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COMPARISON BETWEEN RATIONAL CHEBYSHEV AND MODIFIED GENERALIZED LAGUERRE FUNCTIONS PSEUDOSPECTRAL METHODS FOR SOLVING LANE–EMDEN AND UNSTEADY GAS EQUATIONS

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In this paper we provide a pseudospectral method for Lane–Emden equation which models many phenomena in mathematical physics and astrophysics. We also use this method for solving unsteady gas equation which model unsteady flow of a gas through a semi-infinite porous medium. This approach is based on some orthogonal functions which will be defined. Pseudospectral method reduces the solution of these problems to the solution of systems of algebraic equations. We also compare this work with some other numerical results.

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1. Introduction

In the study of stellar structure [1] an important mathematical model described by the second-order ordinary differential equation

$$xy'' + 2y' + xg(y) = 0\,, \qquad x > 0\,, \tag{1}$$

arises, where $g(y)$ is some given function of y. One of the most popular forms of $g(y)$ is

$$g(y) = y^m\,, \tag{2}$$

subject to the conditions

$$y(0) = 1\,, \qquad y'(0) = 0\,. \tag{3}$$

This equation is standard Lane–Emden equation. It was first proposed by Lane [2] and studied in more detail by Emden [3]. The Lane–Emden equation describes a variety of phenomena in theoretical physics and astrophysics, including aspects of stellar structure, the thermal history of a spherical cloud of gas, isothermal gas spheres, and thermionic currents [4]. This equation is one of the basic equations in the theory of stellar structure and has been the focus of many studies [5–14]. It describes the temperature variation of a spherical gas cloud under the mutual attraction of its molecules and subject to the laws of classical thermodynamics. The polytropic theory of stars essentially follows out of thermodynamic considerations, that deal with the issue of energy transport, through the transfer of material between different levels of the star. The physically interesting range of m is $0 \leq m \leq 5$. Numerical and perturbation approaches to solve equation (2) with $g(y) = y^m$ have been considered by various authors. It has been claimed in the literature that only for $m = 0$, 1 and 5 the solutions of the Lane–Emden equation (also called the polytropic differential equations) could be given in closed form [15].

In fact, for $m = 5$, only a 1-parameter family of solutions is presented. The so-called generalized Lane–Emden equation of the first kind have been looked at in Goenner [16] and Havas [17].

1.1. Lane–Emden equation

Recently, many analytical methods have been used to solve Lane–Emden equation, the main difficulty arises in the singularity of the equation at $x = 0$. Currently, most techniques in use for handling the Lane–Emden-type problems are based on either series solutions or perturbation techniques.

Bender *et al.* [7], proposed a new perturbation technique based on an artificial parameter δ, the method is often called δ-method.

Mandelzweig *et al.* [12] used quasilinearization approach to solve Lane–Emden equation. This method approximates the solution of a nonlinear differential equation by treating the nonlinear terms as a perturbation about the linear ones, and unlike perturbation theories is not based on the existence of some kind of a small parameter. He showed that the quasilinearization method gives excellent results when applied to different nonlinear ordinary differential equations in physics, such as the Blasius, Duffing, Lane–Emden and Thomas–Fermi equations.

Shawagfeh [13] applied a nonperturbative approximate analytical solution for the Lane–Emden equation using the Adomian decomposition method. His solution was in the form of a power series. He used Padé approximations method to accelerate the convergence of the power series.

Seidov *et al.* [18] approximated analytical solution of Lane–Emden equation of index 5 with rational function and discussed accuracy of this solution. Seidov [19] also calculated a well-known series solution for Lane–Emden equation as:

$$y(x) = \sum_{k=0}^{\infty} a_k x^{2k} \,,$$

where $a_0 = 1$ and $a_1 = -1/6$ and

$$a_k = \frac{1}{(k-1)(k)(2k+1)} \sum_{j=1}^{k-1} (jm + j - k + 1)(k - j)(2k - 2j + 1) a_j a_{k-j} \,,$$

where $k \geq 2$.

In [14], Wazwaz employed the Adomian decomposition method with an alternate framework designed to overcome the difficulty of the singular point. It was applied to the differential equations of Lane–Emden type. Further in [20] he used the modified decomposition method for solving analytical treatment of nonlinear differential equations such as Lane–Emden equation. The modified method accelerates the rapid convergence of the series solution, dramatically reduces the size of work, and provides the solution by using few iterations only without any need to the so-called Adomian polynomials.

Liao [21] provided a reliable, easy-to-use analytical algorithm for Lane–Emden type equations. This algorithm logically contains the well-known Adomian decomposition method. Different from all other analytical techniques, this algorithm itself provides us with a convenient way to adjust convergence regions even without Padé technique.

He [22] employed Ritz method to obtain an analytical solution of the problem. By the semi-inverse method, a variational principle is obtained for the Lane–Emden equation, which he gave much numerical convenience when applied to finite element methods or Ritz method.

Parand *et al.* [23, 24] presented two numerical techniques to solve higher ordinary differential equations such as Lane–Emden. Their approach was based on rational Chebyshev and rational Legendre tau method. They presented the derivative and product operational matrices of rational Chebyshev and rational Legendre functions.

These matrices together with the tau method were utilized to reduce the solution of these physical problems to the solution of systems of algebraic equations.

Ramos [25–28] solved Lane–Emden equation through different methods. In [26] he presented linearization methods for singular initial-value problems in second-order ordinary differential equations such as Lane–Emden. These methods result in linear constant-coefficients ordinary differential equations which can be integrated analytical, thus yielding piecewise analytical solutions and globally smooth solutions. Later, he [27] developed piecewiseadaptive decomposition methods for the solution of nonlinear ordinary differential equations. Piecewise-decomposition methods provide series solutions in intervals which are subject to continuity conditions at the end points of each interval, and their adaption is based on the use of either a fixed number of approximants and a variable step size, a variable number of approximants and a fixed step size or a variable number of approximants and a variable step size.

In [28], series solutions of the Lane–Emden equation have been obtained by writing this equation as a Volterra integral equation and assuming that the nonlinearities are sufficiently differentiable. These series solutions have been obtained by either working with the original differential equation or transforming it into an ordinary differential equation that does not contain first-order derivatives. It has been shown that these approaches provide exactly the same solutions as those based on Adomian's decomposition techniques that make use of either a different differential operator that overcomes the singularity at $x = 0$, or a new dependent variable, and Liao's homotopy analysis technique. Series solutions to the Lane–Emden equation have also been obtained by working directly on the original differential equation or transforming it into a simpler one.

Yousefi [29] presented a numerical method for solving the Lane–Emden equation as singular initial value problems. Using integral operator and convert Lane–Emden equations to integral equations and then applying Legendre wavelet approximations. He presented Legendre wavelet properties at first. Then utilized these properties together with the Gaussian integration method to reduce the integral equations to the solution of algebraic equations.

In [30], Chowdhury *et al.* presented a reliable algorithm based on the homotopy-perturbation method (HPM) to solve singular IVPs of time-independent equations. They obtained the approximate and/or exact analytical solutions of the generalized Emden–Fowler type equations. This method is a coupling of the perturbation method and the homotopy method. The HPM is a novel and effective method which can solve various nonlinear equations. The main feature of the HPM is that it deforms a difficult problem into a set of problems which are easier to solve. In this work, HPM yields solutions in convergent series forms with easily computable terms.

Aslanov [31] introduced a further development in the Adomian decomposition method to overcome the difficulty at the singular point of non-homogeneous, linear and non-linear Lane–Emden-like equations, and constructed a recurrence relation for the components of the approximate solution and investigated the convergence conditions for the Emden–Fowler type of equations. He improved the previous results on the convergence radius of the series solution.

Recently, Dehghan and Shakeri [32] applied an exponential transformation to the Lane–Emden equation to overcome the difficulty of a singular point at $x = 0$, and then solved the resulting nonsingular problem by the variational iteration method.

Bataineh *et al.* [33] presented a reliable algorithm based on HAM to obtain the exact and/or approximate analytical solutions of the singular IVPs of the Emden–Fowler type.

The HAM, first proposed by Liao in his Ph.D. dissertation [21], is a promising method for linear and non-linear problems. HAM contains an auxiliary parameter $\hbar$ which provides us with a simple way to adjust and control the convergence region and the rate of convergence of the series solution. In this paper, we aim to employ the pseudospectral method to a singular form of Lane–Emden type initial value problem and an ordinary differential equation that describes the unsteady flow of gas through a semi-infinite porous medium directly.

One of the most accurate analytical solutions are found to Lane–Emden equation of arbitrary index n, by Seidov [34], he used Picard type iteration scheme and rational Padé approximants.

1.2. Unsteady gas equation

In the study of the unsteady flow of a gas through a semi-infinite porous medium [35,36] initially filled with gas at a uniform pressure $p_0 \geq 0$, at time $t = 0$, the pressure at the outflow face is suddenly reduced from p_0 to $p_1 \geq 0$ ($p_1 = 0$ is the case of diffusion into a vacuum) and is, thereafter, maintained at this lower pressure. The unsteady isothermal flow of gas is described by a nonlinear partial differential equation. The nonlinear partial differential equation that describes the unsteady flow of gas through a semi-infinite porous medium has been derived by Muskat [37] in the form

$$\nabla^2 \left(P^2\right) = \frac{2\Phi\mu}{k}\frac{\partial P}{\partial t}\,, \tag{4}$$

where P is the pressure within porous medium, Φ the porosity, μ the viscosity, k the permeability, and t the time. New variables were introduced by Kidder [35] and Davis [38] to transform the nonlinear partial differential equation (4) to the nonlinear ordinary differential equation. The nonlinear ordinary differential equation due to Kidder [35] given by (unsteady gas equation)

$$y'' + 2xy'/(1-\alpha y)^{1/2} = 0\,, \qquad x > 0\,, \qquad 0 < \alpha < 1\,. \tag{5}$$

The typical boundary conditions imposed by the physical properties are

$$y(0) = 1\,, \qquad y(\infty) = 0\,. \tag{6}$$

A substantial amount of numerical and analytical work has been invested so far [35, 39] on this model. The main reason of this interest is that the approximation can be used for many engineering purposes. As stated before, the problem (5) was handled by Kidder [35] where a perturbation technique is carried out to include terms of the second order. Recently, Wazwaz [40] has applied the modified decomposition method for solving this nonlinear equation. The base of his approach is modification of the Adomian decomposition method. The diagonal Padé approximations are effectively used in the analysis to capture the essential behavior of $y(x)$ and to determine the initial slope $y'(0)$.

2. Rational Chebyshev functions

This section is devoted to introducing rational Chebyshev functions and expressing some basic properties of them. At the end, we applied rational Chebyshev approximation to solve Lane–Emden and unsteady gas equations.

Rational Chebyshev functions denoted by $R_n(x)$ are generated from well-known Chebyshev polynomials by using the algebraic mapping $\phi(x) = (x - L)/(x + L)$ [41–43]

$$R_n(x) = T_n\left(\phi(x)\right) , \tag{7}$$

where L is a constant parameter and $T_n(y)$ is the Chebyshev polynomial of degree n.

The constant parameter L sets the length scale of the mapping. Boyd [44–47] offered guidelines for optimizing the map parameter L where $L > 0$.

Numerical results deponed smoothly on constant parameter L, and therefore are not very sensitive to L, so the error varies very slowly with L around the minimum. A little trial and error is usually sufficient to find a value that is nearly optimum. In general, there is no way to avoid a small amount of trial and error in choosing L when solving problems on an unbounded domain. Experience and the asymptotic approximations of Boyd [46] can help, but some experimentation is always necessary as he explains in his book [44].

Using properties of Chebyshev polynomials, we have

$$\begin{aligned} R_n(x) \; &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i 2^{n-2i} \binom{n-i}{i} \left(\frac{x-L}{x+L}\right)^{n-2i} \\ &- \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^i 2^{n-2i-1} \binom{n-i-1}{i} \left(\frac{x-L}{x+L}\right)^{n-2i} . \end{aligned} \tag{8}$$

Other properties of rational Chebyshev functions and a complete discussion on approximating functions by rational Chebyshev functions are given in [42, 43].

2.1. Rational Chebyshev functions approximation

Let

$$\mathfrak{R}_N = \mathrm{span}\big\{R_0, R_1, ..., R_N\big\}\,. \tag{9}$$

We define $P_N : L^2_w(\Lambda) \to \mathfrak{R}_N$ by

$$P_N u(x) = \sum_{k=0}^{N} a_k R_k(x)\,. \tag{10}$$

To obtain the order of convergence of rational Chebyshev approximation, we define the space

$$H^r_{w,A}(\Lambda) = \{v : v \text{ is measurable and } \|v\|_{r,w,A} < \infty\}\,, \tag{11}$$

where the norm is induced by

$$\|v\|_{r,w,A} = \left(\sum_{k=0}^{r} \left\|(x+1)^{\frac{r}{2}+k}\frac{d^k}{dx^k}v\right\|_w^2\right)^{\frac{1}{2}}, \tag{12}$$

and A is the Sturm–Liouville operator as follows:

$$Av(x) = -w^{-1}(x)\frac{d}{dx}\left(w^{-1}(x)\frac{d}{dx}v(x)\right)\,. \tag{13}$$

$w(x)$ is the weight function and $w(x) = \sqrt{L}/(\sqrt{x}(x+L))$. We have the following theorem for the convergence:

Theorem 1. *For any $v \in H^r_{w,A}(\Lambda)$ and $r \geq 0$,*

$$\|P_N v - v\|_w \leq cN^{-r}\|v\|_{r,w,A}\,. \tag{14}$$

Proof 1. A complete proof is given by Guo *et al.* [42].

This theorem shows that the rational Chebyshev approximation has exponential convergence.

2.2. Solving Lane–Emden equation by rational Chebyshev functions

In the first step of our analysis, we apply P_N operator on the function $y(x)$ as follows:

$$P_N y(x) = \sum_{k=0}^{N} a_k R_k(x)\,. \tag{15}$$

Then, we construct the residual function by substituting $y(x)$ by $P_N y(x)$ in the Lane–Emden equation:

$$\mathrm{Res}(x) = \frac{d^2}{dx^2} P_N y(x) + \frac{2}{x}\frac{d}{dx} P_N y(x) + (P_N y(x))^m \,. \tag{16}$$

The equations for obtaining the coefficients a_ks come from equalizing $\mathrm{Res}(x)$ to zero at rational Chebyshev–Gauss–Radau points plus two boundary conditions:

$$\mathrm{Res}(x_j) = 0\,, \qquad j = 1, 2, \ldots, N-1\,, \tag{17}$$

$$P_N y(0) = 1\,, \tag{18}$$

$$\left.\frac{d}{dx} P_N y(x)\right|_{x=0} = 0\,. \tag{19}$$

Solving the set of equations we have the approximating function $P_N y(x)$.

2.3. Solving unsteady gas equation by rational Chebyshev functions

Solving unsteady gas equation is the same as solving Lane–Emden equation. So the residual function is

$$\mathrm{Res}(x) = \frac{d^2}{dx^2} P_N y(x) + \frac{2x}{(1-\alpha y)^{1/2}}\frac{d}{dx} P_N y(x)\,, \tag{20}$$

and the set of equations for obtaining the coefficients a_ks are as follows:

$$\mathrm{Res}(x_j) = 0\,, \qquad j = 1, 2, \ldots, N-1\,, \tag{21}$$

$$P_N y(0) = 1\,, \tag{22}$$

$$\lim_{x\to\infty} P_N y(x) = 0\,. \tag{23}$$

3. Modified generalized Laguerre functions

The Laguerre approximation has been widely used for numerical solutions of differential equations on infinite intervals. $L_n^\alpha(x)$ (generalized Laguerre polynomial) is the nth eigenfunction of the Sturm–Liouville problem [48, 49]:

$$x\frac{d^2}{dx^2} L_n^\alpha(x) + (\alpha + 1 - x)\frac{d}{dx} L_n^\alpha(x) + n L_n^\alpha(x) = 0\,,$$

$$x \in I = [0, \infty)\,, \qquad n = 0, 1, 2, \ldots .$$

The generalized Laguerre polynomials are defined with the following recurrence formula:

$$L_0^\alpha(x) = 1\,, \qquad L_1^\alpha(x) = 1 + \alpha - x\,,$$

$$nL_n^\alpha(x) = (2n-1+\alpha-x)L_{n-1}^\alpha(x) - (n+\alpha-1)L_{n-2}^\alpha(x)\,,$$

these are orthogonal polynomials for the weight function $w_\alpha = x^\alpha e^{-x}$. We define Modified generalized Laguerre functions (which we denote MGLF) ϕ_j as follows:

$$\phi_j(x) = \exp(-x/(2L))L_j^1(x/L)\,, \qquad L>0\,. \tag{24}$$

This system is an orthogonal basis [50, 51] with weight function $w(x) = \frac{x}{L}$ and orthogonality property:

$$\langle \phi_n, \phi_m\rangle_{w_L} = \left(\frac{\Gamma(n+2)}{L^2 n!}\right)\delta_{nm}\,,$$

where δ_{nm} is the Kronecker function.

3.1. Function approximation with Laguerre functions

A function $f(x)$ defined over the interval $I = [0,\infty)$ can be expanded as

$$f(x) = \sum_{i=0}^{\infty} a_i\phi_i(x)\,, \tag{25}$$

where

$$a_i = \frac{\langle f, \phi_i\rangle_w}{\langle \phi_i, \phi_i\rangle_w}\,. \tag{26}$$

If the infinite series in Eq. (25) is truncated with N terms, then it can be written as

$$f(x) \simeq \sum_{i=0}^{N-1} a_i\phi_i(x) = A^T\phi(x)\,, \tag{27}$$

with

$$A = [a_0, a_1, a_2, \ldots, a_{N-1}]^T\,, \tag{28}$$

$$\phi(x) = [\phi_0(x), \phi_1(x), \ldots, \phi_{N-1}(x)]^T\,. \tag{29}$$

3.2. Modified generalized Laguerre functions collocation method

Laguerre–Gauss–Radau points and generalized Laguerre–Gauss-type interpolation were introduced by [52–54].

Let

$$\mathfrak{R}_N = \mathrm{span}\{1, x, \ldots, x^{2N-1}\}\,,$$

we choose the collocation points relative to the zeroes of the functions

$$p_j(x) = \phi_j(x) - ((j+1)/j)\phi_{j-1}(x)\,. \tag{30}$$

Let $w(x) = \frac{x}{L}$ and x_j, $j = 0, 1 \ldots, N-1$, be the N MGLF-Radau points. The relation between MGLF orthogonal systems and MGLF integrations is as follows [55]:

$$\int\limits_0^{+\infty} f(x)w(x)dx = \sum_{j=0}^{N-1} f(x_j)w_j + \left(\frac{\Gamma(N+2)}{(N)!(2N)!}\right) f^{2N}(\xi)e^{\xi}\,,$$

where $0 < \xi < \infty$ and $w_j = x_j\Gamma(N+2)/(L(N+1)![(N+1)\phi_{N+1}(x_j)]^2)$, $j = 0, 1, 2, \ldots, N-1$. In particular, the second term on the right-hand side vanishes when $f(x)$ is a polynomial of degree at most $2N-1$. We define

$$I_N u(x) = \sum_{j=0}^{N} a_j\phi_j(x)\,, \tag{31}$$

such that $I_N u(x_j) = u(x_j),\ j = 0, \ldots, N-1$. $I_N u$ is the orthogonal projection of u upon $\Re_N$ with respect to the discrete inner product and discrete norm as:

$$\begin{aligned} \langle u, v\rangle_{w,N} &= \sum_{j=0}^{N-1} u(x_j)v(x_j)w_j\,, \\ \| u \|_{w,N} &= \langle u, u\rangle_{w,N}^{1/2}\,, \end{aligned}$$

thus for the MGLF Gauss–Radau interpolation we have

$$\langle I_N u, v\rangle_{w,N} = \langle u, v\rangle_{w,N}\,, \qquad \forall u.v \in \Re_N\,.$$

3.3. Solving Lane–Emden equation with modified generalized Laguerre functions

To apply scaled Laguerre collocation method to the standard Lane–Emden Equation introduced in Eq. (1) and Eq. (2) with boundary conditions Eq. (3), at first we expand $y(x)$, as follows:

$$I_N y(x) = \sum_{j=0}^{N} a_j\phi_j(x)\,. \tag{32}$$

To find the unknown coefficients a_j's, we substitute the truncated series into the Eq. (1) with $g(y)$ introduced in Eq. (2) and boundary conditions in Eq. (3). So we have

$$x\sum_{j=0}^{N} a_j\phi_j''(x) + 2\sum_{j=0}^{N} a_j\phi_j'(x) + x\left(\sum_{j=0}^{N} a_j\phi_j(x)\right)^m = 0\,, \tag{33}$$

$$\sum_{j=0}^{N} a_j\phi_j(0) = 1\,, \qquad \sum_{j=0}^{N} a_j\phi_j'(0) = 0\,. \tag{34}$$

By replacing x in Eq. (33) with the $N-1$ collocation points which are roots of functions $\frac{d}{dx}L_N^{\alpha}$, we have $N-1$ equations that generates a set of $N+1$ nonlinear equations with boundary equations in Eq. (3).

Table I shows the comparison of the first zero of y, between Padé approximation used by [7] and rational Chebyshev pseudospectral method (*i.e.* RCM) and modified generalized Laguerre functions method (*i.e.* MGLFM) for $m = 3/2, 2, 3, 4$.

TABLE I

The comparison of the first zero of y, between [7] and the present methods for $m = 3/2, 2, 3, 4$.

m	N	MGLFM	RCM	Bender	Exact value
3/2	6	3.65637555	3.65987637	—	3.65375373
2	6	4.35251508	4.35280120	4.3603	4.35287460
3	6	6.89658947	6.89201052	7.0521	6.89684862
4	6	14.9715138	14.9715334	17.967	14.9715463

Tables II and III show the approximations of $y(x)$ for standard Lane–Emden with $m = 3/2, 4$ obtained respectively by the methods proposed in this paper, and those obtained by Horedt [56].

TABLE II

The comparison of $y(x)$ for present methods and solutions of Horedt [56] for $m = 3/2$.

x	MGLFM	Error (MGLFM)	RCM	Error (RCM)	Solutions of Horedt
0.00	1.0000000000	0.0000000000	1.0000000000	0.0000000000	1.0000000000
0.10	0.9971426000	0.0011920000	0.9964563409	0.0018782591	0.9983346000
0.50	0.9590036100	0.0001002900	0.9537462726	0.0053576274	0.9591039000
1.00	0.8435829000	0.0015869000	0.8434562598	0.0017135402	0.8451698000
3.00	0.1587468000	0.0001108000	0.1589386000	0.0000810000	0.1588576000
3.60	0.0112354500	0.0001444600	0.0111649900	0.0000710000	0.0110909900

TABLE III

The comparison of $y(x)$ for present methods and solutions of Horedt [56] for $m = 4$.

x	MGLFM	Error (MGLFM)	RCM	Error (RCM)	Solutions of Horedt
0.0	1.0000000000	0.0000000000	1.0000000000	0.0000000000	1.0000000000
0.1	0.9983130000	0.0000237000	0.9983163409	0.0000203591	0.9983367000
0.2	0.9938112140	0.0004250140	0.9937462726	0.0003600726	0.9933862000
0.5	0.9609840230	0.0006731230	0.9610174048	0.0007065048	0.9603109000
1.0	0.8602647800	0.0005490200	0.8606462598	0.0001675402	0.8608138000
10.0	0.0607033550	0.0010306150	0.0701785255	0.0105057855	0.0596727400
14.0	0.0091557100	0.0008251830	0.0100712416	0.0017407146	0.0083305270
14.9	0.0005981000	0.0000216811	0.0006992396	0.0001228207	0.0005764189

3.4. Solving unsteady gas equation with modified generalized Laguerre functions

Solving this equation (5) is the same as solving the Lane–Emden equation. Let

$$\mathrm{Res}(x) = \frac{d^2}{dx^2}\sum_{j=0}^{N} a_j\phi_j(x) + \frac{2x}{(1-\alpha\sum_{j=0}^{N} a_j\phi_j)^{1/2}}\frac{d}{dx}\sum_{j=0}^{N} a_j\phi_j(x)\,, \qquad (35)$$

and the set of equations for obtaining the coefficients a_ks are as follows:

$$\mathrm{Res}(x_j) = 0\,, \qquad j = 1,2,\ldots,N-1\,, \qquad (36)$$

$$\sum_{j=0}^{N} a_j\phi_j(0) = 1\,, \qquad (37)$$

it is clear that $\lim_{x\to\infty}\sum_{j=0}^{N} a_j\phi_j(x) = 0$. It seems that MGLF has solutions like Wazwaz method [40] with Padé [3,3] and RCM has solutions like the perturbation method used by [35].

Table IV shows the comparison of the $y'(0)$, obtained by MGLFM and RCM for $N = 6, 7$ and Padé approximation used by [40].

TABLE IV

The comparison of initial slope $y'(0)$ for $\alpha = 0.5$.

N	MGLFM	RCM	Padé [2,2]	Padé [3,3]
6	−1.36417503	−1.10805718	−1.37317809	−1.02552970
7	−1.38213483	−1.26259357	−1.37317809	−1.02552970

Table V shows the approximations of $y(x)$ for standard unsteady gas with $\alpha = 0.5$ obtained by the methods proposed in this paper for $N = 7$, the perturbation method used by [35], and Padé approximation by Wazwaz [40].

TABLE V

The values of $y(x)$ for $\alpha = 0.5$, for $x = 0.1$ to 1.0.

x	MGLFM	RCM	Perturbation	Padé [2,2]	Padé [3,3]
0.1	0.89035366	0.88042558	0.88165883	0.86330606	0.89791670
0.3	0.70810133	0.66402932	0.65653800	0.60330541	0.70411297
0.5	0.53394788	0.46068185	0.46136503	0.37616039	0.53705338
0.7	0.40743340	0.30320332	0.30559765	0.18968434	0.40624260
0.9	0.31070888	0.19666414	0.19046237	0.04323673	0.31799666
1.0	0.28306464	0.15835106	0.15876898	0.01646751	0.29002550

4. Conclusions

The fundamental goal of this paper was to construct an approximation to the solution of some well known nonlinear ordinary differential equations which model many phenomena in mathematical physics and astrophysics. A set of orthogonal functions were proposed to provide an effective but simple way to improve the convergence of the solution by pseudospectral method. We solved Lane–Emden with this method, the findings showed that the present solutions with MGLFM and RCM were highly accurate.

We also used pseudospectral method to solve unsteady gas equation. It seems that MGLFM has solutions like Wazwaz method [40] with Padé [3,3], and RCM has solutions like the perturbation method used by [35].

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