OPPORTUNITIES FOR SAMING REALIZATIONS IN DIFFERENT TASKS

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The Realization Tree Assessment (RTA) tool offers a graphical presentation of the mathematical ideas students got exposed to and engaged with through a certain task. It depicts the mathematical object together with its different realizations, the extent to which opportunities for saming those different realizations were given to students during a lesson and the extent to which students had authority to produce narratives about the object. Five mathematics lessons which were based on a middle-school task dealing with linear functions were analysed using the RTA. The results were compared to RTAs of lessons based on a pattern generalization task. We discuss the similarities and differences between the RTAs in terms of opportunities for explorative participation as well as exhausting the potential of a task.

INTRODUCTION AND THEORETICAL BACKGROUND

The Realization Tree Assessment (RTA) tool (Weingarden & Heyd-Metzuyanim, 2017, 2018; Weingarden, Heyd-Metzuyanim, & Nachlieli, 2017) is designed to explicate the mathematical object that appeared through engagement with a task together with its different realizations. Based on ideas from the commognitive framework (Sfard, 2008), the RTA displays graphically the extent to which opportunities for saming different realizations of the mathematical object were given to students during a lesson. Uniquely from other evaluations tools, the RTA allows exposure to the mathematical content of the lesson, including the mathematical objects that students could be exposed to through engaging with the task. However, the uniqueness of the mathematical content of each task means that each lesson needs a new and unique "skeleton" of an RTA. This "skeleton" includes the main mathematical object that could be exposed by the task, as well as its different realizations. This exclusiveness enables a relatively straightforward comparison between lessons based on the same task, yet it limits the comparison between lessons based on different tasks. This limitation is crucial when one wishes to use the RTA for evaluating various lessons, and planning and analyzing lessons together with teachers.

Until now, we applied the RTA only on lessons based on one particular task – the Hexagons task, to examine how different lessons offer different opportunities for saming realizations of a mathematical object (Weingarden & Heyd-Metzuyanim, 2017, 2018). In the present study, we continue the development of the tool. Our goal is to examine the similarities and differences of RTAs of lessons based on two different tasks.

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THE RTA

The Realization Tree Assessment (RTA) tool (Weingarden et al., 2017) has been developed in the context of the TEAMS (Teaching Exploratively for All Mathematics Students) professional development program. The RTA was developed as an answer to the insufficiency of previous tools to capture the mathematical aspects of explorative instruction – instruction that offers students opportunities for explorative participation.

In explorative participation, the learner establishes new mathematical narratives based on formerly established ones, while objectifying – talking about mathematical objects as existing by themselves (Sfard, 2008). In order to objectify, the student needs to 'same' the different realizations of the mathematical object. The process of *saming* is described by Sfard (2008) as "assigning [the signifier] to a number of things that, so far, have not been considered as in any way 'the same' but are mutually replaceable in a certain closed set of narratives" (p. 170).

The RTA was initially developed to examine students' opportunities for explorative participation, while focusing on the opportunities for saming different realizations of the mathematical object. This is done by graphically illustrating (1) the different realizations of the mathematical objects that are presented during the lesson; (2) the extent to which links between realizations are made; and (3) the extent of students' authority (who produces the mathematical narratives).

In our former works with the RTA, we applied it on one particular task (the Hexagons task) and used it to analyse 10 lessons of different teachers (Weingarden & Heyd-Metzuyanim ,2017, 2018). From these studies we learned about the various ways in which teachers implement the Hexagons task, the different levels of exposure to the mathematical objects afforded to students in different classrooms, and the connections between level of explorations and characteristics such as grade level and track.

In the Hexagons task students are asked to describe the perimeter of a general "train" in a pattern of hexagons "trains" (see Figure 1).

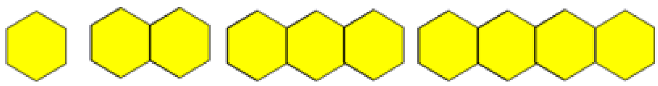

Figure 1: The Hexagons Pattern

This task's richness lies in its affordance to connect different algebraic expressions to a single visual mediator (the perimeter of the Hexagons' train), as there are different algebraic expressions that express how this perimeter can be counted. Therefore, the task provides opportunities for saming the different algebraic expressions that describe different procedures for counting the Hexagon's sides. For example, 4x+2 describes counting upper and bottom sides, then the 2 edges external edges; 6x-2(x-1) describes counting all the hexagon edges, then omitting the internal ones; and there are many more.

The comparison of 10 different lessons that were all based on the Hexagons task revealed interesting differences in the extent to which students got exposed to the various realizations of the "algebraic expression" object. Some lessons included only very little exposure to different realizations, yet those realizations that were mentioned, were explained by students. Other lessons contained many realizations, yet most were authored by the teacher. The lessons we evaluated as most "explorative" were those that contained multiple realizations, multiple links drawn between them, and highest students' authority.

Comparing the "level of exploration" in a lesson is useful for several reasons. First, it enables studying the effects of interventions such as professional development. Second, it may offer insights into factors that contribute to explorative instruction (such as teachers' expertise, school/curriculum variables, tracking and more). Finally, it can serve as a tool for teachers to reflect upon lessons that have implemented a cognitively demanding task.

However, all these potential benefits will only be possible if the RTA can be flexibly applied to a variety of tasks. Therefore, our research question is: to what extent can the opportunities for students' explorative engagement be identified, compared and contrasted by the RTAs of lessons implementing two different tasks?

METHOD

As indicated above, the study reported here was performed in the context of the TEAMS project for training Israeli teachers to implement explorative instructional practices in middle school mathematics classrooms. As part of the professional development, the teachers were asked to implement and videotape a lesson based on a task they had experienced as learners in the professional development. One of those tasks is the Calling-Plans task (originally designed by the Institute for Learning, University of Pittsburgh) which deals with the intersection of two linear functions (see Figure 2). The Calling-Plans task's richness lies in its potential to expose, discuss and link the four realizations of the function object taught in middle-school: verbal, algebraic, graphic and ordered-pairs. This provides opportunities for saming the different realizations of the "intersection of functions" object. In particular, since the Calling-Plans scenario includes two different "calling plans" (each which can be described as a linear function), we denoted as the main object at the root of the RTA to be: intersection of two functions.

> Long Distance Company A charges a base rate of 45 NIS (equivalent of dollars) per month plus 5 agoras (equivalent of cents) per minute that you are on the phone. Long Distance Company B charges a base rate of only 20 NIS per month but they charge you 30 agoras per minute used. Which company would you choose, why?

Figure 2: The Calling-Plans task

The analysis of lessons using the RTA is based on watching only the whole-classroom discussion. It does not require transcription, but rather a careful design of the "skeleton" of the tree, based on theoretical knowledge of the potential of the task. This knowledge is derived through discussion with mathematicians and mathematics educators. The design of the "skeleton" is followed by shading and marking realizations and links whenever they are spotted during the lesson. In the case of the Calling-Plans task, the skeleton of the RTA depicts the mathematical object "intersection of two functions" at the top of the tree and its different realizations as nodes in the tree (see Figure 3). We code the tree according to two criteria: (1) Coloring the realizations that were exposed to students during the lessons based on who articulated the realization (dark color = student; light color = teacher.) (2) Drawing arches between the realizations that were linked during the discussion (continuous line = link made by students; dashed line = link made by the teacher).

Two elements of explorative lessons are examined by the RTA. The first – opportunities for saming realizations - describes the extent to which the lesson exposed students to the different realizations and offered opportunities to same them, that is, view them as representing the same mathematical object. This element is pictured in the RTA by the fullness of the tree: multiple realizations are shaded and multiple links between realizations are drawn. The second element of explorative lessons – students' authority, describes the extent to which *students* (rather than the teacher) articulate the realizations and links. Students' authority is identified in the RTA by the darkness of the tree and by the continuity of the lines.

FINDINGS

Level of explorations in the RTAs of the Calling-Plans task

We start by describing the RTAs of three lessons based on the Calling-Plans task. This is done to exemplify the method and to display contrasting levels of opportunities for explorative participation. Due to space limitations, we only present full RTAs of one Calling-Plans and one Hexagons lesson. However, our analysis is based on unique RTAs drawn for each lesson. The first lesson took place in 8^{th} grade and is called Calling-Plans lesson 1 (CP-Ls1, see Figure 3). In this lesson, students articulated an algebraic realization of the Calling-Plans problem: "y=0.05x+45 , y=0.3x+20", and made some connections to the possible verbal realizations (underlined in the verbal realization box). They described briefly how they found that x=100 ("I subtracted [the two functions]", "I compared them [the two functions] to find the intersection") and mentioned the verbal realizations when they explained: "[when the calling time is] less than 100 minutes, it's more profitable to choose Company B… and above 100 it's better to choose Company A". No other realizations were made explicit during CP-Ls1 and no other connections to the various realizations of the object were drawn. CP-Ls1 is thus identified by a high level of students' authority but opportunities for saming realizations were scarce.

Quite a different picture of explorative participation was found in the second lesson (CP-Ls2). Here, opportunities for saming realizations were found throughout the lesson, but the level of students' authority was low. In CP-Ls2, three types of realizations were exposed to students during the whole-classroom discussion: the verbal realization, several algebraic realizations and the ordered-pairs realization. In addition, links between the realizations were made. However, the light color of most of the realizations (teacher-authored realizations) and specifically, the dashed arches between realizations (teacher-authored links) show that although students were exposed to different realizations of the mathematical object and were provided with opportunities for saming realizations during the lesson, they did not author narratives about the links between the realizations and, consequently, no new narratives about the main mathematical objects were constructed by the students.

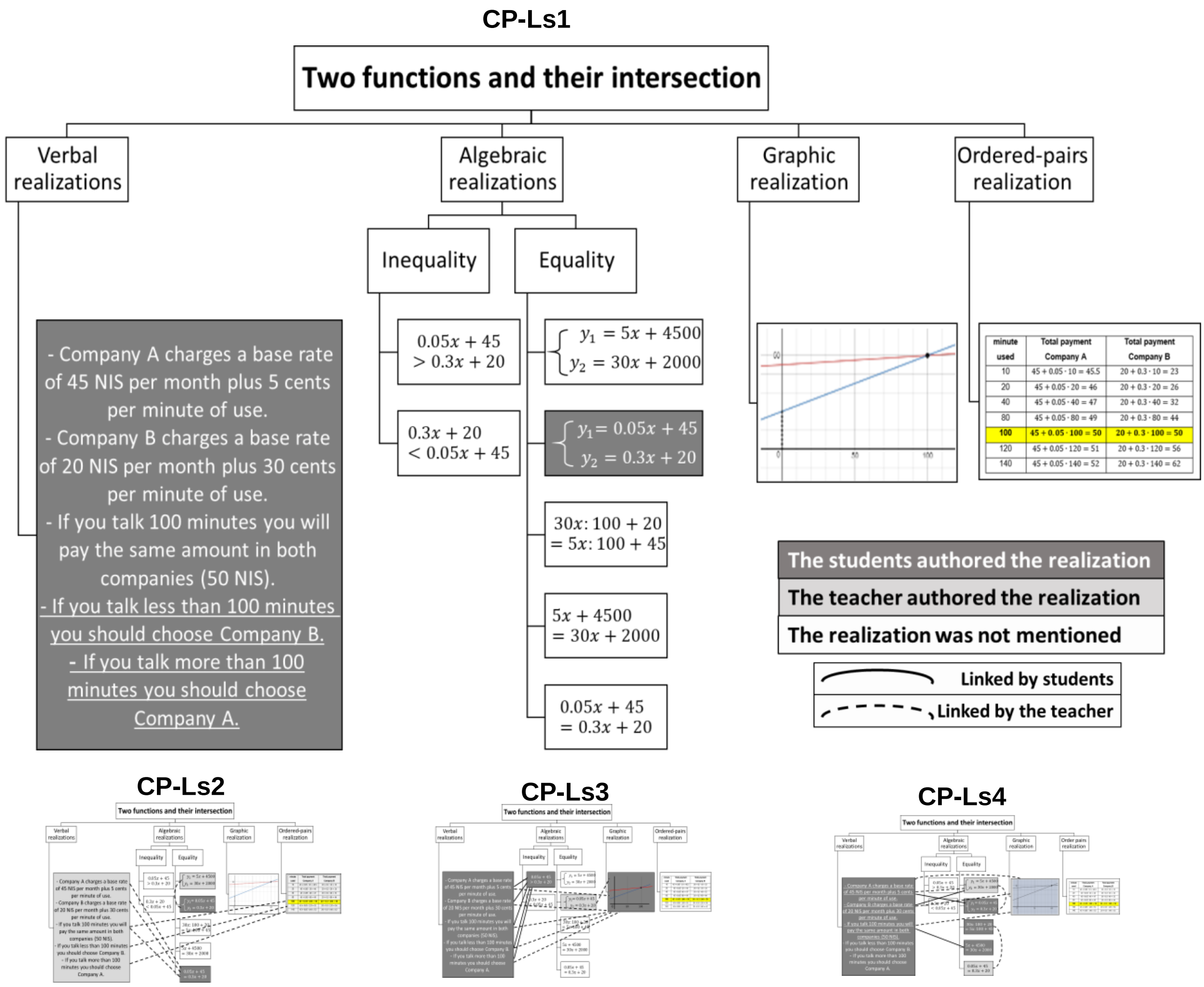

Figure 3: Calling-Plans RTAs

In the 3rd lesson (CP-Ls3), both elements of explorations were found: opportunities for saming realizations and students' authority. Students articulated all three types of realizations that were discussed during the lesson: verbal, graphic and inequality algebraic realizations. One student explained the inequality that she built as she was making links between the verbal realization (underlined) and the inequality algebraic realization (in bold):

"we did the rate of Company A that is 45 NIS (equivalent of dollars) per month and 5 agoras and then we basically did: **45 + 0.05x**. And then we actually did (an) **inequality** of the rate of Company B, which is 20 NIS per month and 30 agoras per call [minutes], which is: **0.3x+20**. And then actually you do the inequality and then find the [intersection]"

Similarities and differences between the RTAs of the two tasks

We now move to compare and contrast the Calling-Plans RTAs with the Hexagons-Task RTAs. At first glance, the skeleton of the Hexagons RTA (Figure 4) is significantly different from the Calling-Plans RTA. The main difference is that each task offers engagement with a different mathematical object.

While the Calling-Plans task deals with the 'intersection of two functions' object, the Hexagons task deals with the 'perimeter' object (perimeter of the Hexagon's general train). Each task affords opportunities for highlighting and saming its main object's different realizations.

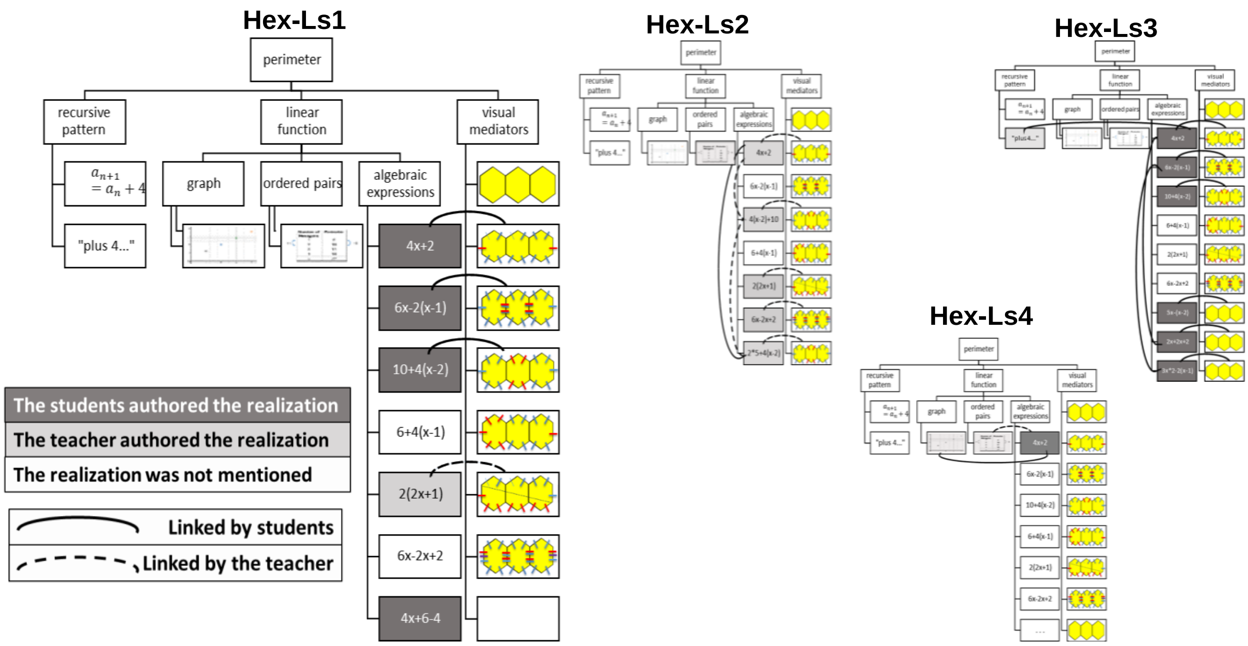

Figure 1: The Hexagons' RTA

Despite these differences, more general similarities can be found between the RTAs of the two tasks. For example, in each of the RTAs (CP and Hex), there are lessons that can be characterized as relatively "weak" or "strong" in students' authority. This is relatively simple to observe. For example, the RTAs depicting CP-Ls3 and Hex-Ls3 are strong in students' authority, while CP-Ls2 and Hex-Ls2 are weak. Another observation that can be made, across the tasks, relates to the opportunities for saming different realizations. Unlike CP-Ls3 and Hex-Ls 3, which contain multiple realizations and multiple links, CP-Ls1 and Hex-Ls1 contain relatively little opportunities for saming different realizations. In CP-Ls1, there are only 2 realizations shaded, and no links. In Hex-Ls1, there are several realizations but very few links, and none of these relate to the saming of the algebraic expressions. We call lessons depicted by such RTAs "show and tell" lessons, where multiple students present their different solutions, but no links are made between them.

The comparison of the two RTAs (CP and Hex) highlights that not all links are necessarily productive for achieving the mathematical goal of the lesson. Take for example the links drawn in different RTAs of the CP task. In CP-Ls4, one teacher devoted much attention to the question of how to convert NIS (equivalent of $) to agoras (cents) and what that would mean for the algebraic expressions y=0.05x+45 which would turn into y=5x+4500. This shifted the discussion to the two *instantiations* of the function objects (y=0.05x+45 & y=0.3x+20; y=5x+4500 & y=30x+2000). Although these two pairs of functions are useful for solving the same problem, this issue was peripheral to the object of the lesson, which was saming the different realizations of the two functions and their intersection.

A similar situation occurred in some Hexagons lessons (e.g Hex-Ls4), where links were made between the algebraic, ordered-pairs and graphic realizations of the 4x+2 expression. Paradoxically, such links would have been appropriate for the Calling-Plans task. In the Hexagons' task, however, the main object to be samed were the algebraic expressions, as embedded in the visual realization of the perimeter of the trains (being "the same" for all the various ways in which it can be counted and expressed algebraically).

DISCUSSION AND CONCLUSION

Our goal for this paper was to examine the extent to which opportunities for students' explorative engagement can be identified, compared and contrasted by the RTAs of lessons implementing two different tasks, the Calling-Plans and the Hexagons task.

Our findings indicate similarity in certain elements of the RTAs and differences in others. The element of students' authority can easily be identified in the RTAs of both tasks. This element is identified by the darkness of the tree (students' realizations) and by the continuity of the lines (students' links between realizations). However, the fullness of the tree (multiple realizations and multiple links), does not necessarily indicate the level of explorative instruction. There are instances where multiple realizations appear in a lesson, yet the lesson does not amount to a substantial mathematical idea. These occur in two main types of lessons: (1) "Show and tell" lessons (Stein, Engle, Smith, & Hughes, 2008), where multiple realizations are shaded, yet no links are drawn between them. (2) "Concepts-gone-wrong" lessons, where multiple links are made, yet they are not the important links that should be highlighted by the task.

The important conclusion drawn from the addition of examining the RTAs of the Calling-Plans task is that links do not always produce opportunities for saming. In certain cases, classroom discussions diverge into making links that do not same realizations of the central mathematical object underlying the task. We claim that the

RTA, especially when applied to different tasks, is useful for exposing the relation between the goal of the task and its enactment. This should be useful not only for researchers attempting to understand classroom instruction, but also as a "representation of practice" (Grossman et al., 2009) by which teacher educators could discuss with teachers the various opportunities for engaging students with objectification that arise from a task.

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