Generalized Quasi Variational-Type Inequalities

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ABSTRACT

In this paper, we define the concepts of \((\eta,h)\)-quasi pseudo-monotone operators on compact set in locally convex Hausdorff topological vector spaces and prove the existence results of solutions for a class of generalized quasi variational type inequalities in locally convex Hausdorff topological vector spaces.

Keywords: Generalized Quasi Variational Type Inequalities (GQVTI); \((\eta,h)\)-Quasi Pseudo-Monotone Operator; Locally Convex Hausdorff Topological Vector Spaces; Compact Sets; Bilinear Functional; Lower Semicontinuous; Upper Semicontinuous

1. Introduction

Variational inequality theory has appeared as an effective and powerful tool to study and investigate a wide class of problems arising in pure and applied sciences including elasticity, optimization, economics, transportation, and structural analysis, see for instance [1,2]. In 1966, Browder [3] first formulated and proved the basic existence theorems of solutions to a class of nonlinear variational inequalities. In 1980, Giannessi [1] introduced the vector variational inequality in a finite dimensional Euclidean space. Since then Chen et al. [4] have intensively studied vector variational inequalities in abstract spaces and have obtained existence theorems for their inequalities.

The pseudo-monotone type operators was first introduced in [5] with a slight variation in the name of this operator. Later these operators were renamed as pseudo-monotone operators in [6]. The pseudomonotone operators are set-valued generalization of the classical pseudo-monotone operator with slight variations. The classical definition of a single-valued pseudo-monotone operator was introduced by Brezis, Nirenberg and Stampacchia [7].

In this paper we obtained some general theorems on solutions for a new class of generalized quasi variational type inequalities for \((\eta,h)\)-quasi pseudo-monotone operators defined as compact sets in topological vector spaces. We have used the generalized version of Ky Fan’s minimax inequality [8] due to Chowdhury and Tan [9].

Let \(X\) and \(Y\) be the topological spaces, \(T : X \to 2^Y\) be the mapping and the graph of \(T\) is the set \(G(T) = \{(x,y) \in X \times Y : y \in T(x)\}\). In this paper, \(\Phi\) denotes either the real field \(\mathbb{R}\) or the compact field \(\mathbb{C}\). Let \(E\) be a topological vector space over \(\Phi\), \(F\) be a vector space over \(\Phi\) and \(\langle \cdot, \cdot \rangle : F \times E \to \Phi\) be a bilinear functional.

For each nonempty subset \(A\) of \(E\) and \(\varepsilon > 0\), let \(W(x_0;\varepsilon) = \{y \in F : \langle y, x_0 \rangle < \varepsilon\}\) and \(U(A;\varepsilon) = \{y \in F : \sup_{x \in A} \langle y, x \rangle < \varepsilon\}\) for \(x_0 \in E\). Let \(\sigma(F,E)\) be the (weak) topology on \(F\) generated by the family \(\{W(x;\varepsilon) : x \in E\text{ and } \varepsilon > 0\}\) as a subbase for the neighbourhood system at 0 and \(\delta(E,F)\) be the (strong) topology on \(F\) generated by the family \(\{U(A;\varepsilon) : A\text{ is a nonempty bounded subset of } E\text{ and } \varepsilon > 0\}\) as a base for the neighbourhood system at 0. The bilinear functional \(\langle \cdot, \cdot \rangle : F \times E \to \Phi\) separates points in \(F\), i.e., for each \(0 \neq y \in F\), there exists \(x \in E\) such that \(\langle y, x \rangle \neq 0\), then \(F\) also becomes Hausdorff. Furthermore, for a net \(\{y_{\alpha}\}_{\alpha \in \mathbb{N}}\) in \(F\) and for \(y \in F\),

1) \(y_{\alpha} \to y\) in \(\sigma(F,E)\) if and only if \(\langle y_{\alpha}, x \rangle \to \langle y, x \rangle\) for each \(x \in E\) and
2) \(y_{\alpha} \to y\) in \(\delta(F,E)\) if and only if \(\langle y_{\alpha}, x \rangle \to \langle y, x \rangle\) uniformly for \(x \in A\) for each nonempty bounded subset \(A\) of \(E\).

Given a set-valued map \(S : X \to 2^X\) and two set valued maps \(M, T : X \to 2^E\), the generalized quasi variational type inequality (GQVTI) problem is to find \(\hat{y} \in X\) and \(\hat{w} \in T(\hat{y})\) such that \(\hat{y} \in S(\hat{y})\) and

\[
\text{Re}\{f - \hat{w}, \eta(\hat{y}, x)\} \leq 0,
\]

for all \(x \in S(\hat{y})\) and \(f \in M(\hat{y})\),

where \(\eta : X \times X \to E\).

If \(\eta(\hat{y}, x) = \hat{y} - x\), then generalized quasi variational type inequality (GQVI) is equivalent to generalized quasi variational inequality (GQVI).
Find \( \hat{y} \in X \) and \( \hat{w} \in T(\hat{y}) \) such that \( \hat{y} \in S(\hat{y}) \) and 
\[
\text{Re}(f - \hat{w}, \hat{y} - x) \leq 0 \quad \text{for all } x \in S(y)
\]
and \( f \in M(\hat{y}) \) was introduced by Shih and Tan [10] in 1989 and later was stated by Chowdhury and Tan in [11].

**Definition 1.** Let \( X \) be a nonempty subset of a topological vector space \( E \) over \( \Phi \) and \( F \) be a topological vector space over \( \Phi \), which is equipped with the \( \sigma(F,E) \)-topology. Let \( \langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi \) be a bilinear functional. Suppose we have the following four maps:

1) \( h : X \times X \rightarrow \mathbb{R} \)
2) \( \eta : X \times X \rightarrow E \)
3) \( M : X \rightarrow 2^F \)
4) \( T : X \rightarrow 2^F \).

1) Then \( T \) is said to be an \((\eta,h)\)-quasi pseudo-monotone type operator if for each \( y \in X \) and every net \( \{y_j\}_{\alpha \in \Gamma} \) in \( X \) converging to \( y \) (or weakly to \( y \)) with 
\[
\limsup_{\alpha} \left[ \inf_{f \in M(y_j)} \inf_{u \in T(y_j)} \text{Re}(f - u, \eta(y_j, y)) + h(y_j, y) \right] \leq 0.
\]

We have 
\[
\limsup_{\alpha} \left[ \inf_{f \in M(y)} \inf_{u \in T(y)} \text{Re}(f - u, \eta(y, y)) + h(y, y) \right] 
\geq \inf_{f \in M(y)} \inf_{u \in T(y)} \text{Re}(f - u, \eta(y, y)) + h(y, y),
\]
for all \( x \in X \);

2) \( T \) is said to be \( h \)-quasi pseudo-monotone operator if \( T \) is \((\eta,h)\)-quasi pseudo-monotone operator with \( \eta(x, y) = x - y \) and for some \( h' : X \rightarrow \mathbb{R} \), 
\[
h(x, y) = h'(x) - h'(y) \quad \text{for all } x, y \in X.
\]

3) a quasi pseudo monotone operator if \( T \) is an \( h \)-quasi pseudo-monotone operator with \( h = 0 \).

**Remark 1.** If \( M = 0 \) and \( T \) is replaced by \(-T\), then \( h \)-quasi pseudo monotone operator reduces to the \( h \)-pseudo monotone operator, see for example [5]. The \( h \)-pseudo monotone operator defined in [5] is slightly more general than the definition of \( h \)-pseudo monotone operator given in [12]. Also we can find the generalization of quasi pseudo monotone operator in [11] and for more detail see [13].

**Theorem 1.** [8] Let \( E \) be a topological vector space, \( X \) be a nonempty convex subset of \( E \) and 
\[
f : X \times X \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}
\]
be such that 
1) for each \( A \in F(X) \) and each fixed \( x \in \text{co}(A) \), 
\[
y \rightarrow f(x, y) \quad \text{is lower semicontinuous on } \text{co}(A);
\]
2) for each \( A \in F(X) \) and each \( y \in \text{co}(A) \), 
\[
\min_{x \in A} f(x, y) \leq 0;
\]
3) for each \( A \in F(X) \) and each \( x, y \in \text{co}(A) \), every net \( \{y_\alpha\}_{\alpha \in \Gamma} \) in \( X \) converging to \( y \) with 
\[
f(tx + (1-t)y, y_\alpha) \leq 0 \quad \text{for all } \alpha \in \Gamma \quad \text{and all } t \in [0,1]
\]
we have 
\[
f(x, y) \leq 0;
\]
4) there exist a nonempty closed compact subset \( K \) of \( X \) and \( x_0 \in K \) such that 
\[
f(x_0, y) > 0 \quad \text{for all } y \in X \setminus K.
\]

Then there exists \( \hat{y} \in K \) such that 
\[
f(x, \hat{y}) \leq 0 \quad \text{for all } x \in X.
\]

**2. Preliminaries**

In this section, we shall mainly state some earlier work which will be needed in proving our main results.

**Lemma 1.** [14] Let \( X \) be a nonempty subset of a Hausdorff topological vector space \( E \) and \( S : X \rightarrow 2^E \) be an upper semicontinuous map such that \( S(x) \) is a bounded subset of \( E \) for each \( x \in X \). Then for each continuous linear functional \( p \) on \( E \), the map 
\[
f_p : X \rightarrow \mathbb{R} \quad \text{defined by}
\]
\[
f_p(y) = \sup_{x \in S(y)} \text{Re}(p, x)
\]
for each \( \lambda \in \mathbb{R} \),

the set \( \left\{ y \in X : f_p(y) = \sup_{x \in S(y)} \text{Re}(p, x) < \lambda \right\} \) is open in \( X \).

**Lemma 2.** [15] Let \( X, Y \) be topological spaces, \( f : X \rightarrow \mathbb{R} \) be non-negative and continuous and \( g : Y \rightarrow \mathbb{R} \) be lower semicontinuous. Then the map 
\[
F : X \times Y \rightarrow \mathbb{R}, \quad \text{defined by} \quad F(x, y) = f(x)g(y)
\]
for all \( (x, y) \in X \times Y \), is lower semicontinuous.

**Lemma 3.** [11] Let \( E \) be a topological vector space over \( \Phi \), \( X \) be a nonempty compact subset of \( E \) and \( F \) be a Hausdorff topological vector space over \( \Phi \). Let 
\[
\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi
\]
be a bilinear functional and 
\[
T : X \rightarrow 2^F \quad \text{be an upper semicontinuous map such that each } T(x) \text{ is compact. Let } M \text{ be a nonempty compact subset of } F, \ x_0 \in X \quad \text{and } \ h : X \rightarrow \mathbb{R} \text{ be continuous. Define } g : X \rightarrow \mathbb{R} \text{ by}
\]
\[
g(y) = \left[ \inf_{f \in M(y)} \inf_{u \in T(y)} \text{Re}(f - w, y - x_0) \right] + h(y)
\]
for each \( y \in X \).

Suppose that \( \langle \cdot, \cdot \rangle \) is continuous on the (compact) subset 
\[
\left[ M - \bigcup_{y \in X} T(y) \right] \times X \quad \text{of } F \times E
\]
Then \( g \) is lower semicontinuous on \( X \).

**Lemma 4.** [11] Let \( E \) be a topological vector space over \( \Phi \), \( F \) be a vector space over \( \Phi \) and \( X \) be a nonempty convex subset of \( E \). Let 
\[
\langle \cdot, \cdot \rangle : F \times E \rightarrow \Phi
\]
be a bilinear functional, equip $F$ with the $\sigma(F,E)$ -
topology. Let $h: X \times X \to \mathbb{R}$ be convex with second argument and
$h(x,x) = 0$ for all $x \in X$. Let $M : X \to F$ be lower semicontinuous along line segments in $X$ to the $\sigma(F,E)$ -
topology on $F$. Let $S : X \to 2^X$ and $T : X \to 2^F$ be two maps. Let the continuous map $\eta : X \times X \to E$ be convex with second argument, $\eta(x,x) = 0$ for every $x \in X$. Suppose that there exists $\hat{y} \in X$ such that $\hat{y} \in S(\hat{y})$, $S(\hat{y})$ is convex and
\[ \inf_{f \in M(x)} \inf_{x \in T(y)} \Re\{f - w, \eta(\hat{y}, x)\} + h(x, \hat{y}) \leq 0 \]
for all $x \in S(\hat{y})$.

Then
\[ \inf_{f \in M(y)} \inf_{x \in T(y)} \Re\{f - w, \eta(\hat{y}, x)\} + h(x, \hat{y}) \leq 0 \]
for all $x \in S(\hat{y})$.

**Theorem 2.** [16] Let $X$ be a nonempty convex subset of a vector space and $Y$ be a nonempty compact convex subset of a Hausdorff topological vector space. Suppose that $f$ is a real-valued function on $X \times Y$ such that for each fixed $x \in X$, the map $y \to f(x, y)$, i.e., $f(x, \cdot)$ is lower semicontinuous and convex on $Y$ and for each fixed $y \in Y$, the map $x \to f(x, y)$, i.e., $f(\cdot, y)$ is concave on $X$. Then
\[ \min_{y \in Y} \sup_{x \in X} f(x, y) = \sup_{x \in X} \min_{y \in Y} f(x, y). \]

**3. Existence Result**

In this section, we prove the existence theorem for the solutions to the generalized quasi variational type inequalities for $(\eta, h)$-quasi-pseudo monotone operator with compact domain in locally convex Hausdorff topological vector spaces.

**Theorem 3.** Let $E$ be a locally convex Hausdorff topological vector space over $\Phi$, $X$ be a nonempty compact convex subset of $E$ and $F$ a Hausdorff topological vector space over $\Phi$. Let $\langle \cdot, \cdot \rangle : F \times E \to \Phi$ be a bilinear continuous functional on compact subset of $F \times X$. Suppose that

1) $S : X \to 2^X$ is upper semicontinuous such that each $S(x)$ is closed and convex;
2) $h : X \times X \to \mathbb{R}$ is convex with second argument, $h(x,x) = 0$ for all $x \in X$;
3) $\eta : X \times X \to E$ is convex with second argument, $\eta(x,x) = 0$ for all $x \in X$;
4) $T : X \to 2^F$ is an $(\eta, h)$-quasi-pseudo-monotone operator and is upper semicontinuous such that each $T(x)$ is compact, convex and $T(X)$ is strongly bounded;
5) $M : X \to F$ is a linear and upper semicontinuous map in $X$ such that each $M(x)$ is (weakly) compact convex;
6) the set
\[ \Sigma = \left\{ y \in X : \sup_{x \in S(y)} \inf_{w \in T(y)} \Re\{M(x) - w, \eta(y, x)\} + h(y, x) - h(x, x) > 0 \right\} \]
is open in $X$.

Then there exists $\hat{y} \in X$ such that
a) $\hat{y} \in S(\hat{y})$ and
b) there exists $\hat{w} \in T(\hat{y})$ with
\[ \Re\{M(\hat{y}) - \hat{w}, \eta(\hat{y}, x)\} + h(\hat{y}, x) - h(x, x) \leq 0 \]
for all $x \in S(\hat{y})$.

Moreover if $S(x) = X$ for all $x \in X$, $E$ is not required to be locally convex and if $T \neq 0$, the continuity assumption on $\langle \cdot, \cdot \rangle$ can be weakened to the assumption that for each $f \in F$, the map $x \to \langle f, x \rangle$ is continuous on $X$.

**Proof.** We divide the proof into three steps.

**Step 1.** There exists $\hat{y} \in X$ such that $\hat{y} \in S(\hat{y})$ and
\[ \sup_{x \in S(y)} \inf_{w \in T(y)} \Re\{M(x) - w, \eta(y, x)\} + h(y, x) - h(x, x) \leq 0. \]

Contrary suppose that for each $y \in X$, either $y \in S(y)$ or there exists $x \in S(y)$ such that
\[ \inf_{w \in T(y)} \Re\{M(x) - w, \eta(y, x)\} + h(y, x) - h(x, x) > 0, \]
that is for each $y \in X$ either $y \in S(y)$ or $y \in \Sigma$. If $y \in S(y)$, then by a Hahn-Banach separation theorem for convex sets is locally convex Hausdorff topological vector spaces, there exists $p \in E^*$ such that
\[ \Re\langle p, y \rangle - \sup_{x \in S(y)} \Re\langle p, x \rangle > 0. \]

For each $p \in E^*$, set
\[ V_p = \left\{ y \in X : \Re\langle p, y \rangle - \sup_{x \in S(y)} \Re\langle p, x \rangle > 0 \right\}. \]

Then $V_p$ is open in $X$ by Lemma 1 and $\Sigma$ is open in $X$ by hypothesis. Now $X = \Sigma \cup \bigcup_{p \in E^*} V_p$ and $\left\{ \Sigma, V_p : p \in E^* \right\}$ is an open covering for $X$. Since $X$ is compact subset of $E$, there exists $p_1, p_2, \ldots, p_n \in E^*$ such that $X = \Sigma \cup \bigcup_{i=1}^n V_{p_i}$ for $i = 1, 2, \ldots, n$. Let
\[ V_i = V_{p_i} \quad \text{for } i = 1, 2, \ldots, n \]
and $\left\{ \beta_0, \beta_1, \ldots, \beta_n \right\}$ be a continuous partition of unity on $X$ subordinated to the
covering \( \{V_i, V_1, \ldots, V_n\} \). Then \( \beta_0, \beta_1, \ldots, \beta_n \) are continuous non-negative real valued functions on \( X \) such that \( \beta_i \) vanishes on \( X \setminus V_i \) for each \( i = 0, 1, \ldots, n \) and \( \sum_{i=0}^n \beta_i(x) = 1 \) for all \( x \in X \) (see [17] p.83).

Define \( \varphi : X \times X \to \mathbb{R} \) by

\[
\varphi(x, y) = \beta_0(y) \left[ \inf_{w \in \mathcal{C}(y)} \text{Re}(M(x) - w, \eta(y, x)) + h(y, x) - h(x, x) \right] + \sum_{i=1}^n \beta_i(y) \text{Re}(p_i, \eta(y, x))
\]

for each \( x, y \in X \). Then we have,

1) \( E \) is Hausdorff for each \( A \in F(X) \) and each fixed \( x \in \text{co}(A) \) the map

\[ y \to \inf_{w \in \mathcal{C}(y)} \text{Re}(M(x) - w, \eta(y, x)) + h(y, x) - h(x, x) \]

is lower semicontinuous on \( \text{co}(A) \) by Lemma 3 and the fact that \( h \) is continuous on \( \text{co}(A) \), therefore the map

\[ \beta_0(y) \left[ \inf_{w \in \mathcal{C}(y)} \text{Re}(M(x) - w, \eta(y, x)) + h(y, x) - h(x, x) \right] + \sum_{i=1}^n \beta_i(y) \langle p_i, \eta(y, x) \rangle > 0. \]

So that

\[ 0 = \varphi(x, y) = \beta_0(y) \left[ \inf_{w \in \mathcal{C}(y)} \text{Re}(M(x) - w, \eta(y, x)) + h(y, x) - h(x, x) \right] + \sum_{i=1}^n \beta_i(y) \text{Re}(p_i, \eta(y, x)) \]

\[ = \beta_0(y) \left[ \inf_{w \in \mathcal{C}(y)} \text{Re}(M(x) - w, \eta(y, x)) + h(y, x) - h(x, x) \right] + \sum_{i=1}^n \beta_i(y) \text{Re}(p_i, \eta(y, x)) \]

\[ \geq \sum_{i=1}^n \beta_i(y) \left[ \inf_{w \in \mathcal{C}(y)} \text{Re}(M(x) - w, \eta(y, x)) + h(y, x) - h(x, x) \right] + \sum_{i=1}^n \beta_i(y) \text{Re}(p_i, \eta(y, x)) > 0 \]

which is a contradiction.

Thus we have \( \min \varphi(x, y) \leq 0 \) for each \( A \in F(X) \)

and each \( y \in \text{co}(A) \) if \( x \in F(X) \).

3) Suppose that \( A \in F(X) \), \( x, y \in \text{co}(A) \) and \( \{y_a\}_{a \in \Gamma} \) is a net in \( X \) converging to \( y \) with

\[ \beta_0(y) \left[ \inf_{w \in \mathcal{C}(y)} \text{Re}(M(x) - w, \eta(y_a, x)) + h(y_a, x) - h(x, x) \right] \leq 0 \]

for all \( \alpha \in \Gamma \), \( t \in [0, 1] \).

**Case 1.** \( \beta_0(y_a) = 0 \).

Note that \( \beta_0(y_a) \geq 0 \) for each \( \alpha \in \Gamma \) and \( \beta_0(y_a) \to 0 \). Since \( T(X) \) is strongly bounded and \( \{y_a\}_{a \in \Gamma} \) is a bounded net, therefore

\[ \limsup_{a} \beta_0(y_a) \left[ \min_{w \in \mathcal{C}(y_a)} \text{Re}(M(x) - w, \eta(y_a, x)) + h(y_a, x) - h(x, x) \right] = 0. \]

Also

\[ \beta_0(y) \left[ \min_{w \in \mathcal{C}(y)} \text{Re}(M(x) - w, \eta(y, x)) + h(y, x) - h(x, x) \right] = 0. \]
Thus
\[
\limsup_a \left[ \beta_h(y_a) \left( \min_{w \in T(y_a)} \Re \{ M(x) - w, \eta(y_a, x) \} + h(y_a, x) - h(x, x) \right) + \sum_{i=1}^{n} \beta(y) \Re \{ p_i, \eta(y, x) \} \right]
\]
\[
= \sum_{i=1}^{n} \beta(y) \Re \{ p_i, \eta(y, x) \} \quad \text{by (1)}
\]
\[
= \beta_h(y) \left( \min_{w \in T(y)} \Re \{ M(x) - w, \eta(y, x) \} + h(y, x) - h(x, x) \right) + \sum_{i=1}^{n} \beta(y) \Re \{ p_i, \eta(y, x) \}.
\]
When \( t = 1 \), we have \( \varphi(x, y_a) \leq 0 \) for all \( \alpha \in \Gamma \) i.e.,
\[
\beta_h(y_a) \left( \min_{w \in T(y_a)} \Re \{ M(x) - w, \eta(y_a, x) \} + h(y_a, x) - h(x, x) \right) + \sum_{i=1}^{n} \beta(y_a) \Re \{ p_i, \eta(y_a, x) \} \leq 0
\] (3)
for all \( \alpha \in \Gamma \).

Therefore by (3), we have
\[
\limsup_a \left[ \beta_h(y_a) \left( \min_{w \in T(y_a)} \Re \{ M(x) - w, \eta(y_a, x) \} + h(y_a, x) - h(x, x) \right) + \sum_{i=1}^{n} \beta(y_a) \Re \{ p_i, \eta(y_a, x) \} \right]
\]
\[
\leq \limsup_a \left[ \beta_h(y_a) \min_{w \in T(y_a)} \Re \{ M(x) - w, \eta(y_a, x) \} + h(y_a, x) - h(x, x) + \sum_{i=1}^{n} \beta(y_a) \Re \{ p_i, \eta(y_a, x) \} \right] \leq 0.
\]
Thus
\[
\limsup_a \left[ \beta_h(y_a) \left( \min_{w \in T(y_a)} \Re \{ M(x) - w, \eta(y_a, x) \} + h(y_a, x) - h(x, x) \right) + \sum_{i=1}^{n} \beta(y_a) \Re \{ p_i, \eta(y_a, x) \} \right] \leq 0. \quad (4)
\]
Hence by (2) and (4), we have \( \varphi(x, y) \leq 0 \).

**Case 2.** \( \beta_h(y) > 0 \).

Since \( \beta_h(y_a) \to \beta_h(y) \), there exists \( \lambda \in \Gamma \) such that \( \beta_h(y_a) > 0 \) for all \( \alpha \geq \lambda \). When \( t = 0 \), we have
\[
\beta_h(y_a) \left( \inf_{w \in T(y_a)} \Re \{ M(y) - w, \eta(y_a, y) \} + h(y_a, y) - h(y, y) \right) + \sum_{i=1}^{n} \beta(y_a) \Re \{ p_i, \eta(y_a, y) \} \leq 0
\]
for all \( \alpha \in \Gamma \).

Thus
\[
\limsup_a \left[ \beta_h(y_a) \left( \inf_{w \in T(y_a)} \Re \{ M(y) - w, \eta(y_a, y) \} + h(y_a, y) - h(y, y) \right) + \sum_{i=1}^{n} \beta(y_a) \Re \{ p_i, \eta(y_a, y) \} \right] \leq 0. \quad (5)
\]
Hence
\[
\limsup_a \left[ \beta_h(y_a) \left( \inf_{w \in T(y_a)} \Re \{ M(y) - w, \eta(y_a, y) \} + h(y_a, y) - h(y, y) \right) + \sum_{i=1}^{n} \beta(y_a) \Re \{ p_i, \eta(y_a, y) \} \right]
\]
\[
\leq \limsup_a \left[ \beta_h(y_a) \left( \inf_{w \in T(y_a)} \Re \{ M(y) - w, \eta(y_a, y) \} + h(y_a, y) - h(y, y) \right) + \sum_{i=1}^{n} \beta(y_a) \Re \{ p_i, \eta(y_a, y) \} \right] \leq 0 \quad \text{(by (5)).}
\]
Since
\[
\liminf_a \left[ \sum_{i=1}^{n} \beta(y_a) \Re \{ p_i, \eta(y_a, y) \} \right] = 0,
\]
we have
\[ \limsup_{\alpha} \left[ \beta_0(y_\alpha) \left( \min_{w \in T(y_\alpha)} \text{Re}(M(y) - w, \eta(y_\alpha, y)) + h(y_\alpha, y) - h(y, y) \right) \right] \leq 0. \] (6)

Since \( \beta_0(y_\alpha) > 0 \) for all \( \alpha > \lambda \). It follows that
\[
\beta_0(y_\alpha) \limsup_{\alpha} \left[ \min_{w \in T(y_\alpha)} \text{Re}(M(y) - w, \eta(y_\alpha, y)) + h(y_\alpha, y) - h(y, y) \right]
= \limsup_{\alpha} \left[ \beta_0(y_\alpha) \left( \min_{w \in T(y_\alpha)} \text{Re}(M(y) - w, \eta(y_\alpha, y)) + h(y_\alpha, y) - h(y, y) \right) \right].
\] (7)

Since \( \beta_0(y) > 0 \) by (6) and (7), we have
\[
\limsup_{\alpha} \left[ \min_{w \in T(y_\alpha)} \text{Re}(M(y) - w, \eta(y_\alpha, y)) + h(y_\alpha, y) - h(y, y) \right] \leq 0.
\]

Since \( T \) is \((\eta, h)\)-quasi pseudomonotone operator, we have
\[
\limsup_{\alpha} \left[ \min_{w \in T(y_\alpha)} \text{Re}(M(x) - w, \eta(y_\alpha, x)) + h(y_\alpha, x) - h(x, x) \right] \geq \min_{w \in T(y)} \text{Re}(M(x) - w, \eta(y, x)) + h(y, x) - h(x, x) \text{ for all } x \in X.
\]

Since \( \beta_0(y) > 0 \), we have
\[
\beta_0(y) \left[ \limsup_{\alpha} \left( \min_{w \in T(y_\alpha)} \text{Re}(M(x) - w, \eta(y_\alpha, x)) + h(y_\alpha, x) - h(x, x) \right) \right] \geq \beta_0(y) \left[ \min_{w \in T(y)} \text{Re}(M(x) - w, \eta(y, x)) + h(y, x) - h(x, x) \right].
\]

Thus
\[
\beta_0(y) \left[ \limsup_{\alpha} \left( \min_{w \in T(y_\alpha)} \text{Re}(M(x) - w, \eta(y_\alpha, x)) + h(y_\alpha, x) - h(x, x) \right) \right] + \sum_{i=1}^{\alpha} \beta_i(y) \text{Re}(p_i, \eta(y, x))
\geq \beta_0(y) \left[ \min_{w \in T(y)} \text{Re}(M(x) - w, \eta(y, x)) + h(y, x) - h(x, x) \right] + \sum_{i=1}^{\alpha} \beta_i(y) \text{Re}(p_i, \eta(y, x)).
\] (8)

When \( \tau = 1 \), we have \( \varphi(x, y_\alpha) \leq 0 \) for all \( \alpha \in \Gamma \), i.e.,
\[
\beta_0(y_\alpha) \left[ \min_{w \in T(y_\alpha)} \text{Re}(M(x) - w, \eta(y_\alpha, x)) + h(y_\alpha, x) - h(x, x) \right] + \sum_{i=1}^{\alpha} \beta_i(y_\alpha) \text{Re}(p_i, \eta(y_\alpha, x)) \leq 0
\]
for all \( \alpha \in \Gamma \).

Thus
\[
0 \geq \limsup_{\alpha} \left[ \beta_0(y_\alpha) \left( \min_{w \in T(y_\alpha)} \text{Re}(M(x) - w, \eta(y_\alpha, x)) + h(y_\alpha, x) - h(x, x) + \sum_{i=1}^{\alpha} \beta_i(y_\alpha) \text{Re}(p_i, \eta(y_\alpha, x)) \right) \right]
\geq \limsup_{\alpha} \left[ \beta_0(y_\alpha) \left( \min_{w \in T(y_\alpha)} \text{Re}(M(x) - w, \eta(y_\alpha, x)) + h(y_\alpha, x) - h(x, x) \right) + \frac{\text{Re}(p_i, \eta(y_\alpha, x))}{\sum_{i=1}^{\alpha} \beta_i(y_\alpha)} \right]
= \beta_0(y) \left[ \limsup_{\alpha} \left( \min_{w \in T(y_\alpha)} \text{Re}(M(x) - w, \eta(y_\alpha, x)) + h(y_\alpha, x) - h(x, x) \right) \right] + \sum_{i=1}^{\alpha} \beta_i(y) \text{Re}(p_i, \eta(y, x))
\geq \beta_0(y) \left[ \min_{w \in T(y)} \text{Re}(M(x) - w, \eta(y, x)) + h(y, x) - h(x, x) \right] + \sum_{i=1}^{\alpha} \beta_i(y) \text{Re}(p_i, \eta(y, x)) \text{ (by (8)).}
\]
Hence, we have \( \varphi(x,y) \leq 0 \).

Since \( X \) is a compact subset of the Hausdorff topological vector space \( E \), it is also closed. Now if we take \( K = X \), then for any \( x_0 \in K = X \), we have

\[ \varphi(x_0, y) > 0 \text{ for all } y \in X \setminus K (= X \setminus X = \emptyset). \]

Thus \( \varphi \) satisfies all the hypothesis of Theorem 1. Hence by Theorem 1, there exists \( \hat{y} \in K \) such that

\[ \varphi(x, \hat{y}) \leq 0 \text{ for all } x \in X, \]

\[ \beta_0(\hat{y}) \inf_{w \in T(\hat{y})} \Re \{ M(\hat{y}) - w, \eta(\hat{y}, x) \} + h(\hat{y}, x) - h(x, x) \]

\[ + \sum_{j=1}^{n} \beta_j(\hat{y}) \Re \{ p_j, \eta(\hat{y}, x) \} \leq 0 \text{ for all } x \in X. \]

Thus, \( \beta_0(\hat{y}) \inf_{w \in T(\hat{y})} \Re \{ M(\hat{y}) - w, \eta(\hat{y}, x) \} + h(\hat{y}, x) - h(x, x) \leq 0 \)

for all \( x \in S(\hat{y}) \).

From Step 1, we have \( \hat{y} \in S(\hat{y}) \) and

\[ \inf_{w \in T(\hat{y})} \Re \{ M(\hat{y}) - w, \eta(\hat{y}, x) \} + h(\hat{y}, x) - h(x, x) \leq 0 \]

for all \( x \in S(\hat{y}) \).

Since \( S(\hat{y}) \) is a convex subset of \( X \) and \( M \) is linear, continuous along line segments in \( X \), by Lemma 4 we have

\[ \inf_{w \in T(\hat{y})} \Re \{ M(\hat{y}) - w, \eta(\hat{y}, x) \} + h(\hat{y}, x) - h(x, x) \leq 0 \]

for all \( x \in S(\hat{y}) \).

**Step 2.**

\[ \inf_{w \in T(\hat{y})} \Re \{ M(\hat{y}) - w, \eta(\hat{y}, x) \} + h(\hat{y}, x) - h(x, x) \leq 0 \]

for all \( x \in S(\hat{y}) \).

**Step 3.** There exists \( \hat{w} \in T(\hat{y}) \) with

\[ \Re \{ M(\hat{y}) - \hat{w}, \eta(\hat{y}, x) \} + h(\hat{y}, x) - h(x, x) \leq 0 \]

for all \( x \in S(\hat{y}) \).

By Step 2 and applying Theorem 2 as proved in Step 3 of Theorem 1 in [11], we can show that there exists \( \hat{w} \in T(\hat{y}) \) such that

\[ \Re \{ M(\hat{y}) - \hat{w}, \eta(\hat{y}, x) \} + h(\hat{y}, x) - h(x, x) \leq 0 \]

for all \( x \in S(\hat{y}) \).

We observe from the above proof that the requirement that \( E \) be locally convex is needed when and only when the separation theorem is applied to the case \( y \notin S(\hat{y}) \). Thus if \( S : X \to 2^X \) is the constant map \( S(x) = X \) for all \( x \in X \), \( E \) is not required to be locally convex.

Finally, if \( T = 0 \), in order to show that for each \( x \in X \), \( y \to \varphi(x,y) \) is lower semicontinuous, Lemma 3 is no longer needed and the weaker continuity assumption as \( \langle \cdot, \cdot \rangle \) that for each \( f \in E \), the map \( x \to \langle f, x \rangle \) is continuous on \( X \) is sufficient. This completes the proof.

**REFERENCES**


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