Optimization of Supply Chain Systems with Price Elasticity of Demand

Uğur Kaplan, Metin Türkay
Department of Industrial Engineering, Koç University, 34450 Istanbul, Turkey, ukaplan@ku.edu.tr, mturkay@ku.edu.tr, http://systemslab.ku.edu.tr/

Bülent Karasören
Department of Mathematics and Institute of Applied Mathematics, Middle East Technical University, 06531 Ankara, Turkey, bulent@metu.edu.tr

Lorenz T. Biegler*
Department of Chemical Engineering, Carnegie Mellon University, Pittsburgh, PA 15213, USA, lb01@andrew.cmu.edu

A centralized multi-echelon, multi-product supply chain network is presented in a multi-period setting with products that show varying demand against price. An important consideration in such complex supply chains is to maintain system performance at high levels for varying demands that may be sensitive to product price. In order to examine the price-centric behavior of the customers, the concept of price elasticity of demand is addressed. The proposed approach includes many realistic features of typical Supply Chain Systems such as production planning and scheduling, inventory management, transportation delay, transportation cost and transportation limits. In addition, the proposed system can be extended to meet unsatisfied demand in future periods by backordering. Effects of the elasticity in price demand in production and inventory decisions are also examined. The supply chain model is formulated as a convex mixed-integer nonlinear programming problem. Reformulations are presented to make the problem tractable. The differential equations are reformulated as difference equations and the unbounded derivative problem occurred due to the non-linear objective function is dealt with an approximation. The approach is illustrated on a multi-echelon, multi-product supply chain network.

Key words: mixed-integer nonlinear programming, supply chain management, smoothing, price elasticity of demand

1. Introduction
Companies continuously seek innovation in their decision making processes to decrease their operational cost, and provide satisfactory service to their customers for survival in the globally competitive market. Firms establish operationally efficient supply chain systems (SCS) to reach this goal (Shapiro (2004)). The supply chain is defined as the set of activities, workers, technological and physical infrastructures and policies concerned with procurement of raw materials, conversion of these raw materials to finished and semi-finished product, and logistics of these products (Hasan (2006)). Although there are many factors that affect the performance in such a complicated system, flow of information and material remain very important. The amounts of inventory held and accumulated orders in the system, which are classified as indicators of performance of SCS, are the explicit results of these flows.

An important consideration in the analysis of SCS is to express the relationship between the price of the product and demand for the product. The product price is strongly affected by the costs associated with actions in the SCS including procurement of raw materials, production costs, transportation costs and inventory holding costs. These decisions are strongly affected by the quantities of materials procured, produced and transported, due to economies of scale favoring larger amounts. However, larger quantities of product in the market do not result in a linear increase in system revenue. Instead, the relationship between the price of the product and the demand
to that product should be modeled using the concept of price elasticity of demand (Lysons and Gillingham (2003)).

A typical SCS includes suppliers, plants, warehouses, distribution centers and retailers as a physical entity is shown in Fig. 1. More than one instance of each supply chain entity can exist in the system; therefore, overall system can be classified as a supply chain network (SCN). In this network, suppliers provide raw materials, plants produce final products, warehouses and distribution centers handle storage, and the retailers generate revenue for the system. During the execution of these tasks, the inter-relationship between suppliers, plants, warehouses, distribution centers and retailers operates with a pull policy. Here upstream nodes of SCN have no authority on decisions of downstream nodes (Shapiro (2001)). Instead, the relationship between retailers and customers is established with respect to price elasticity, a function that determines price and demand between retailers and customers (Baumol (1977)).

Demand for the specific product for a given price, and a specific price for a given demand, can be determined by price elasticity with demand, and the elasticity value of a product is not isolated from its environment (Huq and Dynes (1982)). The key contribution of this paper is the consideration of price elasticity on the operational decisions of supply chain systems. To optimize such a complex SCN, traditional methods consider separately the decisions that should be made at each node of SCN instead of integrated consideration. The design of a SCN has been addressed using mathematical programming models. Melachrinoudis and Min (2006) developed an MILP (Mixed Integer Linear Programming) model to determine the consolidation decisions of warehouses. The authors claim that the firms can take advantage of economies of scale in this setting. Hassan (2006) proposes a systems approach for the analysis of the SCN where a supply chain is defined as all of the activities that take place from procurement of raw materials to delivery to end customers. The vertical and horizontal hierarchies that occur in an SCS are classified, and the multi-nested structure is illustrated by investigating ‘systems of systems’ properties in SCN. Analysis of the decisions that optimize a multi-product, multi-echelon supply chain system is also conducted in Perea-Lopez et al. (2000) and Perea-Lopez et al. (2003). In these studies, classical control and model predictive control (MPC) strategies are used to optimize the SCN in this study. More recently, the optimal operation of multi-product, multi-echelon supply chain systems has been modeled as a hybrid system (Mestan et al. (2006)). Here, a comparison of centralized and decentralized SCS with different topologies is made in terms of the bullwhip effect.

In this paper, we consider a multi-echelon, multi-product SCS. The SCS exhibits both continuous and discrete behavior, with continuous equations discretized on the time domain to provide overall model integrity. Moreover, several realistic operational features such as transportation limits, transportation cost, and transportation delay are considered, and the concept of price elasticity is introduced. Although the price elasticity has been studied conceptually in economics, effects of price elasticity on the operational behavior of multi-echelon, multi-product SCS is considered here for the first time.

The resulting optimization model is a mixed-integer nonlinear program (MINLP) with discrete and continuous variables. Moreover, for every realization of the discrete variables, a convex continuous problem results. Therefore, any stationary point of these continuous problems corresponds to their global optimum. Consequently, a global MINLP solution can be found with state-of-the-art MINLP solvers.

Available MINLP solvers include SBB (Brooke et al. (1998)), an NLP branch and bound algorithm, DICOPT (Viswanathan and Grossmann (1990)), based on outer approximation, BONMIN (Bonami et al. (2008)), a general hybrid solver that includes the previous two algorithms as special cases, and BARON (Sahinidis (1996)), a branch and bound strategy for both trees and subspaces, that guarantees global solutions for nonconvex problems. All of these are implemented in the GAMS modeling environment (Brooke et al. (1998)), where a variety of MILP and NLP solvers can be selected for these MINLP solvers.
All of these methods have been proved to converge to the global MINLP solution as long as NLP subproblems are convex. Moreover, because outer approximation is generally more efficient on convex problems that are dominated by linear terms, we consider only the DICOPT and BONMIN solvers. On the other hand, our convex MINLP model also contains nonlinear price elasticity terms with unbounded derivatives at zero. To deal with this challenge, approximation and reformulation of these terms are needed to yield a solvable model that can still obtain the global solution. This is also addressed in this study.

The main contribution of this paper is the simultaneous consideration of operational behavior of multi-echelon, multi-product SCN with the price elasticity of demand. In literature, price elasticity concept has been studied in the context of SCS. In these studies, instead of optimization, some other algebraic methods are used and it is applied for at most two echelon SCSs (Nagurney and Dong (2002)). In this study, the elasticity is not represented as price elasticity of demand. Instead, price is written as a function of the other parameters such as travel time and travel cost. Therefore, unlike our study, systems of equations are solved to find the best solution and Variational Inequalities are used during modeling and solution process. Because in this paper the price elasticity of demand is considered simultaneously with the optimization model of multi-echelon SCN, a convex MINLP system is obtained. Some production schedules cause premature termination of solution process because of unbounded derivative problem which results suboptimal solution. To prevent this problem, an approximation scheme is developed which can be used to deal with unbounded derivative problems. As a result, the response of multi-echelon SCN to the price elasticity of demand is demonstrated in the context of MINLP based optimization models.

The paper comprises five parts. The description and characteristics of the problem are discussed in Section 2. Section 3 provides a description and analysis of a case study example that illustrates the original model and then develops the reformulated model. After model reformulation the illustrative example is revisited in Section 4. Conclusions are provided in Section 5.

2. Problem Description

A supply chain system can be considered as a higher level abstraction of the actions between the conversion of raw materials and delivery of final product to the end customers. Within this abstract structure there are four processes that have to be managed: sourcing, manufacturing, delivery, and return (Council (2005)).

For the customer and service provider the main objective of the system is the maximization of the overall profit without violating operational constraints, because the reason of existence of every SCS is the maximization of system profit with respect to some service criteria. These criteria are represented by the systems constraints. In this way, system objective can be maximized by matching operation criteria. (DELETED: are the generation of large revenue, reduction of operational cost and timely delivery of service). In this paper, an integrated model is developed to achieve these objectives and exact optimization techniques are used. Therefore, any explicit inventory policy does not used in our approach. Because such policies are heuristic techniques, their consideration in the optimization model may cause a suboptimal result. Also, because an exact optimization approach is used with integrated model, the result obtained and cannot be improved without changing the system constraints and objective. The integrated SCS includes suppliers, plants, plant warehouses, distribution centers and retailers as shown in Fig. 1. The demand and revenue are created by the customer. The demand and price relations in previous periods are known. In the model, transportation cost, transportation delay and transportation limit
constraints exist. Because, the plant and plant warehouse are located in the same geographical region, transfer of end product between these two nodes incurs no cost and there is no time delay.

Because an integrated approach is applied, all nodes are considered as a whole and inefficiencies due to decomposition of the system are eliminated. While some of the demand cannot immediately be satisfied in the intermediate nodes of SCS, customer demand needs to be satisfied immediately. Thus, to satisfy the unmet demand at intermediate nodes, accumulated orders are held in each node.

The plant modeled here is a multi-product single stage system. Each plant can have more than one production line, but only one product can be produced during its production time in each production line (no preemption); i.e., only one product can be produced in each assembly line at each time interval. This sort of plant set-up is common in manufacturing industry. There are some products whose production scheme can allow preemption during their production processes. However, the product modeled in this paper does not allow preemption. Therefore, once the production of that products begins, it should finish to make the production line idle to begin to production of other products. A variable production cost incurs for each product produced in the plant and there is a setup cost to switch from one operating mode to another (such as switching from idle mode to producing a product). This production costing is a very common in manufacturing environment. During the production of a product, some cost issues increase in direct proportion with the production amount of product, such as electricity cost, raw material cost. However, some cost issues occur independently from the amount of production. For example, during the painting processes in car manufacturing, switching from one color to other, requires some fixed cleaning and startup cost. These fixed costs are modeled in this paper. Because no preemption policy is applied, switching from manufacturing end phase to manufacturing start phase incurs some fixed cost to system. Moreover, the relationship between product selling price and demand is established through the concept of price elasticity of demand, leading to a nonlinear revenue function. The relation between price and demand is sometimes omitted during the tactical level SCN. However, most of the time, price is the one of the most important parameters that determines demand, and the price elasticity of demand is one of the most established theories that describes this relation (Tellis (1988)).
2.1. Inventory and Order Balance

The modeled SCN holds inventory, and orders can be accumulated in the system to satisfy later demands. The main reasons for maintaining inventory are the uncertainties in the system: the fixed cost of operations, seasonality, and production and flow limits. Because of these factors, the SCS must provide a certain level of inventory in specific nodes of system to maintain a high service level and to maximize system-wide profit. On the other hand, firms have to decrease their inventories for better financial performance. In addition to inventory, orders are accumulated in the system mainly to allow late satisfaction of some demands. All of the nodes in the system hold the unmet demand and information regarding the owner of this demand. The system then decides the time of satisfaction of this unmet demand with respect to performance criteria.

These order and inventory effects are dynamic, as they both depend on state and rate variables in our system. Here the state variable is the current inventory and the rate variable is the increase or decrease of inventory or order accumulation. Therefore, the rate variable represents upstream and downstream flows for inventory, and the upcoming demand and downstream flow of materials for order accumulation. Also, current values of both inventory and accumulated orders depend on their values at previous time periods. This dependence relation is determined by deterministic rules. Using the variables and constants described in the Appendix, the dynamic behavior of inventory and accumulated orders can be described (Chueshov (1999)) with the following equations:

\[
\frac{dI_{mk}(t)}{dt} = -\sum_{k'} y_{mkk'}(t) + \sum_{k'} y_{mk'k}(t - \tau_{k'k}) \quad \forall m,k,t
\]

\[
\frac{dO_{mkk''}(t)}{dt} = u_{mkk''}(t) - y_{mkk''}(t) \quad \forall m,k,k'',t
\]

Direct inclusion of Eq. (1) and Eq. (2) makes the system computationally difficult and some level of discretization is needed. Therefore, these two differential equations are represented by difference equations with consecutive time intervals equal to one. The difference equations for the differential inventory and order balance equation are the following:

\[
I_{mk}(t) = I_{mk}(t-1) - \sum_{k''} y_{mkk''}(t) + \sum_{k'} y_{mk'k}(t - \tau_{k'k}) \quad \forall m,k,t
\]

\[
O_{mkk''}(t) = O_{mkk''}(t-1) + u_{mkk''}(t) - y_{mkk''}(t) \quad \forall m,k,k'',t
\]

Eq. (3) indicates that the rate of change of inventory \( I_{mk} \) of product \( m \) at node \( k \) between time \( t \) and \( t - 1 \) is equal to the difference of total flows of product \( m \) that comes from upstream nodes \( k' \) with an integer time delay of \( \tau_{k'k} \) and total flows of product \( m \) to downstream nodes \( k'' \). Eq. (4) indicates that the rate of change from time \( t - 1 \) to \( t \) of \( O_{mkk''} \) (the accumulated orders of downstream node \( k'' \) from node \( k \) for product \( m \)) equals the difference of \( u_{mkk''}(t) \) (demand of downstream node \( k'' \) for product \( m \) to node \( k \)) and \( y_{mkk''}(t) \) (corresponding product flow to node \( k'' \) from node \( k \)) at time \( t \).

2.2. Production System

While the SCN considered here includes production systems that have multiple production lines, we assume only one product can be produced in each time interval in each production line and preemption is not allowed in the production lines. In addition, the production system operates with batch production policy with fixed (or flexible) production quantities. The model for the production system is given by the following equations (Mestan et al. (2006)):
The terms in the objective function are defined with the following equations:

\[ 0 \leq PR_{mka}(t) \leq PR_{mka}^U(t) \quad \forall m \in M, k \in N_{pr}, a \in A, t \in T \]  \hspace{1cm} (5)

\[ \sum_{m \in M} pr_{mka}(t) \leq 1 \quad \forall k \in N_{pr}, a \in A, t \in T \]  \hspace{1cm} (6)

\[ \sum_{p' \in M \setminus m} \sum_{t' \in \tau} pr_{m'ka}(t') + \sum_{t' = t}^{t_{lim}} \sum_{p' \in M \setminus m} pr_{m'ka}(t') \leq 1 \quad \forall m_i \in M, k \in N_{pr}, a \in A, t \in T \]  \hspace{1cm} (7)

where \( t_{lim} = \min(|T|, t + l_m - 1) \). Eq. (5) models the production quantity in the plant. When production of product \( m \) at plant \( k \), in production line \( a \) starts at period \( t \) (represented by binary variables \( pr_{mka}(t) \)), the production quantity, \( PR_{mka}(t) \) is bounded by \( PR_{mka}^U(t) \) if the plant operates under flexible production quantity policy. The production quantity can be fixed by converting the inequality in Eq. (5) into an equality. The production of single product at each time interval in each production line is enforced by Eq. (6). The production system does not support preemption; therefore, once a production of a product begins, system will finish the production of this product during the predefined time period (i.e., during \( l_m \) periods for product \( m \)). The behavior of this system is modeled with Eq. (7). Once the production of a particular product starts in period \( t \) (indicated by binary variable by variable \( pr_{mka}(t) \)), then it is not possible to start production of any products. The SCS can have multiple production facilities. Since the mathematical models for the production lines and plants are identical, the model allows multiple facilities (index \( k \)) and production lines (index \( a \)).

### 2.3. Limits on Material Flow

The flow of material from one node to another node has a specified upper bound. In real life, this bound is determined with the production and logistic capability of the SCS. Also, all real world SCSs have an upper bound on the satisfaction of the demand. In our system, we represent this capacity by putting an upper bound on the flow between retailer and customer.

### 2.4. Objective Function

The objective of SCS is the maximization of the profit. The system profit is equal to difference of revenue and total cost of operations that are realized in SCS. The only source of the revenue is the customer, created by product sales in the system. Therefore, the SCS should provide service or product to create revenue. In our system, the costs incurred during the production are the following: holding cost for each node, transportation cost that is realized because of the transfer of material between nodes, cost of production including fixed and variable cost for each production unit and raw material cost. With respect to these cost items, formulation of the objective function is the following:

\[ Z = C_{RE} - C_{HO} - C_{TR} - C_{RM} - C_{PF} - C_{PV} \]  \hspace{1cm} (8)

The terms in the objective function are defined with the following equations:

\[ C_{RE} = \sum_{t \in T} \sum_{m \in M} \sum_{k \in N_{rt}} \sum_{k' \in N_{ca}} \sum_{s \in N_{sp}} Req_{ms} \cdot RC_{ka} \cdot pr_{mka}(t) \]  \hspace{1cm} (9)

\[ C_{HO} = \sum_{t \in T} \sum_{m \in M} \sum_{k \in N_{pw}} HC_{mk} \cdot pr_{mka}(t) + \sum_{t \in T} \sum_{m \in M} \sum_{k \in N_{dc}} HC_{mk} \cdot pr_{mka}(t) + \sum_{t \in T} \sum_{m \in M} \sum_{k \in N_{rt}} HC_{mk} \cdot pr_{mka}(t) \]  \hspace{1cm} (10)

\[ C_{TR} = \sum_{t \in T} \sum_{m \in M} \left( \sum_{k \in N_{pw}} \sum_{k' \in N_{dc}} TC_{mk} \cdot pr_{mka}(t) + \sum_{k \in N_{dc}} \sum_{k'' \in N_{rt}} TC_{mk} \cdot pr_{mka}(t) + \sum_{k \in N_{rt}} \sum_{k'' \in N_{ca}} TC_{mk} \cdot pr_{mka}(t) \right) \]  \hspace{1cm} (11)

\[ C_{RM} = \sum_{t \in T} \sum_{m \in M} \sum_{k \in N_{pr}} \sum_{s \in N_{sp}} Req_{ms} \cdot RC_{ka} \cdot pr_{mka}(t) \]  \hspace{1cm} (12)
If the price elasticity concept is not considered in this SCS, then Eq. (9) would be legitimate as the revenue term, with a constant $P_{rc_m}(t)$. When the price elasticity concept is considered in the system, $P_{rc_m}(t)$ must be modified. According to the concept of price elasticity, the demand to a product changes with its price. Therefore, in a system in which the effect of price elasticity is considered, the relationship between the revenue function and the amount of material shipped does not exhibit linear behavior. Detailed analysis of price elasticity as well as revenue function reformulation are provided next.

2.5. Price Elasticity and Revenue

Price elasticity of demand is the concept that determines the relationship between price of product and demand to that product (Lysons and Gillingham (2003)). Most products, except for luxury items, have a negative elasticity value (Nicholson (1992)). Therefore, an increase in product price leads to decrease in product demand. The rate of decrease of demand depends on the price elasticity value of that product. The demand change in response to price change becomes excessive for the product that has a higher elasticity value, and this product type is called highly elastic. The absolute value of price demand elasticity for product $m$ is calculated with the following formula:

$$E_m = \left| \frac{\% \text{ change in demand}}{\% \text{ change in price}} \right|$$

(15)

Here we assume $E_m > 1$ in order to ensure that the revenue terms remain concave. Eq. (15) leads to the following price elasticity function:

$$P_{rc_m}(u_{mkk''}(t)) = h u_{mkk''}(t)^{-1/E_m} \quad k \in N_{rt}, k'' \in N_{cs}, m \in M$$

(16)

where $h$ is the product price that corresponds to demand of one unit (Williams (2003)). The value of $h$ is calculated from the price, demand and elasticity data of previous planning period. Also in using the elasticity data of previous planning period, it should be assumed that the price elasticity values of a product remain stable during its lifetime. This assumption is valid for most commodity products and some other products. Although the model developed in this paper support varying price elasticity value, to limit the scope it is assumed that the elasticity value remains constant during the life time of the product. It is also assumed that retailers will satisfy all of the demand that is submitted to them immediately. Therefore, at any time interval, demand of a product to retailer responds with the same amount of flow of corresponding product. Although continuous immediate satisfaction of demand is not possible in real life, one of the main aims of the existence of SCN is decreasing of response time of system to the customer, and most of the time this response time can be decreased to zero. However, at some point to limit the scope of model, it is assumed that, this response time is equal to zero. Because of this assumption, the revenue function will be the following:

$$C_{RE} = \sum_{t \in T} \sum_{m \in M} \sum_{k \in N_{rt}} \sum_{k'' \in N_{cs}} h u_{mkk''}(t)^{(1-1/E_m)} = \sum_{t \in T} \sum_{m \in M} \sum_{k \in N_{rt}} \sum_{k'' \in N_{cs}} R(u_{mkk''}(t))$$

(17)

and this term now replaces Eq. (9) in the model.
Table 1  Parameter values in the example problem.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( PR_{mka}(t) )</td>
<td>40</td>
</tr>
<tr>
<td>( l_m )</td>
<td>2</td>
</tr>
<tr>
<td>( HC_{mk}(t) )</td>
<td>2</td>
</tr>
<tr>
<td>( TC_{mkk'}(t) )</td>
<td>1</td>
</tr>
<tr>
<td>( RC_m )</td>
<td>1</td>
</tr>
<tr>
<td>( FC_{mk} )</td>
<td>1</td>
</tr>
<tr>
<td>( VC_{mk} )</td>
<td>1</td>
</tr>
<tr>
<td>( SC_{mkk'} )</td>
<td>1</td>
</tr>
<tr>
<td>( \delta_{Pr-Dc} )</td>
<td>3</td>
</tr>
<tr>
<td>( \delta_{Dc-Rt} )</td>
<td>2</td>
</tr>
<tr>
<td>( Pr_{old} )</td>
<td>100</td>
</tr>
<tr>
<td>( u_{mkk'rt} )</td>
<td>2</td>
</tr>
</tbody>
</table>

3. Model Evaluation and Reformulation

To motivate need for reformulation of the integrated optimization approach presented in the previous section, we consider a small multi-echelon, multi-product SCN. We note that while this case study has a limited scope to focus on price demand elasticity, the resulting development applies to the general model developed in the previous section and extends to different types of objective functions (e.g., that focus on service aspects of performance) and relaxation of preemption.

Here the SCS is comprised of a single plant with single assembly line and its warehouse, along with a single distribution center and single retailer in the system. The plant can produce two products, but because it has single production line, it can handle only one product at a time. However, other nodes in the system can handle both products in the same time interval. Also, as the retailer cannot satisfy all of the demand, a flow constraint between retailer and customer exists. The presented SCN is only one example of possible SCNs. Therefore many other network topologies can be presented as an example. However, increase of network complexity makes it difficult to understand system response to the price elasticity. Moreover, even if the network structure is in is simple, there are many real world examples of it. For instance, if the retailers are considered as customers, the car distribution network fits the network in our model. There is a single producer, single distribution center and because the inventory is managed by the vendor, the demand is satisfied immediately. The modeled SCN is modeled and solved in GAMS platform. Optimization is performed over 50 time periods for one planning horizon. The parameters used in this example are presented in Table 1.

3.1. Analysis of the Results.

The production schedule obtained by solving the problem with DICOPT is given in Fig. 2. As seen in the Gantt chart, since there is initial inventory of both products, there is no production in the first period. The production of product 1 and product 2 are evenly split (i.e., both product are produced 11 times with each having a production quantity of 440 units). In order to eliminate end of horizon effects on inventory, there is no production beyond period 44. If production of a product is scheduled beyond this period, the material would not have reached the customers due to lead times during transportation. We have observed the same behavior in large number of runs performed by changing the parameter values. The schedule given in Fig. 2 is generated from the parameter values given in Table 1.

Due to nonlinearity of the objective function, the optimization problem is classified as an MINLP problem. However, from Fig. 3 it is easy to see that Eq.(17) with \( E_m > 1 \) is a concave function;
maximization of revenue with linear constraints and fixed binary variables is therefore a convex problem. On the other hand, for $E_m > 1$ and a given $u_{mkk'}(t) = 0$, the price in (16) and the revenue term in (17) have unbounded derivatives. Consequently, when solving the MINLP, this can lead to an overflow error and premature termination.

To avoid unbounded derivatives, optimization platforms like GAMS impose an upper bound on the largest Jacobian element (this is based on machine precision). Moreover, in the NLP solver, CONOPT, the optimization is stopped if a Jacobian element exceeds a prespecified value determined by machine precision (on most machines it is around $2.d5.$). As a result, difficulties with large Jacobian elements lead to either convergence failure or the incorrect detection of optimal solutions.

For instance, both the DICOPT and BONMIN solvers in GAMS allowed us to find feasible solutions, and we would expect these to be global optima. During the runs CONOPT is used as an NLP solver and CPLEX is used as a MILP solver. However, both BONMIN and DICOPT report different solutions with different objective function values, and we will see later that both results are suboptimal. Fig. 2 presents the schedule found by DICOPT. The solution statistics are represented in the Table 2.

The profit value reported by DICOPT is equal to $Z = 37,789$ and the one with BONMIN is equal to $37,698$. However, the global optimum for this problem is $Z^* = 39,594$, as reported in the next section.

### 3.2. Reformulation of the Revenue Function.

Since the revenue terms are the only nonlinear ones in the system, we investigate these further to overcome suboptimal solutions. In these terms, it can be observed that the demand values (denoted here as $\xi$) are equal to zero in some time slots. At this point the corresponding revenue value also equals zero. An illustrative revenue function for a single time slot is shown in Fig. 3.
Table 2  Model statistics for problem solved sub-optimally with DICOPT and BONMIN.

<table>
<thead>
<tr>
<th></th>
<th>DICOPT</th>
<th>BONMIN</th>
</tr>
</thead>
<tbody>
<tr>
<td># of variables</td>
<td>2,554</td>
<td>2,554</td>
</tr>
<tr>
<td># of 0-1 variables</td>
<td>150</td>
<td>150</td>
</tr>
<tr>
<td># of constraints</td>
<td>2,534</td>
<td>2,534</td>
</tr>
<tr>
<td>Obj. fn. value</td>
<td>37,789</td>
<td>37,698</td>
</tr>
<tr>
<td># of main iterations</td>
<td>14</td>
<td>33</td>
</tr>
<tr>
<td>Total CPU time (sec)</td>
<td>77.35</td>
<td>177.6</td>
</tr>
<tr>
<td># of NLP Iterations</td>
<td>2,117</td>
<td>1,944</td>
</tr>
<tr>
<td>CPU time for NLP (sec)</td>
<td>8.84</td>
<td>103.2</td>
</tr>
<tr>
<td># of MIP iterations</td>
<td>17,453</td>
<td>25,276</td>
</tr>
<tr>
<td>CPU time for MIP (sec)</td>
<td>68.51</td>
<td>74.4</td>
</tr>
</tbody>
</table>

Figure 3  An Illustrative Revenue Function with its hybrid approximation. The solid lines are the valid part of the revenue function.

The main reason for the local optimal solutions found by these solvers is the unbounded derivatives of the revenue function at zero demand. At this point, if the derivatives are calculated, numerical instability occurs in the MINLP solver. As seen in Fig. 3, the term in Eq. (17), $R(\xi)$ exhibits highly steep behavior when its argument $\xi$ is near zero and this is the main reason for the unreliable and suboptimal results above. We address this issue by approximating $R(\xi)$ in Eq. (17) between a threshold value and zero, as illustrated in Fig. 3.

For this reformulation, Eqs. (3)-(14) are incorporated into the following MINLP:

$$
\begin{align*}
\text{Max} & \quad Z(x,z) \\
\text{s.t.} & \quad Ax + Bz \leq b, \ x \in \mathbb{R}^n, \ z \in \{0,1\}^n
\end{align*}
$$

(18)

where $x$ are the continuous variables and $z$ are the binary variables. The only nonlinear terms in (18) are due to the revenue component in the objective function. Otherwise, all of the constraints are linear. Therefore, for fixed values of $z$, a concave maximization is made in a convex feasible region. As noted in Duran and Grossmann (1986), both DICOPT and BONMIN are guaranteed to converge to global optima for this problem, as long as their functions and their gradients remain bounded. However, as seen above, unbounded derivatives (and interventions by NLP solvers) can lead to convergence failure or misleading solutions. In this section we reformulate (18) so that the global convergence properties can be enforced.

As seen in Fig. 3 for $\xi$ between $\epsilon$ and zero, the revenue function in Eq. (17) is approximated by the tangent vector of the revenue function at $\xi = \epsilon$. This is stated as:

$$
\bar{R}(\xi) = \begin{cases} 
\left( \frac{E_m \epsilon^{-1}}{E_m} \right) R(\epsilon) (\xi - \epsilon) + R(\epsilon), & \text{if } \xi < \epsilon \\
R(\xi), & \text{otherwise}
\end{cases}
$$

(19)
According to Eq. (19) a linear approximation is used in place of the revenue function between zero and \( \epsilon \). For \( \xi \geq \epsilon \) the original revenue function is used.

Using this substitution in (18) leads to the MINLP:

\[
\begin{align*}
\text{Max} & \quad \bar{Z}(x, z) \\
s.t. & \quad Ax + Bz \leq b, \ x \in \mathbb{R}^{nx}, z \in \{0, 1\}^{nz}
\end{align*}
\] (20)

with the following properties.

**Property 1:** Given \( \xi \in \mathbb{R} \), \( \bar{R}(\xi) \) is a concave function. Consequently, every NLP subproblem of (20) is convex and the outer approximation algorithm converges to a global solution.

**Proof:** As \( \bar{R}(\xi) \) is differentiable, concavity holds by satisfaction of the inequalities:

\[
\begin{align*}
\bar{R}(\xi_1) + \frac{d\bar{R}(\xi_1)}{d \xi}(\xi_2 - \xi_1) & \geq \bar{R}(\xi_2) \quad \text{(21)} \\
\bar{R}(\xi_2) + \frac{d\bar{R}(\xi_2)}{d \xi}(\xi_1 - \xi_2) & \geq \bar{R}(\xi_1) \quad \text{(22)}
\end{align*}
\]

for all \( \xi_1 \leq \xi_2 \). This inequality is easily seen for \( \xi_1, \xi_2 \leq \epsilon \), where \( \bar{R} \) is linear in \( \xi \), and for \( \xi_1, \xi_2 \geq \epsilon \), where \( \bar{R} \) equals the concave function \( R(\xi) \). For the case where \( \xi_1 \leq \epsilon \leq \xi_2 \), we have:

\[
\begin{align*}
\bar{R}(\xi_1) + \frac{d\bar{R}(\xi_1)}{d \xi}(\xi_2 - \xi_1) & = \bar{R}(\xi_1) + \frac{dR(\epsilon)}{d \xi}[(\xi_2 - \epsilon) + (\epsilon - \xi_1)] \\
& = R(\epsilon) + \frac{dR(\epsilon)}{d \xi}(\xi_2 - \epsilon) \geq R(\xi_2) = \bar{R}(\xi_2) \quad \text{(23)}
\end{align*}
\]

where the last inequality follows by concavity of \( R(\xi) \).

For the second inequality (22), we note that

\[
\frac{dR(\xi_2)}{d \xi} \leq \frac{dR(\xi_1)}{d \xi} \quad \text{for} \ \xi_2 \geq \xi_1,
\] (24)

and by concavity,

\[
R(\xi_2) + \frac{dR(\xi_2)}{d \xi}(\epsilon - \xi_2) \geq R(\epsilon)
\] (25)

Adding \( \frac{d\bar{R}(\xi_2)}{d \xi}(\xi_1 - \epsilon) \) to both sides of (25) leads to:

\[
\begin{align*}
\bar{R}(\xi_2) + \frac{d\bar{R}(\xi_2)}{d \xi}(\xi_1 - \xi_2) & = R(\xi_2) + \frac{dR(\xi_2)}{d \xi}(\xi_1 - \xi_2) \\
& \geq R(\epsilon) + \frac{dR(\xi_2)}{d \xi}(\xi_1 - \epsilon) \\
& \geq R(\epsilon) + \frac{dR(\epsilon)}{d \xi}(\xi_1 - \epsilon) = \bar{R}(\xi_1)
\end{align*}
\]

and Eq. (22) follows. Convexity of the NLP subproblems then follows from the linear constraints in (18). With this property, convergence to global solutions has been shown in Duran and Grossmann (1986). □

**Property 2:** Let \( (x^*, z^*) \) be the (global) solution of (18) and \( (\bar{x}, \bar{z}) \) be the (global) solution of (20). Also, from the solution \( (\bar{x}, \bar{z}) \) define \( K = N_{cs} \cup N_{de} \cup N_{pr} \cup N_{pw} \cup N_{rt} \), the Cartesian product
J = M × K × K × T and the set \( U = \{ j | j \in J, u_{mkk}(t) \in [0, \epsilon] \} \), with cardinality \(|U|\). These solutions lead to the inequality:

\[
\bar{Z}(\bar{x}, \bar{z}) \geq Z(x^*, z^*) \geq \bar{Z}(\bar{x}, \bar{z}) - R(\epsilon)|U|/E_m,
\]

and hence, \(|\bar{Z}(\bar{x}, \bar{z}) - Z(x^*, z^*)| = O(\epsilon^{(1-1/E_m)})\).

**Proof:** Since \((x^*, z^*)\) and \((\bar{x}, \bar{z})\) are both feasible for \(Ax + Bz \leq b, \ z = \{0, 1\}^n\), and are global maximizers for their respective problems, it is clear that \(Z(\bar{x}, \bar{z}) \geq \bar{Z}(x^*, z^*)\) and \(Z(x^*, z^*) \geq Z(\bar{x}, \bar{z})\). Moreover, since \(R(\xi) \geq R(\xi)\), for all values of \(\xi\) we have \(Z(x^*, z^*) \geq Z(x^*, z^*)\). Finally, since \(R(\xi) \geq \bar{R}(\xi) - R(\epsilon)/E_m\) for all values of \(\xi\), we have \(Z(\bar{x}, \bar{z}) \geq \bar{Z}(\bar{x}, \bar{z}) - R(\epsilon)|U|/E_m\). Combining these relations leads to the following set of inequalities:

\[
\bar{Z}(\bar{x}, \bar{z}) \geq \bar{Z}(x^*, z^*) \geq Z(x^*, z^*) \geq Z(\bar{x}, \bar{z}) \geq \bar{Z}(\bar{x}, \bar{z}) - R(\epsilon)|U|/E_m.
\]

and the desired result follows. □

Note that this reformulation introduces an approximation error. As seen in Fig. 3, at \(\xi = 0\) the value of the revenue term equals \(R(\epsilon)/E_m\), not zero. Therefore, to deal with this approximation error, we subtract the term

\[
\sum_m \sum_k \sum_{k''} \sum_t R(\epsilon) * (1 - pr_{mk}(t))/E_m
\]

from the approximate objective function, \(\bar{Z}(x, z)\), along with the constraints \(\sum_a pr_{mka}(t) \geq pr_{mk}(t)\) and \(pr_{mk}(t) \geq pr_{mka}(t), \forall a\). This modification allows us to redefine \(U\) to \(U = \{ j | j \in J, u_{mkk'}(t) \in (0, \epsilon) \}\) and this strengthens Property 2. This property now shows that the global solution of (18) can easily be bounded using the solution of (20). These bounds are affected by the choice of \(\epsilon\) and the number of positive demands with optimal values less than \(\epsilon\). While a detailed convergence analysis to the global optimum is beyond the scope of the paper, it should be noted that this bound on the global optimum is valid for all values of \(\epsilon\) and the gap between the bound and the global optimum is computable; this gap depends on \(|U|\) and vanishes with \(\epsilon\). However, finding an appropriate value for \(\epsilon\) is problem dependent and may require some trial and error choices of \(\epsilon\).

To provide some guidance in the choice of \(\epsilon\), one can invoke Property 2 and choose an \(\epsilon\) large enough to yield a well-conditioned solution. For, instance, if we let \(M\) represent the largest allowable Jacobian element in the optimization algorithm and define a safety factor, say \(K = 2\), then from (19) a reasonable initial value of \(\epsilon\) is given by:

\[
\epsilon \geq K \left( \frac{h(E_m - 1)}{ME_m} \right)^{E_m}
\]

From this point, successive solutions with smaller values of \(\epsilon\) can be found until \(R(\epsilon)|U|/E_m\) is sufficiently small and an acceptable bound on the global solution is obtained, or ill-conditioning is encountered. In either case, the procedure still yields a valid and useful lower bound on the global solution. Moreover, as seen in our examples, judicious choices of \(\epsilon\) often lead to \(|U| = 0\) and \(|\bar{Z}(\bar{x}, \bar{z}) - Z(x^*, z^*)| = 0\).

Finally, the smoothed revenue term can be implemented in a declarative language (such as GAMS) as follows (Kravanja and Grossmann (1990)):

\[
\bar{R} = \max(\min[(E_m - 1)/\epsilon E_m] R(\epsilon)(\xi - \epsilon) + R(\epsilon), R(\epsilon), R(\xi))
\]

Note that, despite the \(\min\) and \(\max\) operators in (28), \(\bar{R}(\xi)\) is still differentiable at \(\xi = \epsilon\) and its derivatives now remain bounded at \(\xi = 0\). Nevertheless, this problem could not be solved by the SBB solver because it does not allow \(\min\) and \(\max\) operators in the problem statement.
Table 3  Model statistics for problem that is solved optimally ($E_1 = E_2 = 4$).

<table>
<thead>
<tr>
<th></th>
<th>DICOPT</th>
<th>BONMIN</th>
</tr>
</thead>
<tbody>
<tr>
<td># of variables</td>
<td>2,705</td>
<td>2,705</td>
</tr>
<tr>
<td># of 0-1 variables</td>
<td>300</td>
<td>300</td>
</tr>
<tr>
<td># of constraints</td>
<td>2,535</td>
<td>2,535</td>
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<tr>
<td>Obj. fn. value</td>
<td>39,594</td>
<td>39,594</td>
</tr>
<tr>
<td># of main iterations</td>
<td>21</td>
<td>4</td>
</tr>
<tr>
<td>Total CPU time (sec)</td>
<td>62.491</td>
<td>14.187</td>
</tr>
<tr>
<td># of NLP Iterations</td>
<td>7,275</td>
<td>195</td>
</tr>
<tr>
<td>CPU time for NLP (sec)</td>
<td>54.20</td>
<td>8.3</td>
</tr>
<tr>
<td># of MIP iterations</td>
<td>17,735</td>
<td>33</td>
</tr>
<tr>
<td>CPU time for MIP (sec)</td>
<td>8.29</td>
<td>5.8</td>
</tr>
</tbody>
</table>

4. Illustrative Example Revisited.

After the approximation (19) is incorporated into the revenue function (17) by setting $\epsilon$ value to 2, the same runs are carried out again. In this case, we see that $|U| = 0$ at every solution and there is no error from the approximation (19). As a result, the global optimum can be verified from Property 2. To assess the performance of the system, consider the optimal schedule illustrated in Fig. 4. The objective function value obtained from this schedule is $Z^* = 39,157$ which is better than the reported former result. The solution statistics are reported in the Table 3.

The production schedule reported by the optimal solution is completely different than the one reported by the suboptimal solution. The production schedule of products 1 and 2 (as shown in Fig. 4) alternate so as to minimize the inventory buildup of products in the SCS. In addition to production schedules, the objective values in Table 3 are also different for sub-optimal solution reported in Table 2 and optimal solution in Table 3. The main reason for this is the schedule difference and the consecutive changes in the operational behavior of the SCS. Since production
schedule is different, production amounts and the inventory amounts are also different. In addition, since price-quantity relationship is non-linear and depends on the price elasticity value, different production schedules yield different revenues and profit values. Also, without the reformulation method developed in this paper, our MINLP solvers failed to find the optimal solutions.

In order to observe the behavior of SCS with the reformulation approach presented in this paper, we first present the case when the elasticity value of product 2 is doubled. The solution statistics for this case are shown in Table 4.

It is interesting to note that the majority of the production switches to product 2 at this higher price elasticity value, as shown in Fig. 5. When, $E_1 = E_2 = 4$, the production between products 1 and 2 is evenly split for 11 production runs, with a total of 440 units produced for each product. When, $E_1 = 4$ and $E_2 = 8$, product 1 is not produced at all while, product 2 is produced in 22 different time periods with a total production volume of 880 units. The production shift from product 1 to product 2 when the price elasticity for product 2 is higher can be explained by examining Fig. 6.

Fig. 6 shows the relationship between the elasticity of the product 2 and the objective function value of the system. As seen in this figure, while the elasticity of product 2 is increasing, the objective function value of the system also increases. Because the profit of SCN is taken as an objective function, it is observed that continuous increase of price demand elasticity for product values leads to continuous increase of the system profit. At first, while the elasticity is increased, system can increase the demand by applying small decreases to the price. However, after a specific value, the system reaches its limits and no additional demand can be satisfied by the system. After this point, increase of the elasticity just leads to increase in price. Therefore, before and after this threshold value, the profit of the system shows continuous increase in response to increase in the elasticity value of the product. This behavior is illustrated in Fig. 6.

Finally, to observe the effect of price elasticity we solved over 50 cases of the reformulated model to generate the results in Fig. 7. All of these results were solve easily with CPU times similar to
Figure 6  Change in the total profit with respect to $E_2$ when $E_1 = 4$.

Figure 7  Average price and satisfied demand of products 1 and 2 (MaxProduction = 40 and $E_1 = 4$).

those in Tables 3 and 4. Fig. 7 illustrates the relationship between elasticity of product 2 when the elasticity of product 1 is constant for the average demand satisfied and average price of products 1 and 2. As seen in Fig. 7-a, while the elasticity of the product 2 is increasing, the average price of product 2 first decreases and then increases. The main reasons of this behavior are the effect of the higher price elasticity and the capacity of the system. At lower elasticity values the price of product 2 decreases because, during this interval decrease of the price causes increase in the demand of product 2. The increased rate of the demand is higher than decreased rate of the price; therefore this behavior of the price causes increase in the profit. Because the objective function is maximization of the profit, this behavior is reasonable. However, as seen in Fig. 7-b after a point, the price of product 2 increases. At this point, system reaches its maximum capacity; in other words, it cannot satisfy any more demand. Once this point identifies the maximum capacity, the amount of satisfied demand remains constant. Therefore, increase of the elasticity causes increase in the price. As seen in Fig. 7-a, the price of product 1 increases and it remains constant at the value of 300. The main reasons for this behavior are also the effect of price elasticity and capacity of the system. As the price of product 1 reaches 300, the demand for product 1 becomes zero. This
shows that, because the elasticity of product 2 is higher than that of product 1, producing and selling product 2 is more profitable. Therefore, system dedicates itself to the production of product 2. This can also be seen in Fig. 7-b as the satisfied demand of product 1 decreases to zero and that of product 2 increases to maximum capacity of the system.

5. Conclusions
In this paper, we present an optimization model for integrated multi-product and multi-echelon SCS including price elasticity. The specification of corresponding SCS that includes suppliers, plants, plant warehouses, distribution centers and retailers, is illustrated on a two product case. In particular, we analyze the characteristics of the nonlinear revenue function with the consideration of price demand elasticity. Since MINLP solvers report suboptimal solutions for this problem, an approximation reformulation is applied to obtain optimal solution.

Effects of changing price elasticity on the optimal price changes and optimal amount of satisfied demand are also presented. It is shown that increase in the price elasticity leads to an increase in the satisfied demand and that the capacity of system plays a crucial role on the average price of the products. In addition, the relationship between price elasticity value of products and total profit relation is illustrated. It is shown that as the elasticity of products increases, the profitability of the system increases and overall better results are obtained.

Acknowledgments
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Appendix. Nomenclature

Indices:
- \(a\) production line \(a\) in production facility \(k\) node \(k\) in the SCN
- \(k'\) upstream node of node \(k\) in the SCN
- \(k''\) downstream node of node \(k\) in the SCN
- \(n, m\) product \(n\) or \(m\)
- \(t\) time \(t\)

Sets:
- \(A\) set of assembly line
- \(N_{cs}\) subset of nodes \(k\) that represent customers
- \(N_{dc}\) subset of nodes \(k\) that represent distribution center
- \(N_{pr}\) subset of nodes \(k\) that represent manufacturing plants
- \(N_{pw}\) subset of nodes \(k\) that represent warehouses at plants
- \(N_{rt}\) subset of nodes \(k\) that represent retailers
- \(M\) set of product \(m\)
- \(T\) set of time intervals \(t\)

Decision Variables:
\[ I_m(t) \] inventory level of product \( m \) in node \( k \) at time \( t \)
\[ O_{mkk''}(t) \] order accumulation level of product \( m \) in node \( k \) from downstream node \( k'' \) at time \( t \)
\[ P_{Rmka}(t) \] production level of product \( m \) in plant \( k \) on line \( a \) at time \( t \)
\[ p_{Rmka}(t) \] binary variable for the start of production of product \( m \) in plant \( k \) on line \( a \) at time \( t \)
\[ u_{mkk''}(t) \] demand from downstream node \( k'' \) for product \( m \) in node \( k \) at time \( t \)
\[ y_{mkk''}(t) \] flow of product \( m \) from node \( k \) to downstream node \( k'' \) at time \( t \)
\[ y_{mk'k}(t) \] flow of product \( m \) from upstream node \( k' \) to node \( k \) at time \( t \)
\[ z \] binary variables
\[ \xi \] argument of revenue term, \( R(\xi) \), and approximation, \( \bar{R}(\xi) \).

**Parameters:**

\[ E_m \] absolute value of price elasticity of product \( m \)
\[ C_{RE} \] total revenue
\[ C_{HO} \] total holding cost
\[ C_{TR} \] total transportation cost
\[ C_{RM} \] total raw material cost
\[ C_{PF} \] total fixed cost of production
\[ C_{PV} \] total variable cost of production
\[ HC_{mk} \] holding cost of 1 unit of product \( m \) during 1 time interval on node \( k \)
\[ TC_{mkk''} \] transportation cost of 1 unit product \( m \) from node \( k \) to downstream node \( k'' \)
\[ RC_{ka} \] cost of 1 unit raw material \( s \) for production node \( k \)
\[ Req_{ms} \] amount of required raw material \( s \) for the production of 1 unit product \( m \)
\[ FC_{mk} \] fixed cost of production of product \( m \) at plant \( k \)
\[ VC_{mk} \] variable cost of production of product \( m \) at manufacturing plant \( k \)
\[ Prc_m(t) \] price of product \( m \) at time \( t \)
\[ \tau_{k'k} \] transportation delay between opstream node \( k' \) and node \( k \)
\[ \iota_m \] production time of product \( m \)
\[ h \] price of end product that makes the demand one
\[ P_{Rmka}(t) \] maximum production of product \( m \) at plant \( k \) on line \( a \) at time \( t \)
\[ u_{mkk''}^{old}(t) \] demand for product \( m \) in node \( k \) from downstream node \( k'' \) at time \( t \) in previous planning period
\[ P_{Rmka}^{old}(t) \] price of product \( m \) at time \( t \) in the previous planning period
\[ R \] revenue term
\[ \bar{R} \] approximate revenue term
\[ MF \] maximum flow constraint

**References**


