Constructing resonance calabashes of Hill’s equations using step potentials

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Abstract

Based on the characterization of periodic eigenvalues using rotation numbers, we analyse the second and the third periodic eigenvalues of one-dimensional Schrödinger operators with certain step potentials. This gives counter-examples to the Alikakos–Fusco conjecture on the second periodic eigenvalues. Using this simple model, we can also construct infinitely many resonance pockets, which are much like calabashes emanating from a cane, of one-parameter Hill’s equations.

1. Introduction

Consider the eigenvalue problems for one-dimensional Schrödinger operators

\[ Lx := -\frac{d^2 x}{dt^2} - q(t)x = \lambda x, \]

i.e.

\[ \ddot{x} + (\lambda + q(t))x = 0, \quad (1.1) \]

where the potential \( q(t) \) is \( 2\pi \)-periodic and \( q \in L^1(0, 2\pi) \). It is known from [9, 14] that (1.1) has a sequence of periodic eigenvalues

\[ \lambda_0(q) < \lambda_1(q) \leq \lambda_2(q) < \cdots < \lambda_{2k-1}(q) \leq \lambda_{2k}(q) < \cdots \]

with respect to the periodic boundary conditions (P): \( x(0) - x(2\pi) = \dot{x}(0) - \dot{x}(2\pi) = 0 \). These eigenvalues, together with the anti-periodic eigenvalues (see Section 2), play the fundamental role in the stability and instability problems of Hill’s equations.

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Meirong Zhang and Shaobo Gan

Let \( q(t) = p^2(t) \) in (1·1), where \( p(t) \) is \( 2\pi \)-periodic function and has mean value one:

\[
\frac{1}{2\pi} \int_0^{2\pi} p(t) \, dt = 1.
\]

In this paper, we will use rotation numbers to study the second and the third periodic eigenvalues of the following eigenvalue problem

\[
\ddot{x} + (\lambda + p^2(t))x = 0.
\]

Such a choice for potentials is motivated by the following conjecture of Alikakos–Fusco, which is the one-dimensional version of a conjecture concerning the second eigenvalues related with the interface of the Allen–Cahn system and certain geometric problems (see [1–4, 10, 11, 15]).

**Conjecture 1** (Alikakos–Fusco [4, p. 114]). Let \( p(t) \) be a position \( 2\pi \)-periodic potential with mean value one. Then (a) the second periodic eigenvalue \( \lambda_1(p^2) \) of (1·3) is not greater than 0 and (b) \( \lambda_1(p^2) = 0 \) if and only if \( p(t) \equiv 1 \).

A stronger version of Conjecture 1 is

**Conjecture 2.** If \( q(t) \) has mean value one, then (a) \( \lambda_1(q) \leq 0 \) and (b) \( \lambda_1(q) = 0 \) if and only if \( q(t) \equiv 1 \).

Both conjectures are affirmative for certain potentials. However, numerical computation shows that, for the potential \( q(t) = 1 + 2\cos t - \cos 2t \), Conjecture 2 fails, while Conjecture 1 is unclear (see [15]).

In order to study Conjecture 1, one may approximate general potentials by step one. We will find in Section 3 that Conjecture 1, and consequently Conjecture 2, fails for certain step potentials.

In fact, when we analyse the periodic eigenvalues of (1·3) for 2-step functions \( p(t) \) (these step functions can be parameterized using 2 parameters such as the heights, when the constraint (1·2) is accounted), we find that the second and the third eigenvalues coincide \( \lambda_1(p^2) = \lambda_2(p^2) \) when the parameters are on a certain critical curve. This result enables us to construct resonance tongues, resonance pockets of one-parameter Hill’s equations (see [5–8]) at one’s option. For example, one can construct infinitely many resonance pockets inside the second resonance region of (1·3) by choosing a suitable parameter curve. The graphs of such a resonance region are like a series of calabashes emanating from a cane. For this reason, we will call them resonance calabashes. Such a phenomenon is quite different from the classical Mathieu equations and the square wave equations. For the latter equations, one can only expect finitely many resonance pockets inside each resonance region. This phenomenon and some further related problems will be considered in Section 4.

Our analysis for (1·3) is based on the characterization (see [12, 16, 17]) of periodic eigenvalues using rotation numbers.

2. Rotation numbers and periodic eigenvalues

We first introduce some notation. For \( q \in L^1(0, 2\pi) \), the mean value is \( \bar{q} = (2\pi)^{-1} \int_0^{2\pi} q(t) \, dt \). For \( q_1, q_2 \in L^1(0, 2\pi) \), we write \( q_1 \succ q_2 \) if \( q_1(t) \geq q_2(t) \) for a.e. \( t \in [0, 2\pi] \) and \( q_1 > q_2 \). Let \( \mathcal{P} \) be the space of \( 2\pi \)-periodic functions \( q(t) \) such that \( q \in L^1(0, 2\pi) \).
Let \( q \in \mathcal{P} \). We rewrite the periodic eigenvalues \( \lambda_k(q) \) of (1-1) as \( \lambda_k^P(q) \). From [9, 14], we know that eigenvalue problem (1-1) has also a sequence

\[
\lambda_1^P(q) \leq \lambda_2^P(q) < \cdots \leq \lambda_{2k-1}^P(q) \leq \lambda_{2k}^P(q) < \cdots
\]

of anti-periodic eigenvalues with respect to the anti-periodic boundary conditions (A): \( x(0) + x(2\pi) = x(0) + x(2\pi) = 0 \). These eigenvalues \( \lambda_k^P(q) \) and \( \lambda_k^A(q) \) are called the characteristic values of (1-1) and they have the following order (see [9, 14]):

\[
\lambda_1^P(q) < \lambda_2^A(q) < \lambda_2^P(q) < \cdots < \lambda_{2k-1}^A(q) \leq \lambda_{2k}^P(q) < \lambda_{2k-1}^A(q) \leq \lambda_{2k}^A(q) < \cdots .
\]

In the following we write characteristic values as

\[
\Lambda_k(q) = \lambda_k^A(q) \quad \text{and} \quad \overline{\lambda}_k(q) = \lambda_k^A(q) \quad \text{if } k \text{ is odd},
\]

\[
\Lambda_k(q) = \lambda_k^P(q) \quad \text{and} \quad \overline{\lambda}_k(q) = \lambda_k^P(q) \quad \text{if } k \text{ is even}.
\]

Now we use rotation numbers to characterize all characteristic values. Set \( y = -\dot{x} \) in (1-1). Then (1-1) is equivalent to the following linear system:

\[
\dot{x} = -y, \quad \dot{y} = (\lambda + q(t))x. \tag{2-1}
\]

Write (2-1) in the polar coordinates: \( x = r \cos \theta, \ y = r \sin \theta \). Then \( \theta \) satisfies

\[
\dot{\theta} = (\lambda + q(t)) \cos^2 \theta + \sin^2 \theta =: \Xi(t, \theta, \lambda). \tag{2-2}
\]

For any \( \theta_0 \in \mathbb{R} \), let \( \Theta(t; \theta_0, \lambda) \) be the unique solution of (2-2) satisfying the initial condition: \( \Theta(t; \theta_0, \lambda) = \theta_0 \). As the vector field \( \Xi(t, \theta, \lambda) \) is 2\( \pi \)-periodic in \( t \) and is \( \pi \)-periodic in \( \theta \),

\[
\Theta(t; \theta_0 + n\pi, \lambda) = \Theta(t; \theta_0, \lambda) + n\pi \tag{2-3}
\]

for all \( t, \theta_0, \lambda \in \mathbb{R} \) and \( n \in \mathbb{Z} \). Thus the rotation number of (2-2)

\[
\rho(\lambda) = \rho(\lambda, q) = \lim_{t \to \infty} \frac{\Theta(t; \theta_0, \lambda)}{t}
\]

exists and is independent of \( \theta_0 \) (see [9, 13]).

For any given \( q \in \mathcal{P} \), the rotation number function \( \rho(\lambda) \) is a continuous and nondecreasing function such that \( \rho(\lambda) = 0 \) for \( \lambda \leq -1 \), and \( \rho(\lambda) \to +\infty \) when \( \lambda \to +\infty \). Using \( \rho(\lambda) \), the following characterization of characteristic values is proved in [16].

**Theorem 2.1.** Let \( q \in \mathcal{P} \). Then

(i) \( \lambda_k(q) = \min \{ \lambda \in \mathbb{R}: \rho(\lambda) = k/2 \} \) for all \( k \in \mathbb{N} \), and \( \lambda_k(q) = \max \{ \lambda \in \mathbb{R}: \rho(\lambda) = k/2 \} \) for all \( k \in \mathbb{Z}^+ \).

(ii) \( \lambda_k(q) = \overline{\lambda}_k(q) = \lambda \) if and only if all solutions of (1-1) are \( 2\pi \)-periodic (respectively, \( 2\pi \)-antiperiodic) when \( k \) is even (respectively, when \( k \) is odd).

(iii) Let \( q_1, q_2 \in \mathcal{P} \). If \( q_1 > q_2 \), then \( \lambda_k(q_1) < \lambda_k(q_2) \) and \( \overline{\lambda}_k(q_1) < \overline{\lambda}_k(q_2) \) for all \( k \).

(iv) \( \lambda_k(q) \) and \( \overline{\lambda}_k(q) \) are continuously dependent on \( q \in \mathcal{P} \), where \( \mathcal{P} \) is endowed with the \( L^1 \)-metric: \( d(q_1, q_2) = \int_0^{2\pi} |q_1(t) - q_2(t)|dt \).

Note that if \( h_\lambda: \mathbb{R} \to \mathbb{R} \) is the Poincaré map of (2-2), i.e. \( h_\lambda(\theta_0) = \Theta(2\pi; \theta_0, \lambda) \) for \( \theta_0 \in \mathbb{R} \), then \( h_\lambda \) satisfies

\[
h_\lambda(\theta + k\pi) \equiv h_\lambda(\theta) + k\pi \tag{2-4}
\]
for all \( \vartheta \in \mathbb{R} \) and all \( k \in \mathbb{Z} \) (see (2.3)). Now the rotation number \( \rho(\lambda) \) is

\[
\rho(\lambda) = \lim_{n \to \infty} \frac{h^n(\vartheta) - \vartheta}{2n\pi}.
\]

In general, if \( h: \mathbb{R} \to \mathbb{R} \) is a homeomorphism satisfying (2.4), the rotation number of \( h(\cdot) \) is defined as

\[
\rho(h) = \lim_{n \to \infty} \frac{h^n(\vartheta) - \vartheta}{2n\pi}
\]

(independent of \( \vartheta \)).

**Proposition 2.2.** Let \( h(\cdot) \) be a homeomorphism of \( \mathbb{R} \) satisfying (2.4) and \( k \) be an integer. Then

(i) \( \rho(h) \geq k/2 \) if and only if \( \max_{\vartheta \in \mathbb{R}} (h(\vartheta) - (\vartheta + k\pi)) \geq 0 \).

(ii) \( \rho(h) \leq k/2 \) if and only if \( \min_{\vartheta \in \mathbb{R}} (h(\vartheta) - (\vartheta + k\pi)) \leq 0 \).

**Proof.** Let us prove (i). Assume that \( h(\vartheta_0) \geq \vartheta_0 + k\pi \) for some \( \vartheta_0 \in \mathbb{R} \). Using (2.4), it can be checked that \( h^n(\vartheta_0) \geq \vartheta_0 + nk\pi \) for all \( n \in \mathbb{N} \). This shows that

\[
\rho(h) = \lim_{n \to \infty} \frac{h^n(\vartheta_0) - \vartheta_0}{2n\pi} \geq \frac{k}{2}
\]

Conversely, let \( M_0 = \max_{\vartheta \in \mathbb{R}} (h(\vartheta) - (\vartheta + k\pi)) \). If \( M_0 < 0 \), we need to prove that \( \rho(h) < k/2 \). Notice that

\[
h(\vartheta) \leq \vartheta + (k\pi + M_0), \quad \forall \vartheta \in \mathbb{R}
\]

implies that

\[
h^n(\vartheta) \leq \vartheta + n(k\pi + M_0), \quad \forall \vartheta \in \mathbb{R}
\]

for all \( n \in \mathbb{N} \). Thus

\[
\rho(h) = \lim_{n \to \infty} \frac{h^n(\vartheta) - \vartheta}{2n\pi} \leq \frac{k}{2} + \frac{M_0}{2\pi} < \frac{k}{2}
\]

Conclusion (ii) can be proved similarly.

Now let \( p(t) \) be a positive \( 2\pi \)-periodic 2-step function with heights \( a_1, a_2 \) and widths \( t_1, t_2 \) respectively, i.e.

\[
p(t) = \begin{cases} a_1, & t \in [0, t_1), \\ a_2, & t \in [t_1, t_1 + t_2) = [t_1, 2\pi). \end{cases}
\]

Assume further that \( p(t) \) satisfies (1.2). Then the positive parameters \( a_1, t_1 \) satisfy \( t_1 + t_2 = 2\pi \) and \( a_1 t_1 + a_2 t_2 = 2\pi \). We need not consider the trivial case \( a_1 = a_2 = 1 \).

Taking \( a_1, a_2 \) as independent parameters, \( (a_1, a_2) \) are in the domain

\[
\mathcal{D} = \{(a_1, a_2) \in \mathbb{R}^2 : \text{either, } 0 < a_1 < 1 \text{ and } a_2 > 1; \text{ or, } a_1 > 1 \text{ and } 0 < a_2 < 1\}
\]

and \( t_1, t_2 \) are given by

\[
t_1 = \frac{2\pi(a_2 - 1)}{a_2 - a_1}, \quad t_2 = \frac{2\pi(1 - a_1)}{a_2 - a_1}.
\]

For convenience, write also \( \Delta_i = a_i t_i, \ i = 1, 2 \). Then \( \Delta_1 + \Delta_2 = 2\pi \).
Constructing resonance calabashes of Hill’s equations

Theorem 2.3. Let \( p(t) = p_{a_0, a_2}(t) \) be given by (2.5). Then the following hold.

(i) If \( \Delta_1 \neq \pi, \) then \( \lambda_f^f(p^2) < 0 < \lambda_f^r(p^2). \)

(ii) If \( \Delta_1 (=- \Delta_2) = \pi, \) then \( \lambda_f^f(p^2) = \lambda_f^r(p^2) = 0. \)

Proof. As \( \Delta_1 + \Delta_2 = 2\pi, \) one of \( \Delta_i \) is not greater than \( \pi. \) Let us first consider the case that \( \Delta_1 < \pi. \)

Let \( \Theta(t; \theta_0, \lambda) \) and \( \rho(\lambda) \) be the solutions and the rotation number function of (2.2), where \( q(t) = p_{a_1, a_2}(t). \)

Our first aim is to prove that \( \rho(0) = 1. \) To this end, let \( \lambda = 0. \) Define \( \Theta_1 = \Theta(t_1; \theta_0, 0) \) and \( \Theta_2 = \Theta(2\pi; \theta_0, 0). \) Then \( \Theta_1 \) and \( \Theta_2 \) are determined by the following two equations:

\[
\int_{\Theta_1}^{\Theta_1 + 2\pi} \frac{d\theta}{\Phi(a_1, \theta)} = t_1, \quad \int_{\Theta_1}^{\Theta_2} \frac{d\theta}{\Phi(a_2, \theta)} = t_2, \tag{2.6}
\]

where \( \Phi(a, \theta) = a^2 \cos^2 \theta + \sin^2 \theta \) for \( a > 0 \) and \( \theta \in \mathbb{R}. \) The functions \( \Theta_i(\theta_0) \) are well defined and satisfy \( \Theta_i(\theta_0 + k\pi) \equiv \Theta_i(\theta_0) + k\pi. \) By Proposition 2.2, the claim that \( \rho(0) = 1 \) is equivalent to

\[
\min_{\theta_0} (\Theta_2 - (\theta_0 + 2\pi)) \leq 0 \leq \max_{\theta_0} (\Theta_2 - (\theta_0 + 2\pi)). \tag{2.7}
\]

Let us introduce the following function

\[
F(\theta_0) = t_2 - \int_{\Theta_1(\theta_0)}^{\Theta_1(\theta_0) + 2\pi} \frac{d\theta}{\Phi(a_2, \theta)}, \quad \theta_0 \in \mathbb{R}.
\]

Then \( F(\theta_0) \) is \( \pi \)-periodic. Note that, if one introduces the function

\[
G(\varphi) = \int_0^{\varphi} (d\varphi / \Phi(a_2, \varphi)),
\]

then

\[
\Theta_2 - (\theta_0 + 2\pi) \equiv G^{-1}(t_2 + G(\Theta_1)) - G^{-1}(t_2 + G(\Theta_1) - F(\theta_0)). \tag{2.8}
\]

Thus (2.7) is equivalent to

\[
\min_{\theta_0} F(\theta_0) \leq 0 \leq \max_{\theta_0} F(\theta_0), \tag{2.9}
\]

because \( G(\varphi) \) is strictly increasing in \( \varphi. \)

Now we are going to find the critical points and critical values of \( F. \) By the first equation in (2.6) we have \( \Theta_1' = d\Theta_1/d\theta_0 = \Phi(a_1, \Theta_1)/\Phi(a_1, \theta_0). \) Now the condition

\[
0 = F'(\theta_0) = \frac{dF(\theta_0)}{d\theta_0} = \frac{\Theta_1'}{\Phi(a_2, \Theta_1)} - \frac{1}{\Phi(a_2, \theta_0 + 2\pi)} = \frac{\Theta_1'}{\Phi(a_1, \theta_0) \Phi(a_2, \Theta_1)} - \frac{1}{\Phi(a_2, \theta_0)}
\]

yields that \( \tan \Theta_1 = \pm \tan \theta_0. \) Thus \( \Theta_1 = \theta_0 + k\pi \) or \( \Theta_1 = k\pi - \theta_0 \) for some \( k \in \mathbb{Z}. \)

By the \( \pi \)-periodicity of \( F(\theta_0), \) we may assume that \( \theta_0 \in [0, \pi). \) Under the condition \( \Delta_1 < \pi, \) only the following two cases can happen: (1) \( \Theta_1 = \pi - \theta_0 \) with \( \theta_0 \in [0, \pi/2); \) or, (2) \( \Theta_1 = 2\pi - \theta_0 \) with \( \theta_0 \in [\pi/2, \pi). \)
In Case (1), by the first equation in (2.6), \( \theta_0 \) satisfies
\[
\frac{\pi}{a_1} - 2 \int_{0}^{\theta_0} \frac{d\theta}{\Phi(a_1, \theta)} = \frac{\pi}{a_1} - \frac{2}{a_4} \arctan \frac{\tan \theta_0}{a_1}.
\]
This implies that \( \theta_0 = \theta_{01} := \arccot (a_1^{-1} \tan (\Delta_1/2)) \)
and
\[
M_1 := F(\theta_{01}) = \frac{2}{a_2} \left[ \frac{\Delta_1}{2} - \arctan \left( \frac{a_1}{a_2} \tan \frac{\Delta_1}{2} \right) \right].
\tag{2.10}
\]
Similarly, Case (2) implies that \( \theta_0 = \theta_{02} := \arccot (-a_1^{-1} \cot (\Delta_1/2)) \)
and
\[
M_2 := F(\theta_{02}) = \frac{2}{a_2} \left[ \frac{\Delta_1}{2} - \arctan \left( \frac{a_2}{a_1} \tan \frac{\Delta_1}{2} \right) \right].
\tag{2.11}
\]
From (2.10) and (2.11), one sees that \( M_1 M_2 < 0 \) because \( a_1 \neq a_2 \). This proves (2.9) and (2.7). Consequently, \( \rho(0) = 1 \).

In fact, the analysis above can give further results. For any \( \lambda \in \mathbb{R} \), let
\[
m(\lambda) = \min_{a_0} (\Theta(2\pi; \theta_0, \lambda) - (\theta_0 + 2\pi)),
M(\lambda) = \max_{a_0} (\Theta(2\pi; \theta_0, \lambda) - (\theta_0 + 2\pi)).
\]
Using the relation (2.8), the conclusion \( M_1 M_2 < 0 \) means that \( m(0) < 0 < M(0) \).
As \( \Theta(2\pi; \theta_0, \lambda) \) is continuous in \( (\theta_0, \lambda) \), we know that \( m(\lambda) < 0 < M(\lambda) \) whenever \( |\lambda| \ll 1 \).
Applying Proposition 2.2 to \( h(\theta_0) = \Theta(2\pi; \theta_0, \lambda) \), we know that \( \rho(\lambda) = 1 \) whenever \( |\lambda| \ll 1 \). Thus it follows from Theorem 2.4(i) that \( \lambda_1^P(p^2) < 0 < \lambda_2^P(p^2) \) in this case.

Now we consider the case that \( \Delta_1 (= \Delta_2) = \pi \). This condition is equivalent to the parameters \( a_1, a_2 \) being on the following hyperbola:
\[
C_0: \frac{1}{a_1} + \frac{1}{a_2} = 2, \quad a_1, a_2 > 0.
\]
In this case, we get from (2.6) that \( \Theta_1 \equiv \theta_0 + \pi \) and \( \Theta_2 \equiv \Theta_1 + \pi \). Thus \( \Theta(2\pi; \theta_0, 0) \equiv \theta_0 + 2\pi \) for all \( \theta_0 \).
Now one sees that all solutions of (1.3) with \( \lambda = 0 \) are \( 2\pi \)-periodic.
Thus \( \lambda_1^P(p^2) = \lambda_2^P(p^2) = 0 \) by Theorem 2.4(ii).

Finally, assume that \( \Delta_1 > \pi \). Then \( \Delta_2 < \pi \). Since the periodic eigenvalues are invariant under translations of the potential, i.e.
\[
\lambda_k^P(q_s) = \lambda_k^P(q) \quad \text{for all } s \in \mathbb{R} \quad \text{and} \quad k \in \mathbb{Z}^+,
\]
where \( q_s(t) \equiv q(t + s) \), one can reduce this case to the case that \( \Delta_1 < \pi \) by a suitable translation of \( p(t) \). Thus the theorem is proved.

3. Counter-examples to the Alikakos-Fusco conjecture

Example 3.1. There are counter-examples to part (b) of Conjecture 1.
Let \( (a_1, a_2) \) be any point on \( C_0 \) such that \( (a_1, a_2) \neq (1, 1) \). Then \( p(t) = p_{a_1, a_2}(t) \) given
by (2-5) is a counter-example to part (b) of Conjecture 1, because \( \lambda_1^P(p^2) = \lambda_2^P(p^2) = 0 \) and \( p(t) \neq 1 \).
Example 3.2. There are counter-examples to Conjecture 2.
Let $p(t) = p_{a_1, a_2}(t)$ be as in Example 3.1. Define a potential
$$q(t) = q_{a_1, a_2}(t) = 1 + 2(p_{a_1, a_2}(t) - 1) = 2p_{a_1, a_2}(t) - 1.$$  
Then $q(t)$ is a 2-step periodic potential with mean value 1. Since $q = 2p - 1 \prec p^2$, it follows from Theorem 2.1(iii) that $\lambda_1^p(q) > \lambda_1^p(p^2) = 0$. Consequently, $q(t) = q_{a_1, a_2}(t)$ gives a counter-example to Conjecture 2.

As the eigenvalues continuously depend upon potentials, one can use these $q_{a_1, a_2}(t)$ to construct smooth potentials $q(t)$ which give negative answer to Conjecture 2.

We see from Theorem 2.3 that there are no counter-examples to part (a) of Conjecture 1 when 2-step functions $p(t)$ are considered. However, when 3-step potentials are considered, one can even construct counter-examples to part (a) of Conjecture 1. The following is one of these.

Example 3.3. Let $p(t)$ be the 3-step function with heights $9, 2, \frac{1}{2}$ and widths $\pi/10, \pi/10, 9\pi/5$ respectively, i.e.
$$p(t) = \begin{cases} 
9, & t \in [0, \pi/10), \\
2, & t \in [\pi/10, \pi/5), \\
\frac{1}{2}, & t \in [\pi/5, 2\pi]. 
\end{cases} \quad (3.1)$$
Then $p(t)$ satisfies condition (1.2). As in the proof of Theorem 2.3, it can be proved that the corresponding solutions $\Theta(t; \theta_0, \lambda)$ satisfy
$$\max_{\theta_0}(\Theta(2\pi; \theta_0, 0) - (\theta_0 + 2\pi)) < 0$$
(see Fig. 3.1). Thus, we know from Proposition 2.2(i) that $\lambda_1^p(p^2) > 0$. Similarly, one
can use this example to construct smooth potentials so that part (a) of Conjecture 1 fails.

4. Constructing resonance calabashes

Let $q \in \mathcal{P}$. Consider the following Hill’s equation

$$\ddot{x} + q(t)x = 0. \quad (4\cdot1)$$

Equation (4·1) is stable in the sense of Lyapunov if any solution $x(\cdot)$ of (4·1) satisfies

$$\sup_{t \in \mathbb{R}}(|x(t)| + |\dot{x}(t)|) < \infty.$$ We say that (4·1) is unstable if (4·1) is not stable.

In this section we given an application of Theorem 2·3 to the instability problem of Hill’s equation. To describe this, we introduce a parameter $\lambda \in \mathbb{R}$ in (4·1) and consider the following equation:

$$\ddot{x} + (\lambda + q(t))x = 0. \quad (4\cdot2)$$

From [9, 14], it is known that the parameter $\lambda$ such that (4·2) is unstable consists of the following intervals of $\lambda$:

$$(-\infty, \bar{\lambda}_0(q)), \quad (\underline{\lambda}_0(q), \bar{\lambda}_1(q)), \quad \ldots, \quad (\underline{\lambda}_k(q), \bar{\lambda}_k(q)), \quad \ldots.$$ 

Now we consider the one-parameter Hill’s equation

$$\ddot{x} + (\lambda + q_\varepsilon(t))x = 0. \quad (4\cdot3)$$

where $q_\varepsilon \in \mathcal{P}$ for each $\varepsilon$ and $q_0(t)$ is constant, e.g. $q_0(t) \equiv 0$. Some typical examples are the Mathieu case

$$q_\varepsilon(t) = \varepsilon \cos t$$

and the square wave case

$$q_\varepsilon(t) = \varepsilon \text{sign cos } t.$$ Note that when $\varepsilon = 0$, (4·3) is stable for all $\lambda > 0$. However, when $\varepsilon$ evolves, generally speaking, some resonance regions (or, instability regions) in the $(\lambda, \varepsilon)$-plane occur. This is the so-called parametric resonance (see [5–8]).

Analytically, for any $k \in \mathbb{N}$, The $k$th resonance region of (4·3) in the $(\lambda, \varepsilon)$-plane is given by

$$R_k = \{ (\lambda, \varepsilon) : \underline{\lambda}_k(q_\varepsilon) < \lambda < \bar{\lambda}_k(q_\varepsilon) \}.$$ For the Mathieu case, it is known that $\underline{\lambda}_k(q_\varepsilon) < \bar{\lambda}_k(q_\varepsilon)$ holds for all $\varepsilon \neq 0$. In this case, $R_k$ is called a resonance tongue (instability tongue, Arnold tongue). For the square-wave case, besides the resonance tongues $R_1$ and $R_2$, in the third resonance region $R_3$ there happens that $\underline{\lambda}_3(q_{\varepsilon_0}) = \bar{\lambda}_3(q_{\varepsilon_0})$ for certain $\varepsilon_0 \neq 0$. This yields a sub-region of $R_3$:

$$\{ (\lambda, \varepsilon) : \underline{\lambda}_3(q_\varepsilon) < \lambda < \bar{\lambda}_3(q_\varepsilon), \quad 0 < \varepsilon < \varepsilon_0 \},$$ which is called a resonance pocket (see [6, 8]). Such pockets also occur in other resonance regions $R_k(k \geq 4)$ for the square-wave case. This phenomenon also happens in the near Mathieu case: $q_{\varepsilon_1, \varepsilon_2}(t) = \varepsilon_1 \cos t + \varepsilon_2 \cos 2t$ (see [7, 8]). A geometrical explanation to resonance pockets using singularity theory is developed in [6, 7].
Let us introduce the Hill’s map 

(These are similar to the argument for equalities (2)·
where \(q\), \(\lambda\) respectively; and (ii) if either,
least when (4·
Then \(\Lambda_1(\lambda, a, b)\) to the Poincaré matrix of (4·
Now the condition \(\Lambda_k(q, a, b) = \lambda_k(q, a, b)\) is equivalent to
\(\mathcal{H}(\lambda, a, b) = \pm I\).
(4·5)

Considering that \(\text{Sp}(1)\) is 3-dimensional, one can only expect that (4·5) has discrete solutions for general two-parameter Hill’s equations.

However, we can choose one-parameter Hill’s equations from the step potentials \(q(t) = p_{a_1, a_2}(t)\) to obtain arbitrary resonance pockets, including finitely many or infinitely many, transversal or non-transversal pockets. Such resonance pockets look like a series of calabashes emanating from a cane. For this reason, we will call these pockets resonance calabashes.

In order to describe resonance calabashes, we give a characterization of characteristic values using solutions of (2·2).

**Theorem 4·1.** Let \(\Theta(t; \theta_0, \lambda)\) be the solutions of (2·2) and \(k\) an integer. Then the following hold.

(i) \(\lambda = \lambda_k(q)\) if and only if \(\max_{\theta_0}(\Theta(2\pi; \theta_0, \lambda) - (\theta_0 + k\pi)) = 0\).
(ii) \(\lambda = \lambda_k(q)\) if and only if \(\min_{\theta_0}(\Theta(2\pi; \theta_0, \lambda) - (\theta_0 + k\pi)) = 0\).

**Proof.** These follow from Theorem 2·1 and Proposition 2·2.

When \(q(t) = p_{a_1, a_2}(t)\), where \(p_{a_1, a_2}(t)\) is given by (2·5), two conditions to determine \(\lambda = \lambda^P_1(p_{a_1, a_2})\) and \(\lambda = \lambda^P_2(p_{a_1, a_2})\) in Theorem 4·1 are as follows:

\[
\sqrt{a_1^2 + \lambda} \cot \frac{t_1 \sqrt{a_1^2 + \lambda}}{2} + \sqrt{a_2^2 + \lambda} \cot \frac{t_2 \sqrt{a_2^2 + \lambda}}{2} = 0
\]  
(4·6)

and

\[
\sqrt{a_1^2 + \lambda} \cot \frac{t_2 \sqrt{a_2^2 + \lambda}}{2} + \sqrt{a_2^2 + \lambda} \cot \frac{t_1 \sqrt{a_2^2 + \lambda}}{2} = 0.
\]  
(4·7)

(These are similar to the argument for equalities (2·10) and (2·11).)

Now \(\lambda^P_1 := \lambda^P_1(p_{a_1, a_2})\) and \(\lambda^P_2 := \lambda^P_2(p_{a_1, a_2})\) are the solutions of (4·6) and (4·7), at least when \((a_1, a_2)\) is near the critical curve \(C_0\). More exactly, (i) if either, \(a_1 < a_2\) and \(\Delta_1 < \pi\), or, \(a_2 < a_1\) and \(\Delta_2 < \pi\), then \(\lambda^P_1\) and \(\lambda^P_2\) are the solutions of (4·6) and (4·7), respectively; and (ii) if either, \(a_1 > a_2\) and \(\Delta_1 < \pi\), or \(a_2 > a_1\) and \(\Delta_2 < \pi\), then \(\lambda^P_1\) and \(\lambda^P_2\) are the solutions of (4·7) and (4·6), respectively.

Let \(\lambda = \Lambda_1(a_1, a_2)\) and \(\lambda = \Lambda_2(a_1, a_2)\) be the solutions of (4·6) and (4·7), respectively. Then \(\Lambda_1(a_1, a_2) = \Lambda_2(a_1, a_2) = 0\) for all \((a_1, a_2)\) on \(C_0\).
Proposition 4.2. When \((a_1, a_2) \in C_0\), both \(\nabla \Lambda_1(a_1, a_2)\) and \(\nabla \Lambda_2(a_1, a_2)\) are not parallel to the tangent vector of \(C_0\) at \((a_1, a_2)\).

Proof. Let \((a_1, a_2) \in C_0\). Then the normal vector of \(C_0\) at \((a_1, a_2)\) is \(v = (a_2^{-2}, a_2^{-2})\). From (4·6) and (4·7),

\[
\begin{align*}
\frac{\partial \Lambda_1}{\partial a_1} &= 2a_2 - 1, \\
\frac{\partial \Lambda_1}{\partial a_2} &= 2a_1 - 1, \\
\frac{\partial \Lambda_2}{\partial a_1} &= -\frac{a_2^2}{a_1^2 - a_1a_2 + a_2^2}, \\
\frac{\partial \Lambda_2}{\partial a_2} &= -\frac{a_1^2}{a_1^2 - a_1a_2 + a_2^2},
\end{align*}
\]

when \((a_1, a_2) \in C_0\). Thus

\[
\nabla \Lambda_1(a_1, a_2) \cdot v = \frac{2(a_1^2 + a_2^2) - (a_1^2 + a_2^2)}{a_1^2a_2^2} > 0
\]

whenever \((a_1, a_2)\) is on \(C_0\). Similarly,

\[
\nabla \Lambda_2(a_1, a_2) \cdot v = -\frac{a_1^2 + a_2^2}{a_1^2a_2^2(a_1^2 - a_1a_2 + a_2^2)} < 0
\]

for all \((a_1, a_2) \in C_0\).

Now we can construct resonance calabashes as follows. Consider any curve

\[
C: a_1 = a_1(\varepsilon), \quad a_2 = a_2(\varepsilon)
\]

in the parameter domain \(\mathcal{D}\) starting at \((1, 1)\), i.e. \(a_1(0) = a_2(0) = 1\). Consider the second resonance region \(R_2(C)\) of

\[
\dot{x} + (\lambda + q_C(t))x = 0,
\]

(4·8)

where \(q_C(t) = q_{a_1(\varepsilon), a_2(\varepsilon)}(t)\). When \(\varepsilon\) increases, if \((a_1(\varepsilon), a_2(\varepsilon))\) meets \(C_0\) then one gets a pocket. One can continue this procedure to find as many pockets in \(R_2(C)\) as the intersecting points of \(C\) with \(C_0\). Moreover, by Proposition 4·2, the boundary curves of \(R_2(C)\) intersect transversally if and only if the parameter curve \(C\) transversally intersects with the critical curve \(C_0\). Thus, by choosing suitable parameter curves \(C\), one can obtain various resonance calabashes in the second resonance region of one-parameter Hill’s equations with two-step potentials.

For example, let the curves \(C_1\) and \(C_2\) be

\[
C_1: a_1 = \frac{a_2}{2a_2 - 1} - \frac{1}{20} \sin \pi \varepsilon = \frac{1 + \varepsilon}{1 + 2 \varepsilon} - \frac{1}{20} \sin \pi \varepsilon, \quad a_2 = 1 + \varepsilon,
\]

and

\[
C_2: a_1 = \frac{a_2}{2a_2 - 1} - \frac{1}{20} \sin^2 \pi \varepsilon = \frac{1 + \varepsilon}{1 + 2 \varepsilon} - \frac{1}{20} \sin^2 \pi \varepsilon, \quad a_2 = 1 + \varepsilon,
\]

where the parameter \(\varepsilon \in [0, \infty)\). Note that both \(C_1\) and \(C_2\) intersect \(C_0\) whenever \(\varepsilon \in \mathbb{N}\). Thus, in the second resonance regions \(R_2(C_1)\) and \(R_2(C_2)\) of the corresponding
Constructing resonance calabashes of Hill’s equations

Fig. 4-1. Transversal calabashes corresponding to $q_{C_1}(t)$.

Fig. 4-2. Non-transversal calabashes corresponding to $q_{C_2}(t)$.
potentials \( q_C(t) \), there are infinitely many of resonance pockets. The shapes of calabashes \( R_2(C_1) \) and \( R_2(C_2) \) are plotted in Figures 4·1 and 4·2, respectively, where only ten of these calabashes are plotted in both cases. From Proposition 4·2, one knows that all pockets in \( R_2(C_1) \) are transversal, while those in \( R_2(C_2) \) are non-transversal. Thus the resonance calabashes are persistent when the equation

\[
\ddot{x} + (\lambda + q_C(t))x = 0
\]

has some small nonlinear perturbations.

We have only considered the second resonance region of (4·8). It is an interesting problem to analyse the other resonance regions.

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