Existence and global attractivity of positive periodic solutions of periodic $n$-species Lotka–Volterra competition systems with several deviating arguments

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Abstract

In this paper, we study the existence and global attractivity of positive periodic solutions of periodic $n$-species Lotka–Volterra competition systems. By using the method of coincidence degree and Lyapunov functional, a set of easily verifiable sufficient conditions are derived for the existence of at least one strictly positive (componentwise) periodic solution of periodic $n$-species Lotka–Volterra competition systems with several deviating arguments and the existence of a unique globally asymptotically stable periodic solution with strictly positive components of periodic $n$-species Lotka–Volterra competition system with several delays. Some new results are obtained. As an application, we also examine some special cases of the system we considered, which have been studied extensively in the literature. Some known results are improved and generalized. © 1999 Elsevier Science Inc. All rights reserved.

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1. Introduction

In recent years, the application of theories of functional differential equations in mathematical ecology has developed rapidly. Various mathematical models have been proposed in the study of population dynamics, ecology and epidemic. Some of them are described as nonautonomous delay differential equations. Many people are doing research on the dynamics of population with delays, which is useful for the control of the population of mankind, animals and the environment. One of the famous models for dynamics of population is the Lotka–Volterra competition system. Owing to its theoretical and practical significance, the Lotka–Volterra systems have been studied extensively [1–3,5,6,8–22]. To consider periodic environmental factors, it is reasonable to study the Lotka–Volterra system with periodic coefficients. A very basic and important ecological problem associated with the study of multispecies population interaction in a periodic environment is the existence and global attractivity of periodic solutions. Such questions arise also in many other situations.

In this paper, we investigate the following periodic $n$-species Lotka–Volterra competition system with several deviating arguments

$$\ddot{y}_i(t) = y_i(t) \left[ r_i(t) - \sum_{j=1}^{n} a_{ij}(t)y_j(t - \tau_{ij}(t)) \right], \quad i = 1, 2, \ldots, n,$$

(1.1)

where $a_{ij} \in \mathbb{C}(\mathbb{R}, [0, \infty))$, $r_i, \tau_{ij} \in \mathbb{C}(\mathbb{R}, \mathbb{R})$, are $\omega$-periodic functions with

$$\bar{r}_i := \frac{1}{\omega} \int_{0}^{\omega} r_i(t) \, dt > 0; \quad \bar{R}_i := \frac{1}{\omega} \int_{0}^{\omega} |r_i(t)| \, dt;$$

$$\bar{a}_{ij} := \frac{1}{\omega} \int_{0}^{\omega} a_{ij}(t) \, dt \geq 0, \quad i, j = 1, 2, \ldots, n.$$

(1.2)

In mathematical ecology, Eq. (1.1) denotes a model of the dynamics of an $n$-species system in which each individual competes with all others of the system for a common resource and the intra-species and inter-species competition involves deviating arguments $\tau_{ij}$. The assumption of periodicity of the parameters $r_i, a_{ij}, \tau_{ij}$ is a way of incorporating the periodicity of the environment (e.g. seasonal effects of weather condition, food supplies, temperature, mating habits etc). The growth functions $r_i$ are not necessarily positive, since the environment fluctuates randomly; in bad conditions, $r_i$ may be negative. The existence and attractivity of periodic solutions of some special cases of Eq. (1.1) has been studied extensively in the literature [1–3,5,8–11,15,16,18,19,21,22]. Shibata and Saito [17] examined a two-species delay Lotka–Volterra competition system. It has been shown that the time delays in a
two-species Lotka–Volterra competition system can lead to chaotic behaviour. To our knowledge, few papers have been published on the existence and global attractivity of periodic solutions of Eq. (1.1).

The main purpose of this paper is to derive a set of ‘easily verifiable’ sufficient conditions for the existence and global attractivity of positive periodic solution of Eq. (1.1). The present paper is organized as follows. In Section 2, we study the existence of periodic solutions with strictly positive components of Eq. (1.1) by using the continuation theorem of coincidence degree theory proposed by Gaines and Mawhin [7]. In Section 3, we investigate the global attractivity of strictly positive periodic solutions of n-species Lotka–Volterra competition system with several delays, which is a special case of Eq. (1.1), by using the method of Lyapunov functional. We define a suitable Lyapunov functional and obtain a set of easily verifiable sufficient conditions. In Section 3, we also examine the effect of the time delays on the uniform persistence. In Section 4, as an application, we study some special cases of Eq. (1.1), which have been studied extensively in the literature. The examples show that our general easily verifiable sufficient conditions differ from the known results and improve and generalize some known results.

2. Existence of positive periodic solutions

In this section, we study the existence of strictly positive (componentwise) periodic solutions of Eq. (1.1). For the reader’s convenience, we shall first summarize below a few concepts and results from Ref. [7] that will be basic for this section.

Let $X, Z$ be normed vector spaces, $L : \text{Dom } L \subset X \to Z$ be a linear mapping, and $N : X \to Z$ be a continuous mapping. The mapping $L$ will be called a Fredholm mapping of index zero if $\dim \text{Ker } L = \text{codim } \text{Im } L < +\infty$ and $\text{Im } L$ is closed in $Z$. If $L$ is a Fredholm mapping of index zero there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\text{Im } P = \text{Ker } L, \text{Ker } Q = \text{Im } L = \text{Im}(I - Q)$. It follows that $L|\text{Dom } L \cap \text{Ker } P : (I - P)X \to \text{Im } L$ is invertible. We denote the inverse of that map by $K_P$. If $\Omega$ is an open bounded subset of $X$, the mapping $N$ will be called $L$-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \to X$ is compact. Since $\text{Im } Q$ is isomorphic to $\text{Ker } L$, there exist isomorphisms $J : \text{Im } Q \to \text{Ker } L$.

In the proof of our existence theorem below, we will use the continuation theorem of Gaines and Mawhin ([7], p. 40).

**Lemma 2.1** (Continuation theorem). Let $L$ be a Fredholm mapping of index zero and let $N$ be $L$-compact on $\bar{\Omega}$. Suppose

(a) For each $\lambda \in (0, 1)$, every solution $x$ of $Lx = \lambda Nx$ is such that $x \not\in \partial \Omega$;
(b) \( QN x \neq 0 \) for each \( x \in \partial \Omega \cap \text{Ker} \, L \) and
\[ \text{deg}\{QN, \Omega \cap \text{Ker} \, L, 0\} \neq 0. \]
Then the equation \( Lx = Nx \) has at least one solution lying in \( \text{Dom} \, L \cap \bar{\Omega} \).

Lemma 2.2. The domain \( \mathbb{R}^n_+ = \{(y_1, \ldots, y_n)^T \mid y_i > 0, i = 1, 2, \ldots, n\} \) is positive invariant with respect to Eq. (1.1).

Proof. Since
\[ y_i(t) = y_i(0) \exp \left\{ \int_0^t \left( r_i(s) - \sum_{j=1}^n a_{ij}(s) y_j(s - \tau_{ij}(s)) \right) \, ds \right\}, \]
i = 1, 2, \ldots, n, \quad (2.1)
the assertion of the lemma follows immediately for all \( t \in [0, +\infty) \) since considering the biological significance of Eq. (1.1), we specify \( y_i(0) > 0 \), that is \( (y_1(0), \ldots, y_n(0))^T \in \mathbb{R}^n_+ \). The proof is complete.

Theorem 2.1. Suppose that \( \tilde{a}_{ii} > 0 \) and
\[ r_i > \sum_{j=1}^n \frac{\tilde{a}_{ij} \tilde{r}_i}{a_{ij}} \exp \{(\tilde{r}_j + \tilde{R}_j)\omega\}, \quad i = 1, 2, \ldots, n \]
hold. Then Eq. (1.1) has at least one positive periodic solution of period \( \omega \) say \( y^*(t) = (y^*_1(t), \ldots, y^*_n(t))^T \) and there exists a positive constant \( B \) such that
\[ \| y^* \| < B, \text{ where } \| y^* \| = \left( \sum_{i=1}^n \left( \max_{t \in [0, \omega]} |y_i(t)| \right)^2 \right)^{1/2}. \]

Proof. By Lemma 4.1.1 in Ref. [10] and the assumptions in Theorem 2.1, it is not difficult to show that the system of algebraic equations
\[ \sum_{j=1}^n \tilde{a}_{ij} \exp \{u_j\} = \tilde{r}_i, \quad i = 1, 2, \ldots, n \]
has a unique solution \( (u_1^*, u_2^*, \ldots, u_n^*)^T \in \mathbb{R}^n \).

Making the change of variable
\[ y_i(t) = \exp \{x_i(t)\}, \quad i = 1, 2, \ldots, n \]
then Eq. (1.1) is reformulated as
\[ \dot{x}_i(t) = r_i(t) - \sum_{j=1}^n a_{ij}(t) \exp \{x_j(t - \tau_{ij}(t))\}, \quad i = 1, 2, \ldots, n. \]

Let
\[ X = Z = \{x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in C(R, \mathbb{R}^n), \quad x(t + \omega) = x(t)\}, \]
(2.5)
\[ \|x\| = \left( \sum_{i=1}^{n} \left( \max_{t \in [0, \omega]} |x_i(t)| \right)^2 \right)^{1/2} \text{ for any } x \in X \text{ (or } Z). \] (2.6)

Then \( X \) and \( Z \) are both Banach spaces when they are endowed with the norm \( \| \cdot \| \). Let

\[
N_x = N \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} r_1(t) - \sum_{j=1}^{n} a_{1j}(t) \exp \{ x_j(t - \tau_{1j}(t)) \} \\ \vdots \\ r_n(t) - \sum_{j=1}^{n} a_{nj}(t) \exp \{ x_j(t - \tau_{nj}(t)) \} \end{pmatrix} \quad \text{for any } x \in X; \]

\[
Lx = \dot{x}, \quad Px = \frac{1}{\omega} \int_0^\omega x(t) \, dt, \quad x \in X; \quad Qz = \frac{1}{\omega} \int_0^\omega z(t) \, dt, \quad z \in Z. \] (2.7)

Obviously,

\[
\ker L = \{ x \mid x \in X, x = h, h \in \mathbb{R}^n \},
\]

\[
\text{im } L = \left\{ z \mid z \in Z, \int_0^\omega z(t) \, dt = 0 \right\} \] (2.8)

and

\[
\dim \ker L = n = \text{codim im } L. \] (2.9)

Since \( \text{im } L \) is closed in \( Z \), \( L \) is a Fredholm mapping of index zero. It is easy to show that \( P \) and \( Q \) are continuous projectors such that

\[
\text{im } P = \ker L, \quad \ker Q = \text{im } L = \text{im } (I - Q). \] (2.10)

Furthermore, the generalized inverse (to \( L \)) \( K_\rho : \text{im } L \to \ker P \cap \text{dom } L \) is given by

\[
K_\rho(z) = \int_0^t z(s) \, ds - \frac{1}{\omega} \int_0^\omega \int_0^t z(s) \, ds \, dt. \] (2.11)

Thus

\[
QN : X \to Z
\]

\[
\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \to \begin{pmatrix} \frac{1}{\omega} \int_0^\omega \left[ r_1(t) - \sum_{j=1}^{n} a_{1j}(t) \exp \{ x_j(t - \tau_{1j}(t)) \} \right] \, dt \\ \vdots \\ \frac{1}{\omega} \int_0^\omega \left[ r_n(t) - \sum_{j=1}^{n} a_{nj}(t) \exp \{ x_j(t - \tau_{nj}(t)) \} \right] \, dt \end{pmatrix}
\]
Clearly, $QN$ and $K_p(I - Q)N : X \rightarrow X$ and $Q^2$ are continuous. Using the Arzela–Ascoli theorem, it is not difficult to show that $K_p(I - Q)N(\bar{\Omega})$ is compact for any open bounded set $\Omega \subset X$. Moreover, $QN(\bar{\Omega})$ is bounded. Thus, $N$ is $L$-compact on $\bar{\Omega}$ with any open bounded set $\Omega \subset X$. The isomorphism $J$ of $\text{Im} \ Q$ onto $\text{Ker} \ L$ can be the identity mapping, since $\text{Im} \ Q = \text{Ker} \ L$.

Now we reach the position to search for an appropriate open, bounded subset $\Omega$ for the application of the continuation theorem. Corresponding to the operator equation $Lx = \lambda Nx$, $\lambda \in (0, 1)$, we have

$$
\dot{x}_i(t) = \lambda \left[ r_i(t) - \sum_{j=1}^{n} a_{ij}(t) \exp \left\{ x_j(t - \tau_{ij}(t)) \right\} \right], \quad \lambda \in (0, 1),
$$

$$
i = 1, 2, \ldots, n.
$$

(2.13)

Assume that $x = x(t) \in X$ is a solution of Eq. (2.13) for a certain $\lambda \in (0, 1)$. Integrating Eq. (2.13) over the interval $[0, \omega]$, we obtain
\[
\int _{0}^{\omega } \left[ r_i(t) - \sum _{j=1}^{n} a_{ij}(t) \exp \{ x_j(t - \tau _{ij}(t)) \} \right] \, dt = 0, \quad i = 1, 2, \ldots , n,
\]
that is,
\[
\sum _{j=1}^{n} \int _{0}^{\omega } a_{ij}(t) \exp \{ x_j(t - \tau _{ij}(t)) \} \, dt = \bar{r}_i \omega , \quad i = 1, 2, \ldots , n.
\] (2.14)

It follows from Eqs. (2.13) and (2.14) that
\[
\int _{0}^{\omega } |\dot{x}_i(t)| \, dt = \lambda \int _{0}^{\omega } \left| r_i(t) - \sum _{j=1}^{n} a_{ij}(t) \exp \{ x_j(t - \tau _{ij}(t)) \} \right| \, dt \\
< \int _{0}^{\omega } |r_i(t)| \, dt + \sum _{j=1}^{n} \int _{0}^{\omega } \left| a_{ij}(t) \exp \{ x_j(t - \tau _{ij}(t)) \} \right| \, dt \\
= (\bar{r}_i + \bar{R}_i) \omega , \quad i = 1, 2, \ldots , n,
\]
that is,
\[
\int _{0}^{\omega } |\dot{x}_i(t)| \, dt < (\bar{r}_i + \bar{R}_i) \omega , \quad i = 1, 2, \ldots , n.
\] (2.15)

Since \( x(t) \in X \), there exist \( \xi _i \in [0, \omega] \) such that
\[
x_i(\xi _i) = \min _{t \in [0, \omega]} x_i(t), \quad i = 1, 2, \ldots , n.
\] (2.16)

From Eqs. (2.14) and (2.16), we see that
\[
\bar{a}_{ii} \omega \exp \{ x_i(\xi _i) \} \leq \int _{0}^{\omega } a_{ii}(t) \exp \{ x_i(t - \tau _{ii}(t)) \} \, dt < \bar{r}_i \omega ,
\]
that is,
\[
x_i(\xi _i) < \ln \left( \frac{\bar{r}_i}{\bar{a}_{ii}} \right), \quad i = 1, 2, \ldots , n.
\] (2.17)

From Eqs. (2.15) and (2.18), one obtains
\[
x_i(t) \leq x_i(\xi _i) + \int _{0}^{\omega } |\dot{x}_i| \, dt < \ln \left( \frac{\bar{r}_i}{\bar{a}_{ii}} \right) + (\bar{r}_i + \bar{R}_i) \omega ,
\]
\[
i = 1, 2, \ldots , n.
\] (2.19)

On the other hand, there also exist \( \eta _i \in [0, \omega] \) such that
From Eqs. (2.14) and (2.20) it follows that
\[ \tilde{r}_i \omega = \sum_{j=1}^{n} \int_{0}^{\omega} a_{ij}(t) \exp \{ x_j(t - \tau_{ij}(t)) \} \, dt \]
\[ \leq \sum_{j=1}^{n} \int_{0}^{\omega} a_{ij}(t) \exp \{ x_j(\eta_j) \} \, dt \]
\[ = \sum_{j=1}^{n} \tilde{a}_{ij} \omega \exp \{ x_j(\eta_j) \}, \quad i = 1, 2, \ldots, n, \]  
then from Eq. (2.19), one obtains
\[ \tilde{a}_{ii} \exp \{ x_i(\eta_i) \} \geq \tilde{r}_i - \sum_{j=1, j \neq i}^{n} \tilde{a}_{ij} \exp \{ x_j(\eta_j) \} \]
\[ \geq \tilde{r}_i - \sum_{j=1, j \neq i}^{n} \tilde{a}_{ij} \frac{\tilde{r}_j}{\tilde{a}_{jj}} \exp \{ (\tilde{r}_j + \tilde{R}_j) \omega \}, \]  
which imply
\[ x_i(\eta_i) \geq \ln \left\{ \frac{\tilde{r}_i - \sum_{j=1, j \neq i}^{n} \tilde{a}_{ij}(\tilde{r}_j/\tilde{a}_{jj}) \exp \{ (\tilde{r}_j + \tilde{R}_j) \omega \}}{\tilde{a}_{ii}} \right\} := B_i, \quad i = 1, 2, \ldots, n. \]  
From Eqs. (2.15) and (2.23), we have
\[ x_i(t) \geq x_i(\eta_i) - \int_{0}^{\omega} |\dot{x}_i| \, dt > B_i - (\tilde{r}_i + \tilde{R}_i) \omega, \quad i = 1, 2, \ldots, n. \]  
Eqs. (2.19) and (2.24) imply
\[ \max_{t \in [0, \omega]} |x_i(t)| < \max \left\{ \ln \left\{ \frac{\tilde{r}_i}{\tilde{a}_{ii}} \right\} + (\tilde{r}_i + \tilde{R}_i) \omega \right\} := H_i. \]  
Clearly, \( H_i \) are independent of \( \lambda \). Set \( H = (\sum_{i=1}^{n} H_i^2)^{1/2} + C \), where \( C \) is taken sufficiently large such that the unique solution of Eq. (2.2) satisfies \( \|(u_1^*, u_2^*, \ldots, u_n^*)^T\| < C \), then \( \|x\| < H \).

Let \( \Omega := \{ x = (x_1, \ldots, x_n)^T \in X | \|x\| < H \} \). It is clear that \( \Omega \) verifies the requirement (a) in Lemma 2.1. When \( x \in \partial \Omega \cap \text{Ker} \, L = \partial \Omega \cap \mathbb{R}^n \), \( x \) is a constant vector in \( \mathbb{R}^n \) with \( \|x\| = H \). Then
Furthermore, in view of the assumption in Theorem 2.1, it is easy to prove that
\[ \deg\{JQN_x, \Omega \cap \text{Ker } L, 0\} \]
\[ = \text{sign}\left\{ (-1)^n (\det(\bar{a}_{ij})_{n \times n}) \exp \left( \sum_{j=1}^{n} u_j \right) \right\} \neq 0. \]  

By now we have proved that \( \Omega \) verifies all the requirements in Lemma 2.1. Hence, Eq. (2.4) has at least one \( \omega \)-periodic solution \( x^*(t) \) in \( \hat{\Omega} \). Set \( y^*_i(t) = \exp\{x^*_i(t)\} \), then by Eq. (2.3) we know that \( y^*(t) = (y^*_1(t), \ldots, y^*_n(t))^T \) is a positive \( \omega \)-periodic solution of Eq. (1.1). The boundedness of \( x^*(t) \) implies the existence of positive constant \( B \) in Theorem 2.1. The proof of Theorem 2.1 is complete.

**Remark 2.1.** Theorem 2.1 is also valid for both the advanced type and the mixed type Lotka–Volterra competition system. From Theorem 2.1, we can see that the deviating arguments \( \tau_{ij}, i, j = 1, 2, \ldots, n \) have no effect on the existence of positive periodic solution of Eq. (1.1).

**Remark 2.2.** Li [15] discussed the existence of positive periodic solutions of periodic \( N \)-species competition system with time delays, which is a special case of Eq. (1.1). Theorem 2.1 improves and extends the results in Ref. [15].

3. **Global asymptotic stability of positive periodic solutions**

We shall assume, henceforth, that \( \tau_{ij}, i, j = 1, \ldots, n \) are non-negative constants. That is, in this section we will confine ourselves to periodic \( n \)-species Lotka–Volterra competition system with several delays, which is a special case of Eq. (1.1).
Definition 3.1. Let \( y^*(t) = (y^*_1(t), \ldots, y^*_n(t))^T \) be a strictly positive (component-wise) periodic solution of Eq. (1.1) and we say \( y^*(t) \) is globally asymptotically stable (or attractive) if any other solution \( y(t) = (y_1(t), \ldots, y_n(t))^T \) of Eq. (1.1) together with the initial condition

\[
y_i(s) = \phi_i(s) \geq 0, \quad s \in [-\tau_i, 0]; \quad \phi_i(0) > 0; \quad i = 1, \ldots, n
\]

where \( \tau_i = \max_{1 \leq j \leq n} \tau_{ji} \) and \( \phi_i(s) \) is continuous on \([-\tau_i, 0]\), has the property

\[
\lim_{t \to +\infty} \sum_{i=1}^n |y_i(t) - y_i^*(t)| = 0.
\]

It is immediate that if \( y^*(t) \) is globally asymptotically stable then \( y^*(t) \) is in fact unique.

Lemma 3.1 ([4]). Let \( f \) be a nonnegative function defined on \([0, +\infty)\) such that \( f \) is integrable on \([0, +\infty)\) and is uniformly continuous on \([0, +\infty)\). Then \( \lim_{t \to +\infty} f(t) = 0 \).

Theorem 3.1. Assume that the conditions in Theorem 2.1 hold. Furthermore, suppose that \( \tau_{ii} \equiv 0, i = 1, 2, \ldots, n \) and

\[
\min_{t \in [0, a]} a_{jj}(t) > \sum_{i=1}^n \left( \max_{t \in [0, a]} a_{ij}(t) \right), \quad j = 1, 2, \ldots, n.
\]

Then system (1.1) has a unique periodic solution with strictly positive components, say \( y^*(t) = (y^*_1(t), \ldots, y^*_n(t))^T \), which is globally asymptotically stable and there exists a positive constant \( B > 0 \) such that \( \|y^*\| \leq B \).

Proof. By Theorem 2.1, there exists a strictly positive (componentwise) periodic solution \( y^*(t) = (y^*_1(t), \ldots, y^*_n(t))^T \) of Eq. (1.1) and there is a positive constant \( B > 0 \) such that \( \|y^*\| \leq B \). To complete the proof of Theorem 3.1, we only need to show that \( y^*(t) \) is globally asymptotically stable. Consider a Lyapunov functional \( v(t) \) defined by

\[
v(t) = \sum_{i=1}^n \left[ |\ln(y_i(t)) - \ln(y^*_i(t))| + \sum_{j=1}^n \int_{t-\tau_{ij}}^t a_{ij}(u+\tau_{ij}) |y_j(u) - y^*_j(u)| \, du \right],
\]

\( t \geq 0 \).

(3.4)

Obviously
\[
v(0) \leq \sum_{i=1}^{n} (|\ln(y_i(0)) - \ln(y_i^*(0))|) + \sum_{j=1}^{n} \left( \max_{i \neq j} \sup_{\tau \in [0, \infty]} |\phi_j(t) - y_j^*(t)| \right) < \infty, \\
v(t) \geq \sum_{i=1}^{n} (|\ln(y_i(t)) - \ln(y_i^*(t))|), \quad t \geq 0.
\]

A direct calculation of the right derivative \(D^+v\) of \(v(t)\) leads to

\[
D^+ v(t) \leq \sum_{i=1}^{n} \left( -a_{ii}(t)|y_i(t) - y_i^*(t)| + \sum_{j=1}^{n} a_{ij}(t + \tau_{ij})|y_j(t) - y_j^*(t)| \right) \\
= \sum_{j=1}^{n} \left( -a_{jj}(t)|y_j(t) - y_j^*(t)| + \sum_{i=1}^{n} a_{ij}(t + \tau_{ij})|y_j(t) - y_j^*(t)| \right) \\
\leq \sum_{j=1}^{n} \left( \max_{i \neq j} \left( \sup_{\tau \in [0, \infty]} |\phi_j(t) - y_j^*(t)| \right) \right) |y_j(t) - y_j^*(t)| \\
\leq -\mu \sum_{j=1}^{n} |y_j(t) - y_j^*(t)|, \quad t \geq 0,
\]

where

\[
\mu = \min_{1 \leq j \leq n} \left( \min_{\tau \in [0, \infty]} a_{jj}(t) - \sup_{\tau \in [0, \infty]} \left( \max_{i \neq j} a_{ij}(t) \right) \right) > 0.
\]

Integrating from 0 to \(t\) on both sides of Eq. (3.7) produces

\[
v(t) + \mu \sum_{j=1}^{n} \int_0^t |y_j(s) - y_j^*(s)| \, ds \leq v(0) < +\infty, \quad t \geq 0,
\]

then

\[
\sum_{i=1}^{n} \int_0^t |y_i(s) - y_i^*(s)| \, ds \leq v(0) / \mu < +\infty, \quad t \geq 0,
\]

and hence \(\sum_{i=1}^{n} |y_i(t) - y_i^*(t)| \in L^1[0, +\infty)\). By Eqs. (3.5)–(3.8), one obtains
\[ \sum_{i=1}^{n} |\ln(y_i(t)) - \ln(y^*_i(t))| \leq v(t) \leq v(0) < +\infty, \quad t \geq 0. \] 

(3.10)

Therefore

\[ |\ln(y_i(t)) - \ln(y^*_i(t))| \leq v(0), \quad i = 1, 2, \ldots, n, \quad t \geq 0. \] 

(3.11)

Furthermore

\[ \left( \min_{t \in [0, \omega]} y^*_i(t) \right) \exp\{-v(0)\} \leq y_i(t) \leq \left( \max_{t \in [0, \omega]} y^*_i(t) \right) \exp\{v(0)\} < +\infty, \]

\[ i = 1, 2, \ldots, n; \quad t \geq 0. \] 

(3.12)

The boundedness of \( y^*(t) \) and Eq. (3.12) imply that \( y_i(t) \) are bounded above and below by positive constants for \( t \geq 0 \). Since \( y_i(t) \) and \( y^*_i(t) \) are bounded for \( t \geq 0 \) with bounded derivatives (from the equations satisfied by them), it will follow that \( \sum_{i=1}^{n} |y_i(t) - y^*_i(t)| \) are uniformly continuous on \( [0, +\infty) \). By lemma 3.1 obtains

\[ \lim_{t \to +\infty} \sum_{i=1}^{n} |y_i(t) - y^*_i(t)| = 0. \] 

(3.13)

Now the proof is complete.

**Corollary 3.1.** Under the conditions in Theorem 3.1, system (1.1) is permanent.

Essentially, condition (3.3) in Theorem 3.1 indicates that the periodic solution is a global attractor if the undelayed intra-species competition dominates over the delayed inter-species competition. This amounts to saying that individuals compete more with those of the same species rather than with those of the other species; in other words, intra-species competition is more significant than the inter-species competition. Such assumptions and results are consistent with the results derived from the autonomous version of competition models. Under the condition in Theorem 3.1, the system is permanent. The delays in interspecific competition are harmless for the permanence of system (1.1). On the other hand, as long as we restrict our consideration to the autonomous case, namely \( r_i(t), a_{ij}(t), i, j = 1, \ldots, n \) are positive constants, then the unique globally asymptotically stable periodic solution will be given by positive equilibrium.

**Remarks 3.1.** We should point out that the results of Theorem 2.1 and Theorem 3.1 remain valid if some or all of the terms with discrete delays in Eq. (1.1) are replaced by continuously distributed finite (or infinite) delays, which have been proved in our other papers.
4. Applications

In order to illustrate some feature of our main results, in the following we will apply Theorems 2.1 and 3.1 to some one- or two-dimensional systems, which have been studied extensively in the literature.

**Application 1.** Consider the delay logistic equation

\[
\frac{dy(t)}{dt} = r(t)y(t)\left(1 - \frac{y(t - \tau(t))}{K(t)}\right), \tag{4.1}
\]

where \(r, K, \tau\) are continuous \(\omega\)-periodic functions with \(\int_0^\omega r(t)\, dt > 0, K(t) > 0\). Noting Eq. (2.22) and applying Theorem 2.1 to Eq. (4.1), we obtain the following theorem, which is an immediate corollary of Theorem 2.1.

**Theorem 4.1.** System (4.1) has at least one strictly positive \(\omega\)-periodic solution.

Zhang and Gopalsamy [23] study a special case of Eq. (4.1) in which the period of the environmental parameters is related to the time delay in the self-regulating negative feedback mechanism. It has been proved that if \(\tau(t) = m\omega, m\) is a positive integer, then there exists a positive \(\omega\)-periodic solution of Eq. (4.1). Obviously, Theorem 4.1 improves the results in Ref. [23]. It is interesting that for periodic logistic equation, the time delay has no effect on the existence of positive periodic solutions.

**Application 2.** Consider a two-species competition system in a periodic environment of the form

\[
\begin{align*}
\frac{dy_1(t)}{dt} &= y_1(t)[r_1(t) - a_{11}(t)y_1(t - \tau_{11}(t)) - a_{12}(t)y_2(t - \tau_{12}(t))],
\frac{dy_2(t)}{dt} &= y_2(t)[r_2(t) - a_{21}(t)y_1(t - \tau_{21}(t)) - a_{22}(t)y_2(t - \tau_{22}(t))],
\end{align*}
\tag{4.2}
\]

where \(r_i, a_{ij}, \tau_{ij}, i, j = 1, 2\) are continuous \(\omega\)-periodic functions with \(\int_0^\omega r_i(t)\, dt > 0, a_{ij}(t) > 0\). Eq. (4.2) is different from the periodic two-species competition system considered in the literature, since \(r_i(t)\) are not always positive. Applying Theorems 2.1 and 3.1 to Eq. (4.2), we have the following results.

**Theorem 4.2.** Suppose that

\[
\begin{align*}
\bar{r}_1\bar{a}_{22} &> \bar{r}_2\bar{a}_{12}\exp\{(\bar{r}_2 + \bar{R}_2)\omega\}, \\
\bar{r}_2\bar{a}_{11} &> \bar{r}_1\bar{a}_{21}\exp\{(\bar{r}_1 + \bar{R}_1)\omega\}
\end{align*}
\tag{4.3}
\]

hold. Then Eq. (4.2) has at least one strictly positive (componentwise) \(\omega\)-periodic solution.

**Theorem 4.3.** Suppose that

\[
\tau_{11}(t) = \tau_{22}(t) = 0; \quad \tau_{12}(t), \tau_{21}(t) \text{ are non-negative constants}
\tag{4.5}
\]
and
\[
\begin{align*}
\bar{r}_1 \alpha_{22} &> \bar{r}_2 \alpha_{12} \exp \{(\bar{r}_2 + \bar{R}_2) \omega\}, \\
\min_{t \in [0,\omega]} a_{11}(t) &> \max_{t \in [0,\omega]} a_{21}(t), \\
\min_{t \in [0,\omega]} a_{22}(t) &> \max_{t \in [0,\omega]} a_{12}(t)
\end{align*}
\] (4.6)
hold. Then Eq. (4.2) has a unique \(\omega\)-periodic solution with strictly positive components, which is globally asymptotically stable.

Gopalsamy [8] also examined a two-species periodic Lotka–Volterra system
\[
\begin{align*}
\frac{dy_1(t)}{dt} &= y_1(t)\{r_1(t) - a_{11}(t)y_1(t) - a_{12}(t)y_2(t - \omega)\}, \\
\frac{dy_2(t)}{dt} &= y_2(t)\{r_2(t) - a_{21}(t)y_1(t - \omega) - a_{22}(t)y_2(t)\},
\end{align*}
\] (4.7)
where \(r_i, a_{ij} \in C(R, (0, +\infty)), i, j = 1, 2\) are \(\omega\)-periodic functions. It has been proved that if
\[
\begin{align*}
\bar{r}_1^l a_{22}^l &> \bar{r}_2^l a_{12}^l, & \bar{r}_2^l a_{11}^l &> \bar{r}_1^l a_{21}^l, \\
\alpha_{11}^l &> \alpha_{21}^l, & \alpha_{22}^l &> \alpha_{12}^l.
\end{align*}
\] (4.8)
Then Eq. (4.7) has a unique positive periodic solution which is globally asymptotically stable, where
\[
\begin{align*}
\bar{r}_i^l &= \min_{t \in [0,\omega]} r_i(t), & \bar{r}_i^l &= \max_{t \in [0,\omega]} r_i(t); \\
\alpha_{ij}^l &= \min_{t \in [0,\omega]} a_{ij}(t), & \alpha_{ij}^l &= \max_{t \in [0,\omega]} a_{ij}(t), & i, j = 1, 2.
\end{align*}
\] (4.9)
Obviously, our results for two-species competition system are different from the results obtained in Ref. [8]. For \(n\)-species competition system, our results are fresh and general.

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References

