Conditions for convergence and subsequential convergence

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Abstract

Let \((u_n)\) be a sequence, regularly generated by another sequence \((\alpha_n)\) where either \((\alpha_n)\) or \((\Delta \alpha_n) = (\alpha_n - \alpha_{n-1})\) is slowly oscillating. We investigate conditions under which the sequence \((u_n)\) converges or converges subsequentially.

Keywords: Slowly oscillating sequences; Moderately oscillatory sequences; Regularly generated sequences; Moderately divergent sequences; Subsequentially convergent sequences

1. Introduction

Let \(L\) be any linear space of sequences and \((u_n)\) be a sequence in \(L\). If there exists a subclass \(A\) of sequences from \(L\) such that for some \(\alpha = (\alpha_n) \in A\) and for all nonnegative integers \(n\)

\[ u_n = \alpha_n + \sum_{k=1}^{n} \frac{\alpha_k}{k}, \tag{1.1} \]

we say that the sequence \((u_n)\) is regularly generated by \(\alpha\) and \(\alpha\) is called a generator of \((u_n)\). The class of all sequences regularly generated by sequences in \(A\) is denoted by \(U(A)\).

The identity

\[ u_n - \sigma_n^{(1)}(u) = V_n(\Delta u) \tag{1.2} \]

where \(\sigma_n^{(1)}(u) = \frac{1}{n+1} \sum_{k=0}^{n} u_k\), \(V_n(\Delta u) = \frac{1}{n+1} \sum_{k=0}^{n} k \Delta u_k\), \(\Delta u_n = u_n - u_{n-1}\) and \(u_{-1} = 0\), is well known as the Kronecker identity and used extensively in the proofs.

Definition 1.1 (\(II\)). A sequence \((u_n)\) is slowly oscillating if

\[ \lim_{\lambda \to 1^+} \lim_{n \to \infty} \max_{n+1 \leq k \leq [\lambda n]} \left| \sum_{j=n+1}^{k} \Delta u_j \right| = 0. \tag{1.3} \]

The class of all slowly oscillating sequences is denoted by \(SO\).

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If a sequence \((u_n)\) converges, then
\[ \Delta u_n = o(1) \quad \text{as} \quad n \to \infty. \] (1.4)

That the converse is not necessarily true is clear from the example of the sequence \((\log n)\), which satisfies (1.4).

Since \(\Delta u_n = o(1)\) as \(n \to \infty\) is a necessary condition for convergence of a sequence \((u_n)\), we pose the following question: Under which condition(s) may we obtain some information on the behavior of \((u_n)\) satisfying (1.4) or its generalizations? In the case where \((u_n)\) is bounded together with (1.4), we only have convergence of some subsequences of \((u_n)\). This suggests a new kind of convergence.

**Definition 1.2 ([2]).** A real sequence \(u = (u_n)\) converges subsequentially if there exists a finite interval \(I(u)\) such that all accumulation points of \(u = (u_n)\) are in \(I(u)\) and every point of \(I(u)\) is an accumulation point of \(I(u)\).

Stanojević [3] proved that if \((S_n(\alpha)) = (\sum_{k=1}^{n} \alpha_k)\) is slowly oscillating, then \((\sum_{k=1}^{n} \frac{\alpha_k}{k})\) converges. The purpose of this work is to weaken the necessary condition for convergence of \((\sum_{k=1}^{n} \frac{\alpha_k}{k})\) and then recover convergence or subsequential convergence of a sequence \((u_n)\) in \(U(SO)\) and \(U(SO_\Delta)\), respectively. The class \(U(SO_\Delta)\) is the class of all sequences regularly generated by the sequences \(\alpha = (\alpha_n)\) where \((\Delta \alpha_n)\) is slowly oscillating.

## 2. The main result

**Definition 2.1 ([1]).** A sequence \((u_n)\) is moderately oscillatory if for \(\lambda > 1\),
\[ \lim_{n \to \infty} \max_{n+1 \leq k \leq \lfloor \lambda n \rfloor} \left| \sum_{j=n+1}^{k} \Delta u_j \right| < \infty. \] (2.1)

The class of all moderately oscillatory sequences is denoted by MO.

**Definition 2.2.** A positive sequence \((u_n)\) is slowly varying if \(\lim_{n} \frac{u(n \lambda^n)}{\lambda^n} = 1\).

**Theorem 2.1.** Let \((u_n) \in U(SO_\Delta)\) be regularly generated by \((\alpha_n)\). If \((S_n(\alpha))\) is moderately oscillatory, then \((u_n)\) converges subsequentially.

We need following lemmas for the proof of this theorem.

**Lemma 2.1.** If \((S_n(\alpha))\) is moderately oscillatory, then \(\sum_{n=1}^{\infty} \frac{\alpha_n}{n}\) converges.

**Proof.** For the sequence \((R(n)) = \left( \exp \left| \sum_{j=1}^{n} \alpha_j \right| \right)\), we have
\[ \frac{R(\lfloor \lambda n \rfloor)}{R(n)} \leq \exp \left| \sum_{j=n+1}^{\lfloor \lambda n \rfloor} \alpha_j \right|. \] (2.2)

Taking the \(\lim\) of both sides of (2.2) gives \(\lim_{n} \frac{R(\lfloor \lambda n \rfloor)}{R(n)} \leq \exp \lim_{n} \left| \sum_{j=n+1}^{\lfloor \lambda n \rfloor} \alpha_j \right|. \) By hypothesis, we necessarily have \(\lim_{n} \left| \sum_{j=n+1}^{\lfloor \lambda n \rfloor} \alpha_j \right| < \infty \) for \(\lambda > 1\). Hence, \((\log R(n))\) is slowly varying [4]. For \(p \in (1, 2]\), \(\left| \sum_{j=1}^{n} \alpha_j \right|^p = \log^p R(n)\), \(n \geq 1\) and \(\sum_{n=1}^{\infty} \left| \sum_{j=1}^{n} \alpha_j \right| n^{-p} < \infty \) for every \(p > 1\). From F. Riesz’s theorem [5], it follows that \(\sum_{n=1}^{\infty} \frac{\alpha_n}{n} < \infty\). \(\square\)

**Lemma 2.2.** Let \((\alpha_n)\) be a sequence of real numbers. If \((\sum_{k=1}^{n} \frac{\alpha_k}{k})\) converges, then \(\sigma_n^{(1)}(\alpha) = o(1)\) as \(n \to \infty\).

**Proof.** Set \(\gamma_n = \sum_{k=1}^{n} \frac{\alpha_k}{k}\). Hence, \(\alpha_n = n \Delta \gamma_n\) and then \(\sigma_n^{(1)}(\alpha) = V_n(\Delta \gamma)\). Since \((\gamma_n)\) converges, it follows from the Kronecker identity that \(V_n(\Delta \gamma) = o(1)\) as \(n \to \infty\). This completes the proof. \(\square\)

**Lemma 2.3 ([2]).** Let \((u_n)\) be a bounded sequence of real numbers. If \(\Delta u_n = o(1)\) as \(n \to \infty\), then \((u_n)\) converges subsequentially.

**Lemma 2.4 ([6]).** Let \((\sigma_n^{(1)}(u))\) converge to \(s\). If \((u_n) \in SO\), then \((u_n)\) converges to \(s\).
3. Proof of Theorem 2.1

Since \((u_n)\) is regularly generated by \((\alpha_n)\), the sequence \((u_n)\) has the representation \((1.1)\). Hence, by Lemma 2.1 the sequence \((\gamma_n) = \left(\sum_{k=1}^{n} \frac{\alpha_k}{k}\right)\) converges and by Lemma 2.2 \(\sigma_n^{(1)}(\alpha) = o(1)\) as \(n \to \infty\). Therefore \((\sigma_n^{(1)}(u))\) converges to the limit of the sequence \((\gamma_n)\). Since \((\Delta \alpha_n)\) is slowly oscillating and \(\frac{n \alpha_n}{n} = o(1)\) as \(n \to \infty\), we have \(\Delta \alpha_n = o(1)\) as \(n \to \infty\) by Lemma 2.4. Using \((1.1)\), we have \(\Delta u_n = \Delta \alpha_n + \frac{\alpha_n}{n}\). From the last identity it follows that \(\Delta u_n = o(1)\) as \(n \to \infty\). To complete the proof, it suffices to show that \(u_n = O(1)\) as \(n \to \infty\). Since \((S_n(\alpha)) \in \text{MO}\), we have \(V_n(\alpha) = O(1)\) as \(n \to \infty\) \([1]\). From \(\alpha_n = \frac{V_n(\alpha)}{n} + V_n(\alpha)\), it follows that \(\alpha_n = O(1)\) as \(n \to \infty\). Finally, we have \(u_n = O(1)\) as \(n \to \infty\). Thus \((u_n)\) converges subsequentially by Lemma 2.3.

Note that in Theorem 2.1 the condition \((\Delta \alpha_n) \in \text{SO}\) can be replaced with \((\Delta V_n(\Delta \alpha))\) in \(\text{SO}\).

Remark 3.1. If \((u_n)\) is regularly generated by \(\alpha = (\alpha_n)\), then \((\sigma_n^{(m)}(u))\) is regularly generated by \(\sigma^{(m)}(\alpha) = (\sigma_n^{(m)}(\alpha))\) for all integers \(m \geq 1\). If \(\left(\sum_{k=1}^{n} \sigma_k^{(m)}(\alpha)\right)\) \(\in \text{MO}\) and \((\sigma_k^{(j)}(\alpha))\) \(\in \text{SO}\) for some \(0 \leq j \leq m\), then \((\sigma_n^{(m)}(u))\) converges by Theorem 2.1.

Corollary 3.2. Let \((u_n) \in U(\text{SO})\) be regularly generated by \((\alpha_n)\). If \((S_n(\alpha))\) is moderately oscillatory, then \((u_n)\) converges to \(\lim_{n \to \infty} \sigma_n^{(1)}(u)\).

Proof. For a sequence \((u_n)\) regularly generated by \((\alpha_n)\), we have

\[
V_n(\Delta u) = V_n(\Delta \alpha) + \sigma_n^{(1)}(\alpha).
\]  

By Lemma 2.2, \(\sigma_n^{(1)}(\alpha) = o(1)\) as \(n \to \infty\). From the definition of slow oscillation, it follows that \((\alpha_n)\) is slowly oscillating if and only if \((V_n(\Delta \alpha))\) is slowly oscillating and bounded \([7]\). Hence, by the use of \((3.1)\), \((V_n(\Delta u))\) is slowly oscillating and bounded. That is, \((u_n)\) is slowly oscillating. Since \((\sigma_n^{(1)}(u))\) converges, \((u_n)\) converges to the same limit, by Lemma 2.4. \(\square\)

We end this section with the following generalization of slow oscillation, called moderate divergence, and then we give a necessary condition for \((C, 1)\) convergence of \((u_n)\) regularly generated by \((\alpha_n)\) in terms of moderately divergent sequences.

Definition 3.1 (\([8]\) ). A positive sequence \((u_n)\) is moderately divergent if for every \(\lambda > 1\), \(u_n = o(n^{\lambda-1})\) as \(n \to \infty\) and \(\sum_{n=1}^{\infty} \frac{u_n}{n^{\lambda}} < \infty\).

The class of all moderately divergent sequences is denoted by \(\text{MD}\). Notice that every positive slowly oscillating sequence is moderately divergent.

Theorem 3.3. Let \((u_n)\) be regularly generated by \((\alpha_n)\). If \((\frac{S_n(\alpha)}{n})\) \(\in \text{MD}\) for \(\gamma \in (0, 1)\), then

(i) \((\sigma_n^{(1)}(u))\) converges, and

(ii) \(u_n = O(\Delta (n^{\gamma} m_n))\) as \(n \to \infty\) for some \((m_n) \in \text{MD}\).

Proof. Applying summation by parts, we have

\[
\sum_{k=1}^{n} \frac{\alpha_k}{k} = \frac{1}{n} \sum_{k=1}^{n} \alpha_k + \sum_{k=1}^{n-1} \frac{S_k(\alpha)}{k(k+1)}.
\]  

Since \((\frac{S_n(\alpha)}{n})\) \(\in \text{MD}\) for \(\gamma \in (0, 1)\), we have \(\alpha_n^{(1)} = n^{\gamma} m_n\) for some \((m_n) \in \text{MD}\). It then follows that \(\sigma_n^{(1)}(\alpha) = o(1)\) as \(n \to \infty\). Since \(\sum_{k=1}^{\infty} \frac{m_k}{k^{\gamma}}\) converges for \(\gamma \in (0, 1)\), the second term on the right side of \((3.2)\) converges. Hence, \((\sigma_n^{(1)}(u))\) converges by Lemma 2.2.

Part (ii) is an easy consequence of part (i). \(\square\)
References