SOME RESULTS OF FIXED POINT THEOREMS IN
CONVEX METRIC SPACES

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Abstract. In this paper we study some fixed point theorems for self-mappings satisfying certain contraction principles on a convex complete metric space. In addition, we also improve and extend some very recently results in [9].

1. INTRODUCTION AND PRELIMINARY

In 1970, Takahashi [11] introduced the notion of convexity in metric spaces and studied some fixed point theorems for nonexpansive mappings in such spaces. A convex metric space is a generalized space. For example, every normed space and cone Banach space is a convex metric space and convex complete metric space, respectively. Subsequently, many mathematicians in [2]-[7], [10, 12] and recently, Moosaei [9] studied fixed point theorems in convex metric spaces.

Our results improve and extend some of Moosaei’s results in [9] and Karapinar’s results in [8] from a cone Banach space to a convex complete metric space. For instance, Karapinar proved that

Theorem 1.1. ([8, Theorem 2.4]) Let C be a closed and convex subset of a cone Banach space X with the norm \( \| x \|_p = d(x, 0) \), and \( T : C \to C \) be a mapping which satisfies the condition

\[
\exists q \in [2, 4), \quad \forall x, y \in C, \quad d(x, Tx) + d(y, Ty) \leq q d(x, y).
\]

Then \( T \) has at least one fixed point.
Letting \( x = y \) in the above inequality, it is easy to see that \( T \) is an identity mapping. In this paper, results in [8, 9] is improved and extended to a convex complete metric space.

**Theorem 1.2.** ([8, Theorem 2.6]) Let \( C \) be a closed and convex subset of a cone Banach space \( X \) with the norm \( \|x\|_p = d(x,0) \), and \( T : C \to C \) be a mapping which satisfies the condition

\[
\exists r \in [2, 5), \quad \forall x, y \in C, \quad d(Tx, Ty) + d(x, Tx) + d(y, Ty) \leq rd(x, y).
\]

Then \( T \) has at least one fixed point.

**Definition 1.3.** ([1]) Let \((X, d)\) be a metric space and \( I = [0, 1] \). A mapping \( W : X \times X \times I \to X \) is said to be a convex structure on \( X \) if for each \((x, y, \lambda) \in X \times X \times I \) and \( u \in X \),

\[
d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y).
\]

A metric space \((X, d)\) together with a convex structure \( W \) is called a convex metric space, which is denoted by \((X, d, W)\).

**Example 1.4.** Let \((X, d, \|\|)\) be a normed space. The mapping \( W : X \times X \times I \to X \) defined by \( W(x, y, \lambda) = \lambda x + (1 - \lambda)y \) for each \( x, y \in X \), \( \lambda \in I \) is a convex structure on \( X \).

**Definition 1.5.** ([1]) Let \((X, d, W)\) be a convex metric space. A nonempty subset \( C \) of \( X \) is said to be convex if \( W(x, y, \lambda) \in C \) whenever \((x, y, \lambda) \in C \times C \times I \).

**Lemma 1.6.** ([9]) Let \((X, d, W)\) be a convex metric space, then the following statements hold:

(i) \( d(x, y) = d(x, W(x, y, \lambda)) + d(y, W(x, y, \lambda)) \) for all \((x, y, \lambda) \in X \times X \times I \).

(ii) \( d(x, W(x, y, \lambda)) = (1 - \lambda)d(x, y) \) for all \( x, y \in X \).

(iii) \( d(y, W(x, y, \lambda)) = \lambda d(x, y) \) for all \( x, y \in X \).

**Proof.** To prove (i) see [9, Lemma 3.1].

By definition, we have

\[
d(x, W(x, y, \lambda)) \leq (1 - \lambda)d(x, y)
\]

and on the other hand

\[
(1 - \lambda)d(x, y) = d(x, y) - \lambda d(x, y)
\]

\[
= [d(x, W(x, y, \lambda)) + d(y, W(x, y, \lambda))] - \lambda d(x, y)
\]

but

\[
d(y, W(x, y, \lambda)) \leq \lambda d(x, y).
\]
Therefore
\[(1 - \lambda)d(x, y) \leq d(x, W(x, y, \lambda)).\]
Thus \(d(x, W(x, y, \lambda)) = (1 - \lambda)d(x, y)\) for all \(x, y \in X\). This completes proof of (ii).
For (iii), by (i) and (ii), we have
\[d(x, y) = d(x, W(x, y, \lambda)) + d(y, W(x, y, \lambda)) = (1 - \lambda)d(x, y) + d(y, W(x, y, \lambda)).\]
So \(d(y, W(x, y, \lambda)) = \lambda d(x, y)\) for all \(x, y \in X\). □

2. Main Results

**Theorem 2.1.** Let \(C\) be a nonempty closed convex subset of a convex complete metric space \((X, d, W)\) and \(T\) be a self-mapping of \(C\). If there exist \(a, b, c, e, f, k\) such that
\[
\frac{b + e - |f|(1 - \lambda) - |c|\lambda}{1 - \lambda} \leq k < \frac{a + b + c + e + f - |c|\lambda - |f|(1 - \lambda)}{1 - \lambda}
\]
\[(2.1)\]
then \(T\) has at least one fixed point.

**Proof.** Fix \(\lambda \in (0, 1)\). Suppose \(x_0 \in C\) is arbitrary. We define a sequence \(\{x_n\}_{n=1}^{\infty}\) in the following way:
\[x_n = W(x_{n-1}, T(x_{n-1}, \lambda)), \quad n = 1, 2, 3, \ldots.\]
As \(C\) is convex, \(x_n \in C\) for all \(n \in \mathbb{N}\). By Lemma 1.6 and above relation, we have
\[d(x_{n+1}, x_n) = (1 - \lambda)d(x_n, Tx_n), \quad n = 1, 2, 3, \ldots. \quad (2.3)\]
\[d(x_n, Tx_{n-1}) = \lambda d(x_{n-1}, Tx_{n-1}) = \frac{\lambda}{1 - \lambda}d(x_n, x_{n-1}). \quad (2.4)\]
By relation (2.3)
\[
\frac{1}{1 - \lambda}d(x_{n+1}, x_n) = d(x_n, Tx_n) \leq d(x_n, Tx_{n-1}) + d(Tx_{n-1}, Tx_n) \quad (2.5)
\]
and
\[
\frac{c}{1 - \lambda}d(x_{n+1}, x_n) - \frac{|c|\lambda}{1 - \lambda}d(x_n, x_{n-1}) \leq cd(Tx_{n-1}, Tx_n). \quad (2.6)
\]
And also by relation (2.3) and triangle inequality we have
\[
\frac{1}{1 - \lambda}d(x_{n+1}, x_n) = d(x_n, Tx_n) \leq d(x_n, x_{n-1}) + d(x_{n-1}, Tx_n) \quad (2.8)
\]
and
\[ \frac{f}{1-\lambda}d(x_{n+1}, x_n) - |f|d(x_n, x_{n-1}) \leq fd(Tx_{n-1}, Tx_n) \]  
for all \( n \in \mathbb{N} \). Now, by substituting \( x \) with \( x_n \) and \( y \) with \( x_{n-1} \) in \((2.2)\), we get
\[ ad(x_n, Tx_n) + bd(x_{n-1}, Tx_{n-1}) + cd(Tx_n, Tx_{n-1}) 
+ ed(x_n, Tx_{n-1}) + fd(x_{n-1}, Tx_{n-1}) \leq kd(x_n, x_{n-1}) \]
so by the relations \((2.3)\),\((2.4)\),\((2.7)\) and \((2.10)\), we obtain
\[ \left( \frac{a + c + f}{1-\lambda} \right) d(x_{n+1}, x_n) + \left( \frac{b - |c|\lambda + e\lambda}{1-\lambda} - |f| \right) d(x_{n-1}, x_n) \leq kd(x_n, x_{n-1}). \]
Thus
\[ d(x_n, x_{n+1}) \leq \left( \frac{k(1 - \lambda) + |f|(1 - \lambda) - b - e + |c|\lambda}{a + c + f} \right) d(x_n, x_{n-1}) \]
for all \( n \in \mathbb{N} \). By the relation \((2.1)\) \( \frac{k(1 - \lambda) + |f|(1 - \lambda) - b - e + |c|\lambda}{a + c + f} \in [0, 1) \) and hence, \( \{x_n\} \subseteq C \) is a contraction sequence. Therefore, it is a Cauchy sequence. Since \( C \) is a closed subset of a complete space, so \( \lim_{n \to \infty} x_n = x^* \) for some \( x^* \in C \). Now by relation \((2.3)\)
\[ \frac{1}{1-\lambda} d(x_{n+1}, x_n) = d(x_n, Tx_n) \leq d(x_n, x^*) + d(x^*, Tx_n) \]
we obtain \( \lim_{n \to \infty} d(x_n, Tx_n) = 0 \) and by
\[ d(x^*, Tx_n) \leq d(x^*, x_n) + d(x_n, Tx_n) \]
we get \( \lim_{n \to \infty} Tx_n = x^* \).
Now, by substituting \( x \) with \( x^* \) and \( y \) with \( x_n \) in relation \((2.2)\), we obtain
\[ ad(x^*, Tx^*) + bd(x_n, Tx_n) + cd(Tx^*, Tx_n) + ed(x^*, Tx_n) + fd(x_n, Tx^*) \leq kd(x^*, x_n). \]
So
\[ (a + c + f)d(x^*, Tx^*) \leq 0. \]
But by relation \((2.1)\) \( a + c + f \geq 0 \) thus \( Tx^* = x^* \). \( \square \)

The following corollary improves and extends \([9, \text{Theorem 3.2}]\).

**Corollary 2.2.** Let \( C \) be a nonempty closed convex subset of a convex complete metric space \((X, d, W)\) and \( T \) be a self-mapping of \( C \). If there exist \( a, b, c \) and \( k \) such that
\[ 2b - |c| \leq k < 2(a + b + c) - |c|, \]  
then
\[ d(x_n, Tx_n) \leq \frac{k}{a + c + f} d(x_n, x_{n-1}) \]
for all \( n \in \mathbb{N} \).
Some results of fixed point theorems in convex metric spaces

\[ ad(x, Tx) + bd(y, Ty) + cd(Tx, Ty) \leq kd(x, y) \]  
(2.12)

for all \( x, y \in C \), then \( T \) has at least one fixed point.

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**References**


