(mod\, n)} \text{ with } i \equiv 0(mod\, 2) \text{ or } i = n, \\ & \text{or } e = uu_n \end{cases}$$

Let $w, v \in V(LC_n)$ and consider the following cases:

Case 1. $deg_{LC_n}(w) = 4$

If $deg_{LC_n}(w) = 4$, then $w = w_i$ for some $i = 1, 2, \ldots, n$. Now, consider the following subcases:

Subcase 1. $i = n$

If $i = n$, then consider the following sub-subcases:

Sub-subcase 1. $v = w_1$ or $v = w_{n-1}$

If $v = w_1$, then $(w_1, w_n,)$ is a rainbow path connecting w and v. Similarly, if $v = w_{n-1}$, then $(w_{n-1}, w_n,)$ is a rainbow path connecting w and v.

Sub-subcase 2. $v = w_k$ with $2 \leq k \leq n-2$

If k is even, then $(w_n, u_{n-1}, u, u_{k-1}, w_k)$ is a rainbow path connecting w and v. If k is odd, then $(w_1, u_1, u, u_{k-1}, w_k)$ is a rainbow path connecting w and v.

Sub-subcase 3. $deg(v) = 3$

If $deg(v) = 3$, then $v = u_j$ for some $j = 1, 2, \ldots, n$. If $i = n$, then $(w_n, u_n,)$ is a rainbow path connecting w and v. If $i = n-1$, then $(w_n, u_{n-1},)$ is a rainbow path connecting w and v. If $1 \leq i \leq n-2$, then (w_n, u_n, u, u_i) is a rainbow path connecting w and v.

Subcase 2. $i \neq n$

If $i \neq n$, then consider the following sub-subcases:

Sub-subcase 1. $v = w_k$ with $k \neq n, j$

If k and i have the same parity, then $(w_i, u_i, u, u_{k-1}, w_k)$ is a rainbow path connecting w and v. If k and i have the different parity, then (w_i, u_i, u, u_k, w_k) is a rainbow path connecting w and v.

Sub-subcase 2. $v = u_j$ with $j = 1, 2, \ldots, n$

If $v = u_i$ for some $i = 1, 2, \ldots, n$, then either (w_i, u_{i-1}, u, u_j) or (w_i, u_i, u, u_j) is a rainbow path connecting w and v.

Sub-subcase 3. $v = u$

If $v = u$, then (w_n, u_n, u) is a rainbow path connecting w and v.

Case 2. $deg_{LC_n}(w) = 3$

If $deg_{LC_n}(w) = 3$, then $w = u_i$ for some $i = 1, 2, \ldots, n$. Consider the following subcases:

Subcase 1. $i = n$

If $i = n$, then consider the following sub-subcases.

Sub-subcase 1. $v = u_k$ with $k \neq n$

If $v = u_k$ with $k \neq n$, then (u_n, u, u_k) is a rainbow path connecting w and v.

Sub-subcase 2. $v = u$

If $v = u$, then (u_n, u) is a rainbow path connecting w and v.

Subcase 2. $i \neq n$

If $i \neq n$, then consider the following sub-subcases.

Sub-subcase 1. $v = u_k$ for some $k \neq i$

If k and i have the same parity, then $(u_i, u, u_{k-1}, w_k, u_k)$ is a rainbow path connecting w and v. If k and i have different parity, then (u_i, u, u_k) is a rainbow path connecting w and v.

Sub-subcase 2. $v = u$

If $v = u$, then (u_i, u) is a rainbow path connecting w and v.

Hence, f_2 is a rainbow 4-coloring of LC_n. Thus, $rc(LC_n) \leq 4$. Since the diameter of LC_n is equal to 4, by Theorem 2.1, $4 \leq rc(LC_n)$. Therefore, $rc(LC_n) = 4$ for $n \geq 9$ and n is odd. □

3. RAINBOW CONNECTION NUMBER OF HELM GRAPH

This section gives an upperbound of the rainbow connection number of helm graph. Theorem 3.1 is due to Chartrand et al. [1].

Theorem 3.1. [1] *Let $W_n = (\{v\}, \varnothing) + [v_1, v_1, \ldots, v_n]$ be a wheel of order $n+1$. Then*

$$rc(W_n) \leq \begin{cases} 1, & \text{if } n = 3 \\ 2, & \text{if } n = 4, 5, 6 \\ 3, & \text{if } n \geq 7 \end{cases}$$

Theorems 3.2, 3.3 and 3.4 are due to Schiermeyer [3]. We shall be needing Theorem 3.1, 3.2 and 3.4 in the proof Theorem 3.6.

Theorem 3.2. [3] *Let G be a graph and H be a subgraph of G. Let G' be the graph obtained from G by contracting H to a singe vertex. Then*

$$rc(G) \leq rc(G') + rc(H)$$

.

Theorem 3.3. [3] *Let G be a connected graph. Then $rc(G)$ is greater than or equal the number of end-vertices.*

Theorem 3.4. [3] *Let G be a graph of order n. Then $rc(G) = n - 1$ if and only if G is a tree.*

The next theorem gives the rainbow connection number of helm H_n for $n = 3, 4, 5, 6$.

Theorem 3.5. *Let $n \leq 6$ and H_n be the Helm obtained from a wheel $W_n = (\{u\}, \varnothing) + [w_1, w_2, \ldots, w_n]$ by attaching pendant edges $v_i w_i$ with $i = 1, 2, \ldots, n$. Then $rc(H_n) = n$.*

Proof. For $n = 3$, define $f : E(H_3) \to \{1, 2, 3\}$ as follows

$$\begin{array}{cccccc} w_1w_2 \mapsto 1, & w_2w_3 \mapsto 2, & w_1w_3 \mapsto 3, & v_1w_1 \mapsto 2, & v_2w_2 \mapsto 3, & v_3w_3 \mapsto 1 \\ uw_1 \mapsto 1, & uw_2 \mapsto 1, & uw_3 \mapsto 2. & & & \end{array}$$

Then f is a rainbow 3-coloring of H_3. Hence, $rc(H_3) \leq 3$. But by Theorem 3.3, $3 \leq rc(H_3)$. Therefore, $rc(H_3) = 3$. For $n = 4$, define $f : E(H_4) \to \{1, 2, 3, 4\}$ as follows

$$\begin{array}{cccccc} w_1w_2 \mapsto 4, & w_2w_3 \mapsto 1, & w_3w_4 \mapsto 1, & w_1w_4 \mapsto 3, & v_1w_1 \mapsto 1, & v_2w_2 \mapsto 2 \\ v_3w_3 \mapsto 3, & v_4w_4 \mapsto 4, & uw_1 \mapsto 2, & uw_2 \mapsto 1, & uw_3 \mapsto 4, & uw_4 \mapsto 3 \end{array}$$

Then f is a rainbow 4-coloring of H_4. Hence, $rc(H_4) \leq 4$. But by Theorem 3.3, $4 \leq rc(H_4)$. Therefore, $rc(H_4) = 4$.

For $n = 5$, define $f : E(H_5) \to \{1, 2, 3, 4, 5\}$ as follows

$$\begin{array}{cccccc} w_1w_2 \mapsto 5, & w_2w_3 \mapsto 4, & w_3w_4 \mapsto 1, & w_4w_5 \mapsto 2, & w_1w_5 \mapsto 3, & v_1w_1 \mapsto 1 \\ v_2w_2 \mapsto 2, & v_3w_3 \mapsto 3, & v_4w_4 \mapsto 4, & v_5w_5 \mapsto 5, & uw_1 \mapsto 3, & uw_2 \mapsto 3 \\ uw_3 \mapsto 1, & uw_4 \mapsto 5, & uw_5 \mapsto 4. & & & \end{array}$$

Then f is a rainbow coloring of H_5. Hence, $rc(H_5) \leq 5$. But by Theorem 3.3, $5 \leq rc(H_5)$. Therefore, $rc(H_5) = 5$. For $n = 6$, define $f : E(H_6) \to \{1, 2, 3, 4, 5, 6\}$ as follows

$$\begin{array}{cccccc} w_1w_2 \mapsto 5, & w_2w_3 \mapsto 6, & w_3w_4 \mapsto 1, & w_4w_5 \mapsto 2, & w_5w_6 \mapsto 3, & w_1w_6 \mapsto 4 \\ v_1w_1 \mapsto 1, & v_2w_2 \mapsto 2, & v_3w_3 \mapsto 3, & v_4w_4 \mapsto 4, & v_5w_5 \mapsto 5, & v_6w_6 \mapsto 6 \\ uw_1 \mapsto 2, & uw_2 \mapsto 3, & uw_3 \mapsto 4, & uw_4 \mapsto 5, & uw_5 \mapsto 6, & uw_6 \mapsto 1 \end{array}$$

Then f is a rainbow coloring of H_6. Hence, $rc(H_6) \leq 6$. But by Theorem 3.3, $6 \leq rc(H_6)$. Therefore, $rc(H_6) = 6$. □

The next theorem gives an upper-bound of the rainbow connection number of helm H_n for $n > 6$. We conjecture that this bound is tight.

Theorem 3.6. *Let $n \geq 7$ and H_n be the Helm obtained from a wheel $W_n = (\{u\}, \varnothing) + [w_1, w_2, \ldots, w_n]$ by attaching pendant edges $v_i w_i$ with $i = 1, 2, \ldots, n$. Then $rc(H_n) \leq n + 3$.*

Proof. Let $n \geq 7$ and H_n be the Helm obtained from a wheel $W_n = (\{u\}, \varnothing) + [w_1, w_2, \ldots, w_n]$ by attaching pendant edges $v_i w_i$ with $i = 1, 2, \ldots, n$. Let $H = W_n$. Then $G' = K_{1,n}$. By Theorem 3.1 $rc(H) = 3$, and by Theorem 3.4 $rc(G') = n$. Thus, by Theorem 3.2

$$rc(G) \leq rc(G') + rc(H) = n + 3$$

. □

4. RAINBOW CONNECTION NUMBER OF SUNFLOWER GRAPH

This section gives the rainbow connection number of sunflower graph.

Theorem 4.1. *Let $W_n = (\{u\}, \varnothing) + [u_1, u_1, \ldots, u_n]$ be a wheel of order $n+1$, and SF_n be the sunflower graph obtained by adding a vertex w_i joined by edges to vertices u_i and $u_{i+1(mod\, n)}$ for every $i = 1, 2, \ldots, n$. Then*

$$rc(SF_n) = \begin{cases} 2, & if\ n = 3 \\ 3, & if\ n = 4, 5 \\ 4, & if\ n \geq 6 \end{cases}$$

Proof. For $n = 3$, we define $f : E(SF_3) \to \{1, 2\}$ as follows

$$\begin{array}{cccccc} u_1w_1 \mapsto 1, & u_2w_1 \mapsto 2, & u_2w_2 \mapsto 1, & u_3w_2 \mapsto 2, & u_3w_3 \mapsto 1, & u_1w_3 \mapsto 2 \\ u_1u_2 \mapsto 2, & u_2u_3 \mapsto 2, & u_3u_1 \mapsto 2, & uu_1 \mapsto 2, & uu_2 \mapsto 2, & uu_3 \mapsto 2 \end{array}$$

Then f is a rainbow 2-coloring of SF_3. Hence, $rc(SF_3) \leq 2$. Since the diameter of SF_3 is equal to 2, by Theorem 2.1, $2 \leq rc(SF_3)$. Accordingly, $rc(SF_3) = 2$. For $n = 4$, define $f : E(SF_4) \to \{1, 2, 3\}$ as follows

$$\begin{array}{cccccc} u_1w_1 \mapsto 1, & u_2w_1 \mapsto 2, & u_2w_2 \mapsto 1, & u_3w_2 \mapsto 3, & u_3w_3 \mapsto 1, & u_4w_3 \mapsto 2 \\ u_4w_4 \mapsto 1, & u_1w_4 \mapsto 3, & u_1u_2 \mapsto 2, & u_2u_3 \mapsto 3, & u_3u_4 \mapsto 2, & u_4u_1 \mapsto 2 \\ uu_1 \mapsto 1, & uu_2 \mapsto 1, & uu_3 \mapsto 2, & uu_4 \mapsto 3. & & \end{array}$$

Then f is a rainbow 3-coloring of SF_4. Hence, $rc(SF_4) \leq 3$.

Accordingly, $rc(SF_4) = 3$.

For $n = 5$, define $f : E(SF_5) \to \{1, 2, 3\}$ as follows

$$\begin{array}{cccccc} u_1w_1 \mapsto 3, & u_2w_1 \mapsto 1, & u_2w_2 \mapsto 3, & u_3w_2 \mapsto 1, & u_3w_3 \mapsto 3, & u_4w_3 \mapsto 1 \\ u_4w_4 \mapsto 3, & u_5w_4 \mapsto 1, & u_5w_5 \mapsto 2, & u_1w_5 \mapsto 1, & u_1u_2 \mapsto 2, & u_2u_3 \mapsto 2 \\ u_3u_4 \mapsto 2, & u_4u_5 \mapsto 3, & u_5u_1 \mapsto 2, & uu_1 \mapsto 3, & uu_2 \mapsto 3, & uu_3 \mapsto 2 \\ uu_4 \mapsto 2, & uu_5 \mapsto 2. & & & & \end{array}$$

Then f is a rainbow 3-coloring of SF_5. Hence, $rc(SF_5) \leq 3$. Since the diameter of SF_5 is equal to 3, by Theorem 2.1, $3 \leq rc(SF_5)$. Accordingly, $rc(SF_5) = 3$.

If $n \geq 6$ is even, then we define $f : E(SF_n) \to \{1, 2, 3, 4\}$ as follows

$$f(e) = \begin{cases} 1, & \text{if } e = uu_i \text{ with } i \equiv 1(mod\,2), \text{ or } e = u_iu_{i+1(mod\,n)} \text{ for all } n \in \mathbb{N} \\ 2, & \text{if } e = uu_i \text{ with } i \equiv 0(mod\,2) \\ 3, & \text{if } e = u_iw_i \text{ or } e = u_iw_{i-1(mod\,n)} \text{ with } i \equiv 1(mod\,2) \\ 4, & \text{if } e = u_iw_i \text{ or } e = u_iw_{i-1} \text{ with } i \equiv 0(mod\,2) \end{cases}$$

Let $w, v \in V(SF_n)$ and consider the following cases:

Case 1. $deg_{SF_n}(w) = 2$

If $deg_{SF_n}(u) = 2$, then $w = w_i$ for some $i = 1, 2, \ldots, n$. Now, consider the following subcases:

Subcase 1. $deg_{SF_n}(v) = 2$

If $deg_{SF_n}(v) = 2$, then $v = w_j$ for some $j = 1, 2, \ldots, n$ with $j \neq i$. Note that either $\left(w_i, u_{i+1(mod\,n)}, u, u_{j+1}, w_j\right)$ or $(w_i, v_i, v, v_{j+1}, w_j)$ is a rainbow path connecting w and v.

Subcase 2. $deg_{SF_n}(v) = 5$

If $deg_{SF_n}(v) = 5$, then $v = u_j$ for some $j = 1, 2, \ldots, n$. Note that either $\left(w_i, u_{i+1(mod\,n)}, u, u_j\right)$ or (w_i, u_i, u, u_j) is a rainbow path connecting w and v.

Subcase 3. $deg_{SF_n}(v) = n$

If $deg_{SF_n}(v) = n$, then $v = u$. Note that (w_i, u_i, u) is a rainbow path connecting w and v.

Case 2. $deg_{SF_n}(w) = 5$

If $deg_{G_n}(w) = 3$, then $w = u_i$ for some $i = 1, 2, \ldots, n$. Consider the following subcases:

Subcase 1. $deg_{SF_n}(v) = 5$

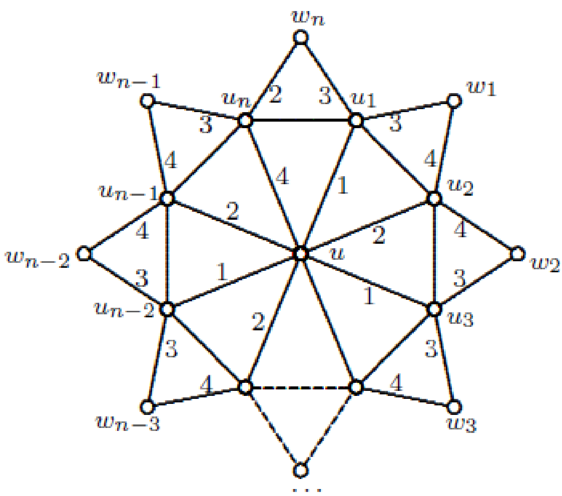

Figure 3: The function f_2 in SF_n

If $deg_{SF_n}(v) = 5$, then $v = u_j$ for some $j = 1, 2, \ldots, n$ with $j \neq i$. Note that either $\left(u_i, u, u_{i-1(mod\, n)}, w_{i-1(mod\, n)}, u_j\right)$ or (u_i, u, u_j) is a rainbow path connecting w and v.

Subcase 2. $deg_{SF_n}(v) = n$

If $deg_{SF_n}(v) = n$, then $v = u$. Note that (u_i, u) is a rainbow path connecting w and v.

Hence, f is a rainbow coloring of SF_n. Thus, $rc(SF_n) \leq 4$. Since the diameter of SF_n is equal to 4 if $n \geq 6$, by Theorem 2.1, $4 \leq rc(G_n)$. Therefore, $rc(SF_n) = 4$ if n is even.

If n is odd, then we define $f : E(SF_n) \to \{1, 2, 3, 4\}$ as follows

$$f(e) = \begin{cases} 1, & \text{if } e = uu_i \text{ with } i \equiv 1(mod\, 2) \text{ and } i \neq n, \\ & \text{or } e = u_i u_{i+1(mod\, n)} \text{ for all } n \in \mathbb{N} \\ 2, & \text{if } e = uu_i \text{ with } i \equiv 0(mod\, 2), \text{ or } e = u_n w_n \\ 3, & \text{if } e = u_i w_i \text{ or } e = u_i w_{i-1(mod\, n)} \text{ with } i \equiv 1(mod\, 2) \text{ and } i \neq n, \\ & \text{or } e = u_n w_{n-1} \\ 4, & \text{if } e = u_i w_i \text{ or } e = u_i w_{i-1} \text{ with } i \equiv 0(mod\, 2), \text{ or } e = uu_n \end{cases}$$

Let $w, v \in V(SF_n)$ and consider the following cases:

Case 1. $deg_{SF_n}(w) = 2$

If $deg_{SF_n}(w) = 2$, then $w = w_i$ for some $i = 1, 2, \ldots, n$. Now, consider the following subcases:

Subcase 1. $i = n - 1$

If $i = n - 1$, then consider the following sub-subcases:

Sub-subcase 1. $deg_{SF_n}(v) = 2$

If $deg_{SF_n}(v) = 2$, then $v = w_j$ for some $j \neq n - 1$. Note that either $\big(w_{n-1}, u_{n-1}, u, u_{j-1(mod\, n)}, w_j\big)$ with $j-1 \equiv 1(modn)$, or $(w_{n-1}, u_{n-1}, u, u_j, w_j)$ with $j \equiv 1(mod\, n)$ is a rainbow path connecting w and v.

Sub-subcase 2. $deg_{SF_n}(v) = 5$

If $deg_{SF_n}(v) = 5$, then $v = u_j$ for some $j = 1, 2, \ldots, n$. Note that (w_{n-1}, u_n, u, u_j) is a rainbow path connecting w and v.

Sub-subcase 3. $deg_{SF_n}(v) = n$

If $deg_{SF_n}(v) = n$, then $v = u$. Note that (w_{n-1}, u_n, u) is a rainbow path connecting w and u.

Subcase 2. $i \neq n - 1$

If $i \neq n - 1$, then consider the following sub-subcases:

Sub-subcase 1. $deg_{SF_n}(v) = 2$

If $deg_{SF_n}(v) = 2$, then $v = w_j$ for some $j \neq i$ and $j \neq n - 1$. Note that either $\big(w_i, u_{i+1(mod\, n)}, u, u_{j+1}, w_j\big)$ or $(w_i, u_i, u, u_{j+1}, w_j)$ is a rainbow path connecting w and v.

Sub-subcase 2. $deg_{SF_n}(v) = 5$

If $deg_{SF_n}(v) = 5$, then $v = u_j$ for some $j = 1, 2, \ldots, n$. Note that either $\big(w_i, u_{i+1(mod\, n)}, u, u_j\big)$ or (w_i, u_i, u, u_j) is a rainbow path connecting w and v.

Sub-subcase 3. $deg_{SF_n}(v) = n$

If $deg_{SF_n}(v) = n$, then $v = u$. Note that (w_i, u_i, u) is a rainbow path connecting w and v.

Case 2. $deg_{SF_n}(w) = 5$

If $deg_{SF_n}(w) = 3$, then $w = v_i$ for some $i = 1, 2, \ldots, n$. Consider the following subcases:

Subcase 1. $i = n$

If $i = n$, then consider the following subcases:

Sub-subcase 1. $deg_{SF_n}(v) = 5$

If $deg_{SF_n}(v) = 5$, then $v = u_j$ for some $j \neq n$. Note that (u_n, u, u_j) is a rainbow path connecting w and v.

Sub-subcase 2. $deg_{SF_n}(v) = n$

If $deg_{SF_n}(v) = n$, then $v = u$. Note that (u_n, u) is a rainbow path connecting w and v.

Subcase 2. $i \neq n$

If $i \neq n$, then consider the following subcases:

Sub-subcase 1. $deg_{SF_n}(v) = 5$

If $deg_{SF_n}(v) = 5$, then $v = u_j$ for some $j \neq n, i$. Note that (u_i, u, u_j) is a rainbow path connecting w and v.

Sub-subcase 2. $deg_{SF_n}(v) = n$

If $deg_{SF_n}(v) = n$, then $v = u$. Note that (u_i, u) is a rainbow path connecting w and v.

Hence, f is a rainbow 4-coloring of SF_n. Thus, $rc(SF_n) \leq 4$. Since the diameter of SF_n is equal to 4 if $n \geq 7$, by Theorem 2.1, $4 \leq rc(G_n)$. Therefore, $rc(SF_n) = 4$ also if n is odd. □

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