Isometric embeddings of subdivided connected graphs into hypercubes

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Abstract

Isometric subgraphs of hypercubes are known as partial cubes. These graphs have first been investigated by Graham and Pollack [R.L. Graham, H. Pollack, On the addressing problem for loop switching, Bell System Technol. J. 50 (1971) 2495–2519; and D. Djoković, Distance preserving subgraphs of hypercubes, J. Combin. Theory Ser. B 14 (1973) 263–267]. Several papers followed with various characterizations of partial cubes. In this paper, we determine all subdivisions of a given configuration which can be embedded isometrically in the hypercube. More specifically, we deal with the case where this configuration is a connected graph of order 4, a complete graph of order 5 and the case of a k-fan Fk (k ≥ 3).

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1. Introduction

Isometric subgraphs of hypercubes, called partial cubes, have first been investigated by Graham and Pollack [5] and Djoković [4]. Later, several characterizations were given using a relation defined on the edge set or by constructive operations. Partial cubes have found various applications, for instance, in [3,7] and [8], interesting applications in chemical graph theory were established.

Clearly, partial cubes are bipartite. A simple way to obtain a bipartite graph is to subdivide every edge of a complete graph of order 4, a complete graph of order 5 and the case of a k-fan Fk (k ≥ 3).

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0012-365X/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
is called principal in $G$ if $u \in H$. We denote by $P(u, v)$ the path that connects the principal vertices $u$ and $v$ in $G$ and does not contain an intermediary principal vertex. This path is said plain if it does not contain any added vertexes of $G$, i.e., the principal vertexes $u$ and $v$ are adjacent in $G$. A principal vertex $u$ in $G$ is said universal if the neighboring set of $u$ in $H$ and the neighboring set of $u$ in $G$ are identical. Given a graph $G$, $S(G)$ is obtained by subdividing each edge of $G$ exactly once. In [9], Klavžar and Lipovec showed that $S(G)$ is a partial cube if and only if every block of $G$ is either a cycle or a complete graph.

We say that two edges $e = xy$ and $f = uv$ of a graph $G$ are in the Džoković–Winkler relation $\theta$ and note $e \theta f$ if $d_G(x, u) + d_G(y, v) \neq d_G(x, v) + d_G(y, u)$ [4, 10]. In a shortest path $P$ of a graph $G$, no two distinct edges are in relation $\theta$. In an isometric cycle $C_{2k}$, ($k \geq 2$), each edge $e$ is in relation with exactly one different edge $f$ with respect to the relation $\theta$. Winkler [10] proved that a bipartite graph is a partial cube if and only if the relation $\theta$ is transitive.

Let $G$ be a connected graph. A proper cover $\{G_1, G_2\}$ consists of two isometric subgraphs $G_1, G_2$ of $G$ such that $G' = G_1 \cup G_2$ and $G_0 = G_1 \cap G_2$ is a nonempty subgraph, called intersection of the cover. The expansion of $G'$ with respect to $G_1$ and $G_2$ is the graph $G$ constructed as follows. Let $G_i$ be an isomorphic copy of $G_i'$ for $i = 1, 2$ and for any vertex $u'$, let $u_i$ be the corresponding vertex in $G_i$, for $i = 1, 2$. Then, $G$ is obtained from the disjoint union $G_1 \cup G_2$, where for each $u'$ in $G_0$, the vertices $u_1$ and $u_2$ are joined by an edge. Chepoi [2] proved that a graph $G$ is a partial cube if and only if $G$ can be obtained from $K_1$ by a sequence of expansions.

In this paper, we study partial cubes as subdivisions of a graph. In Section 2, we characterize the subdivisions of all connected graphs of order 4. These characterizations allow us to determine partial cubes which are subdivisions of a complete graph $K_5$ (Section 3):

**Theorem 1.** Let $G$ be a subdivision of a complete graph $K_5$. Then, $G$ is a partial cube if and only if $G$ is isomorphic to $S(K_5)$ or $G$ is bipartite and contains a universal vertex.

Finally, we conclude with some open questions.

## 2. Graphs of order 4 and fans

If $T$ is a tree, then the subdivided $T$ is also a tree and thus a partial cube. A cycle $C_{2n}$ is a partial cube, too. Let $G$ be a graph constructed by connecting a vertex of an even cycle to a pendant vertex of a path. Such a graph can be embedded in the hypercube by preserving the distance. We just embed the path and the cycle into different dimensions. The graph $G$ is considered as a subdivision of a paw (a triangle $K_3$ with a pendant vertex). Then, a subdivided paw is a partial cube if the subdivided $K_3$ is an even cycle. In literature, we can find the following other results concerning partial cubes as subdivisions of connected graphs:

Let $W_k$ be the $k$-wheel, that is, the graph obtained by connecting one vertex $u$ to all the vertices of the cycle $C_k$. The vertex $u$ is the central vertex of $W_k$ and the remaining vertices are denoted $w_1, \ldots, w_k$. The graph $W_k(m_1, \ldots, m_k; n_1, \ldots, n_k)$ is obtained by subdividing edges of $W_k$, where $m_i$ is the number of vertices added on the edge $w_i w_{i+1}$, and $n_i$ is the number of vertices added on the inner edge $uw_i$. The vertices added on edges $w_i w_{i+1}$ (resp. on edges $uw_i$) are $w_{i,1}, \ldots, w_{i,m_i}$ (resp. $u_{i,1}, \ldots, u_{i,n_i}$) for $m_i > 0$ (resp. for $n_i > 0$).

**Theorem 2 ([6]).** Let $G$ be a subdivisions $k$-wheel ($k \geq 3$). Then, $G$ is a partial cube if and only if $G$ is isomorphic to $W_k(m_1, \ldots, m_k; n_1, \ldots, n_k)$, where $m_i$ is odd for $i = 1, \ldots, k$ and $n_1 = \cdots = n_k = 0$, or $G = W_3(1, 1; 1; 1, 1, 1)$.

An interesting instance of partial cubes mentioned in the last theorem is the subdivided wheel $W_3(1, 1; 1; 0, 0, 0)$. It is defined as the vertex-deleted subgraph $Q_3 - v$ where $v \in Q_3$ and is denoted $Q_3^-$, see Fig. 1. (In all our figures, bold vertices represent the vertices added to a given configuration.)

**Proposition 3 ([9]).** For any $n \geq 1, S(K_n)$ is a partial cube.

For $n = 4, S(K_4)$ which is isomorphic to the subdivided wheel $W_3(1, 1; 1; 1, 1, 1)$ can be embedded in the hypercube $Q_4$. See Fig. 2.

**Lemma 4 ([6]).** Let $G$ be a graph and let $K$ be an isometric subgraph of $G$ which is isomorphic to a subdivision of $K_{2,3}$. Then, $G$ is not a partial cube.
**Theorem 5.** Let $G$ be a subdivision of a diamond $(K_4 - e)$. Then, $G$ is a partial cube if and only if the chord of the diamond is not subdivided and each subdivided triangle $K_3$ of the diamond is an even cycle in $G$.

**Proof.** Let $G$ be a subdivision of the diamond $(K_4 - e)$ induced by vertices $w_1$, $w_2$, $w_3$, $w_4$ such that $w_1w_3$ is its chord.

If $G$ is a partial cube, then $G$ is bipartite. Consequently, each cycle of $G$ is even. Furthermore, the edge $w_1w_3$ does not contain any added vertex, otherwise $G$ is isomorphic to a subdivision of $K_{2,3}$. According to Lemma 4, $G$ is not a partial cube.

Now, suppose that the vertices $w_1$ and $w_3$ are adjacent in $G$, and let us prove that $G$ is a partial cube. The graph $G$ is isomorphic to a combination of two even isometric cycles $C_m$ and $C_n$ having a common edge $w_1w_3$. The equivalence classes of the relation $\theta$ have exactly two edges except the class that intersects $G$ in $w_1w_3$. We can verify that $w_1w_3\theta e$ and $w_1w_3\theta f$ where $e$ and $f$ are edges of $C_m$ and $C_n$, respectively. It is straightforward to see that $e\not\theta f$. Furthermore, each edge other than $e$ in $C_m$ (resp. $f$ in $C_n$) is not in relation with $f$ (resp. with $e$). Then, $\theta$ is a transitive relation in the edge set of $G$ bipartite graph. According to [10], $G$ is a partial cube. □

For $k \geq 3$, the $k$-fan $F_k$ is the graph obtained by connecting a vertex $u$ to all the vertices of a path on $k$ vertices $w_1, \ldots, w_k$.

No graph $G$ having an isometric subgraph which is isomorphic to a subdivision of a $k$-fan $F_k$ such that $uw_2, \ldots, uw_{k-1}$ is subdivided, is a partial cube. We complete this result (proved in [6]) by the following:

**Theorem 6.** Let $G$ be a subdivided $k$-fan $F_k (k \geq 3)$. Then, $G$ is a partial cube if and only if each subdivided triangle of $F_k$ is an even cycle in $G$ and the edge $uw_j$ is subdivided for no $j \in \{2, \ldots, k-1\}$.

**Proof.** Necessary condition, see [6].

Sufficient condition. Let $G$ be a subdivided $k$-fan $F_k (k \geq 3)$ induced by the vertices $u$, $w_1, \ldots, w_k$ such that each subdivided triangle of the fan is an even cycle in $G$. Assume that $uw_2, \ldots, uw_{k-1} \in E(G)$ and let us show that $G$ is a partial cube by induction on $k$.

If $k = 3$, $G$ is a subdivided diamond having no added vertex to its chord. According to Theorem 5, $G$ is a partial cube.

Now, suppose that $k > 3$ and let us show that $G$ is a partial cube. Since the subdivided diamond $u$, $w_{k-2}$, $w_{k-1}$, $w_k$ is isometric and partial cube, there exists two edges $e$ and $f$ in $G$ such that: $e\theta f$, $e\theta uw_{k-1}$ and $f\theta uw_{k-1}$ where $e \in P(u, w_k)$ or $P(w_{k-1}, w_k)$ and $f \in P(w_{k-2}, w_{k-1})$. The subgraph $H$ of $G$ which is isomorphic to a subdivided $(k - 1)$-fan $u$, $w_1, \ldots, w_{k-1}$ is isometric and partial cube by induction hypothesis. For each edge $g \in H; g \neq f$, $g$ and $e$ belong to a shortest path in $G$. Then $e$ is not in relation to $g$ with respect to $\theta$. The $\theta$ class of the other edges of the subdivided triangle $u$, $w_{k-1}$, $w_k$ contains just two edges. Consequently, the transitivity of $\theta$ is preserved in $G$ and then, $G$ is a partial cube. □

### 3. Subdivisions of the complete graph $K_5$

In this section, we deal with some partial cubes as subdivisions of complete graphs. Before studying the subdivisions of a complete graph $K_5$, we prove the following result:

**Lemma 7.** Let $G$ be a subdivided envelope $u$, $w_1$, $w_2$, $w_3$, $w_4$ where $w_3$ and $w_4$ are the vertices of degree 4. Let $H$ be the subgraph of $G$ which is isomorphic to a subdivided $K_4$ induced by $w_1$, $w_2$, $w_3$, $w_4$. Then, $G$ is a partial cube if and only if $H$ is isomorphic to $W_3(m_1, m_2, m_3; 0, 0, 0)$ (if $m_i$ is odd $i \in \{1, 2, 3\}$) with $w_3$ (or $w_4$) is the central vertex of $W_3$, and the total number of added vertices to the edges $uw_2$ and $uw_4$ is odd.

**Proof.** Necessary condition. Assume that $G$ is a partial cube. We show that $w_3$ (or $w_4$) is a universal vertex in $H$.

Suppose that $P(w_3, w_4)$ is not isometric and put $K$ the subgraph of $G$ induced by all its vertices except the inner vertices of $P(w_3, w_4)$. The subgraph $K$ is isometric and isomorphic to a subdivision of a 3-wheel $W_3$ (consider $w_1, w_2, w_3, w_4$ its principal vertices). Since $G$ is a partial cube, $K$ is a partial cube, too. Then, according to Theorem 2, the subgraph $K$ is isomorphic either to $S(K_4)$ or to $W_3(m_1, m_2, m_3; 0, 0, 0)$ with $m_i$ is odd $i \in \{1, 2, 3\}$. 

![Fig. 2. The subdivision $S(K_4)$ and its isometric embedding in $Q_4$.](image-url)
If $K$ is isomorphic to $S(K_4)$, the subdivided diamond $u, w_1, w_3, w_4$ is isometric. Its chord is subdivided, in contradiction with Theorem 5.

If $K$ is isomorphic to $W_3(m_1, m_2, m_3; 0, 0, 0)$ with $m_i$ is odd $i \in \{1, 2, 3\}$, the central vertex of the wheel is reached in $w_2$ or $w_1$ since $d_K(w_3, w_4) \geq 2$. If the vertex $w_2$ (resp. $w_1$) is the universal vertex of such a wheel, the graph $G$ contains an isometric subdivided diamond $w_2, w_3, w_4, u$ (resp. $w_1, w_3, w_4, u$). Then, by virtue of Theorem 5, $P(w_3, w_4)$ has to be plain. This is a contradiction, since $P(w_3, w_4)$ is not isometric.

We conclude that $P(w_3, w_4)$ is isometric and by consequence, the subgraph $H$ is isometric, too. The subgraph $H$ is not isomorphic to $S(K_4)$ otherwise the subdivided diamond $w_1, w_3, u, w_4$ having a subdivided chord is isometric. According to Theorem 5, $G$ is not a partial cube. Then, $H$ is isomorphic to $W_3(m_1, m_2, m_3; 0, 0, 0)$ with $m_i$ odd $i \in \{1, 2, 3\}$. The central vertex of this subdivided wheel is reached in $w_3$ (or $w_4$), otherwise the subdivided diamond induced by $w_1, w_3, u, w_4$ (or $w_2, w_3, u, w_4$) is isometric. Its chord is subdivided. Then, $G$ is not a partial cube, contradiction.

Consequently, the subgraph $H$ contains a universal vertex $w_2$ or $w_4$. Furthermore, since $G$ is a bipartite graph, the total number of added vertices to the edges $uw_3$ and $uw_4$ is odd.

**Sufficient condition.** Suppose that $w_3$ is a universal vertex in $H$ and $G$ bipartite and let us prove that $G$ is a partial cube. See Fig. 3. The subgraph $H$ and the subdivided diamonds induced respectively by $w_1, w_3, u, w_4$ and $w_2, w_3, u, w_4$ are partial cubes. The subdivided 4-fan induced by $u, w_1, w_2, w_3, w_4$ is a partial cube too. Since all these subgraphs of $G$ are isometric and partial cubes, the transitivity of the relation $\equiv$ is preserved. Consequently, $G$ is a partial cube. 

**Lemma 8.** Let $G$ be a subdivision of a complete graph $K_n(n \geq 4)$ where each path $P(w_i, w_j)$ with $i, j \in \{1, \ldots, n\}$ is isometric. Then, $G$ is a partial cube if and only if $G$ is isomorphic to $S(K_n)$ or $G$ contains a universal vertex $w_{00}$ and for each $i \neq j, i, j \neq 00$, the length of the path $P(w_i, w_j)$ is two.

**Proof.** Let $G$ be a subdivision of a complete graph $K_n$ induced by $w_1, \ldots, w_n$. Suppose that all the paths $P(w_i, w_j)$ with $i \neq j; i, j \in \{1, \ldots, n\}$ are isometric. Assume that $G$ is a partial cube, and let us show by induction on $n$ that either $G$ is isomorphic to $S(K_n)$ or $G$ contains a universal vertex. If $n = 4$, $G$ is a subdivided 3-wheel $W_3$. According to Theorem 2, $G$ is either $W_3(1, 1, 1; 1, 1, 1)$ or $W_3(0, 1, 1; 0, 0, 0)$.

Assume that $n > 4$, and let $H$ be the subgraph of $G$ which is isomorphic to a subdivision of a complete graph $w_1, \ldots, w_{n-1}$. This subgraph is isometric and consequently a partial cube. By the induction hypothesis, $H$ has a universal vertex or $H$ is isomorphic to $S(K_{n-1})$. In the first case, we assume that $w_1$ is universal in $H$. For each $i \neq j; i, j \in \{2, \ldots, n-1\}$, the subdivided complete graph $w_1, w_n, w_i, w_j$ is an isometric subgraph then a partial cube. Therefore, $w_1$ is universal in each one of these subdivisions denoted $K_{ij}$ for all $i \neq j; i, j \in \{2, \ldots, n-1\}$. Then, $w_1$ is universal in $G$. In the second case, the subdivided complete graph $K_{ij}$ is isomorphic either to $S(K_2)$ or to $W_3(1, 1, 1; 0, 0, 0)$ (Theorem 2). Consequently, either the vertex $w_1$ is universal in $G$ or $G$ is isomorphic to $S(K_n)$.

Since $S(K_n)$ is a partial cube, the rest of the proof consists of showing that each subdivided complete graph $K_n$ which has a universal vertex and exactly one added vertex to the other edges of $K_n$ is a partial cube.

Let us now denote by $u$ the universal vertex of $G$, and $w_1, w_2, \ldots, w_{n-1}$ the other principal vertices of the subdivided complete graph $K_n$. For each $i \neq j; i, j \in \{1, \ldots, n-1\}$, $w_{ij}$ is the vertex belonging to $P(w_i, w_j)$. We can isometrically embed $G$ in the hypercube $Q_{n-1}$ as follows. We put the vertex $u$ in the origin. All the $w_i$ components are null except the $i$-th one for each $i \in \{1, \ldots, n-1\}$. The $i$-th and the $j$-th component of $w_{ij}$ are equals to 1 and the others are null for all $i \neq j; i, j \in \{1, \ldots, n-1\}$.

Let us consider $G$ a subdivision of a complete graph $K_3$. Here we prove that $G$ is partial cube if and only if $G$ is isomorphic to $S(K_3)$ or $G$ is bipartite and contains a universal vertex (Theorem 1).
**Proof of Theorem 1**

**Necessary condition.** Let $G$ be a subdivided $K_5$ and let $u$ be a principal vertex such that there exists at least a principal vertex $a$ in $G$ different from $u$ such that $P(u, a)$ is not geodesic. The case where all the paths joining a principal vertex are isometric, has been treated above (Lemma 8). We denote by $H$ the isometric subgraph of $G$ resulting from deleting the inner vertices of all non geodesic paths $P(u, a)$. The subgraph $H$ is isomorphic to a subdivided $K_5 - K_{1,p}$ with $1 \leq p \leq 3$. Let $\{K; L; u\}$ be a partition of the principal vertex set of $G$ into $K = \{x; P(u, x) \text{ isometric in } G\}$ and $L = \{a; P(u, a) \text{ non-geodesic in } G\}$. It is straightforward to note that the sets $K$ and $L$ are not empty. Suppose that $G$ is a partial cube and let us show that $G$ contains a universal vertex $x$ such that $x \in K$.

Case 1. Suppose that $|K| = 3$ and $|L| = 1$. Put $K = \{x, y, z\}$ and $L = \{a\}$. See Fig. 4(a). We assume that the au-geodesic goes through the vertex $x$. If there exists two vertices $x$ and $y$ in $K$ such that the xy-geodesic is $P(x, z) \cup P(z, y)$ with $z \in K$ then, the subdivided complete graph induced by $u, x, y, z$ is isometric in $G$. The subgraph is a partial cube and is not isomorphic to $S(K_4)$. Then, it contains a universal vertex. Since $P(x, z) \cup P(z, y)$ is a xy-geodesic, the vertex $z$ is universal in the subdivided $K_4$. Consequently, the subdivided complete graph induced by $x, y, z$ is isometric. Thus, the vertex $z$ is universal in $G$.

Now, suppose that for each pair of vertices $\{x, y\}$ in $K$, there is no $xy$-geodesic going through $z$. Now, suppose that $G$ is not such a geodesic. Let us show that there exists a plain path joining $x$ and another principal vertex in $K$. If $P(a, y)$ and $P(a, z)$ are not isometric, then the subdivided diamond induced by $u, x, y, a$ is isometric in $G$. According to Theorem 5, the vertex $x$ is adjacent to the vertex $y$. If $P(a, y)$ is isometric, then the subdivided diamond induced by $x, a, y, u$ or $a, x, z, u$ is isometric. Therefore, the vertex $x$ is adjacent to the vertex $y$ or the vertex $z$.

Since there exists a vertex $y \in K$ such that $P(x, y)$ is plain, the subdivision of a complete graph $x, y, a, u$ is isometric in $G$. Thus, the vertex $x$ (or $y$) is universal in $G$. Then, $G$ contains a geodesic relying two vertices of $K$ and going throw another vertex of $K$. This is in contradiction with the hypothesis.

Case 2. Suppose that $|K| = |L| = 2$. Put $K = \{x, y\}$ and $L = \{a, b\}$. See Fig. 4(b). Since the subgraph $H$ is an isometric subdivided envelope, and due to Lemma 7, the vertex $x$ (or $y$) is universal in the subdivided complete graph $a, b, x, y$. The subdivided $K_4$ induced by $u, x, y, a$ is isometric. Therefore, the vertex $x$ is universal in $G$.

Case 3. Suppose that $|K| = 1$ and $|L| = 3$. Put $K = \{x\}$ and $L = \{a, b, c\}$. See Fig. 4(c). The subgraph of $G$ which is isomorphic to a subdivision of $K_4$ induced by $a, b, c$ is isometric. This one is not isomorphic to $S(K_4)$ otherwise, there exists a subdivided diamond having a subdivided chord. If the subdivided $K_4$ contains a universal vertex, this one is not reached in $L$ (for instance $a$). If it is the case, the isometric subdivided $K_4$ induced by $a, b, c, u$ is isometric. Then the vertex $a$ is adjacent to $u$ (Theorem 5). The path $P(u, a)$ is not geodesic, contradiction. Then, the vertex $x$ is universal in the subdivided $K_4$. Finally, by considering the isometric subdivided complete graph on $u, x, a, b$, we conclude that the vertex $x$ is universal in $G$.

**Sufficient condition.** Now, let us suppose that $G$ is a subdivided complete graph $K_5$ such that $G$ contains a universal vertex $u$ and the other edges of $K_5$ not adjacent to $u$, noted $w_i(w_j \ (i \neq j, 1 \leq i,j \leq 4)$, are oddly added in $G$. We put $n$ the total number of added vertices to these edges. Let $u$ us show by induction on $n$ that $G$ is a partial cube. It is clear that $n \geq 6$. For $n = 6$, $G$ is a subdivided $K_5$ where each path connecting two principal vertices in $G$ is isometric. According to Lemma 8, $G$ is a partial cube. Now, let us suppose that $n > 6$. Consider $G'$ a bipartite subdivided $K_5$ having a universal vertex $u$ such that $G'$ is obtained from $K_5$ by adding $n - 2$ vertices to $w_i(w_j$ with $i \neq j, 1 \leq i,j \leq 4$. Then, there exists two paths $P(w_1, w_2) = (w_1, w_{1,1}, w_{1,2}, \ldots, w_{1,2k+1}, w_2)$ and $P'(w_1, w_2) = (w_1, w'_{1,1}, w'_{1,2}, \ldots, w'_{1,2k-1}, w_2)$ connecting the vertices $w_1$ and $w_2$ in $G$ and $G'$ respectively with $k \geq 1$. Let $G'_1$ and $G'_2$ be the isometric subgraphs of $G'$ respectively induced by $V(G'_1) = \{w'_{1,k-1}, w'_{1,k}, \ldots, w'_{1,2k-1}, w_2\}$ and $V(G'_2) = V(G') \setminus \{w'_{1,k}, w'_{1,k+1}, \ldots, w'_{1,2k-1}\}$. The graph $G'$ is a partial cube by the induction hypothesis. The expansion over the proper cover $\{G'_1, G'_2\}$ of $G'$ is a partial cube. The resulting graph is isomorphic to $G$. We note that if $k = 1$, we put $V(G'_1) = \{w_1, w_{1,1}, w_2\}$ and $V(G'_2) = V(G') \setminus \{w_{1,1}\}$.

Fig. 4. A non isometric path which links the vertex $u$ to another principal vertex in $G$ is represented by a dashed curve.
4. Open questions

In this paper, we have contributed to the characterization of partial cubes as subdivisions of a given configuration. We have established certain results however more questions remain still open:

The subdivisions of a complete graphs $K_n$ that are partial cubes have been determined for an order $n \in \{4, 5\}$. Using Theorem 1, we will be able to generalize this result for $n \geq 4$ (see [1]).

Let us state another interesting related question. A generalized envelope $E_{m,n}$ is the graph obtained by connecting a vertex $u$ to $m$ vertices of a complete graph $K_{m+n}$ with $m \geq 2$ and $n \geq 2$. The neighborhood of $u$ is noted $K$ and $L = V(E_{m,n}) \setminus (K \cup \{u\})$. The graph $E_{2,2}$ is isomorphic to the envelope and we have provided all subdivisions of the envelope which are partial cubes. Here, we propose the following conjecture for the generalized envelope subdivisions:

Conjecture 9. Let $G$ be a bipartite subdivided $E_{m,n}$ ($n \geq 2; m \geq 2$). Then, $G$ is a partial cube if and only if:

- for $m = 2$, there exists a vertex $x \in K$ such that $x$ is adjacent to $y$ in $G$ for all $y \in K \cup L$.
- for $m > 2$, $G$ has a universal vertex $x$ with $x \in K$.

Using our result in [1], we can prove this conjecture for $m = 2$.

On the other side, within this paper, we have mentioned some subdivisions which are not partial cubes for instance the subdivided $K_{2,3}$. So, it would be interesting to find a characterization of a partial cube by providing a list of forbidden isometric subgraphs if it exists.

References