Abstract—While random linear network coding is a powerful tool for disseminating information in communication networks, it is highly susceptible to errors caused by various sources. Due to error propagation, errors greatly deteriorate the throughput of network coding and seriously undermine both reliability and security of data. Hence error control for network coding is vital. Recently, constant-dimension codes (CDCs), especially Kötter–Kschischang (KK) codes, have been proposed for error control in random linear network coding. KK codes can also be constructed from Gabidulin codes, an important class of rank metric codes. Rank metric decoders have been recently proposed for both Gabidulin and KK codes, but they have high computational complexities. Furthermore, it is not clear whether such decoders are feasible and suitable for hardware implementations. In this paper, we reduce the complexities of rank metric decoders and propose novel decoder architectures for both codes. The synthesis results of our decoder architectures for Gabidulin and KK codes with limited error-correcting capabilities over small fields show that our architectures not only are affordable, but also achieve high throughput.

Index Terms—Constant-dimension codes (CDCs), Decoding, Error correction coding, Gabidulin codes, Galois fields, Integrated circuits, Kötter–Kschischang codes, Network coding, Rank metric codes, Subspace codes.

I. INTRODUCTION

Network coding [1] is a promising candidate for a new unifying design paradigm for communication networks [2]. The basic idea of network coding is simple yet powerful. Traditionally, an intermediate node in a network simply stores and forwards each packet it receives. Instead, in network coding, an intermediate node combines several incoming packets into one or several outgoing packets and forwards. This simple change has a significant ramification: the capacity of a multicast (the minimum cut between the source node and any destination node) can be achieved by network coding, but not by traditional networks without network coding [1]. In addition to the throughput improvement, network coding is robust to nonergodic network failures and ergodic packet erasures, can be done in a distributed manner with low complexity, and improves mobility and management of wireless networks. Furthermore, implementations of network coding in both wired and wireless environments demonstrate its practical benefits (see, for example, [3], [4]). Due to these advantages, network coding is already used or considered in gossip-based data dissemination [5], 802.11 wireless ad hoc networking [4], peer-to-peer networks [3], and mobile ad hoc networks (MANETs) [6].

Random linear network coding (RLNC) [7] is arguably the most important class of network coding. Instead of using network coding operations centrally designed to achieve the maximum throughput based on the network topology, RLNC treats all packets as vectors over some finite field and forms an outgoing packet by linearly combining incoming packets using random coefficients. Due to its random linear operations, RLNC not only achieves network capacity in a distributed manner, but also provides robustness to changing network conditions. Unfortunately, it is highly susceptible to errors caused by various reasons, such as noise, malicious or malfunctioning nodes, or insufficient min-cut [8]. Since linearly combining packets results in error propagation, errors greatly deteriorate the throughput of network coding and seriously undermine both reliability and security of data. Thus, error control for random linear network coding is critical and has received growing attention recently (see, for example, [8], [9]).

Error control schemes proposed for RLNC assume two types of transmission models. The schemes of the first type depend on and take advantage of the underlying network topology or the particular linear network coding operations performed at various network nodes. The schemes of the second type [8], [9] assume that the transmitter and receiver have no knowledge of such channel transfer characteristics. The two transmission models are referred to as coherent and noncoherent network coding, respectively.

It has been recently shown [8] that an error control code for noncoherent network coding, called a subspace code, is a set of subspaces (of a vector space), and information is encoded in the choice of a subspace as a codeword; a set of packets that generate the chosen subspace is then transmitted [8]. In contrast, a code in classical coding theory is a set of vectors, and information is encoded in the choice of a vector as a codeword. A subspace code is called a constant-dimension code (CDCs) if its subspaces are of the same dimension. CDCs
are of particular interest since they lead to simplified network protocols due to the fixed dimension. A class of asymptotically optimal CDCs have been proposed in [8], and they are referred to as the KK codes. A decoding algorithm based on interpolation for bivariate linearized polynomials is also proposed in [8] for the KK codes. It was shown that KK codes correspond to lifting [9] of Gabidulin codes [10], [11], a class of optimal rank metric codes. Gabidulin codes are also called maximum rank distance (MRD) codes, since they achieve the Singleton bound of rank metric [10], as RS codes achieve the Singleton bound of Hamming distance. Gabidulin codes have found applications in storage systems [11], space-time coding [12] as well as cryptography [13], and various decoding algorithms have been proposed for Gabidulin codes [14]–[18]. Due to the connection between Gabidulin and KK codes, the decoding of KK codes can be viewed as generalized decoding of Gabidulin codes, which involves deviations as well as errors and erasures [9]. Gabidulin codes are significant in themselves. For coherent network coding, the error correction capability of error control schemes is succinctly described by the rank metric [19], thus error control codes for coherent network coding are essentially rank metric codes.

The benefits of network coding above come at the price of additional operations needed at the source nodes for encoding, at the intermediate nodes for linear combining, and at the destination node(s) for decoding. In practice, the decoding complexities at destination nodes are much greater than the encoding and combining complexities. The decoding complexities of RLNC are particularly high when large underlying fields are assumed and when additional mechanisms such as error control are accounted for. Clearly, the decoding complexities of RLNC are critical to both software and hardware implementations. Furthermore, area/power overheads of their VLSI implementations are important factors in system design. Unfortunately, prior research efforts have mostly focused on theoretical aspects of network coding, and complexity reduction and efficient VLSI implementation of network coding decoders have not been sufficiently investigated so far. For example, although the decoding complexities of Gabidulin and KK codes were analyzed in [20], [21] and [8], [9], respectively, but they do not reflect the impact of the size of the underlying finite fields. To ensure high probability of success for RLNC, a field of size $2^8$ or $2^{16}$ is desired [22]. However, these large field sizes will increase decoding complexities and hence complicate hardware implementations. Finally, to the best of our knowledge, hardware architectures for these decoders have not been investigated in the open literature.

In this paper, we fill this significant gap by investigating complexity reductions and efficient hardware implementations for decoders in RLNC with error control. This effort is significant to the evaluation and design of network coding for several reasons. First, our results evaluate the complexities of decoders for RLNC as well as the area, power, and throughput of their hardware implementations, thereby helping to determine the feasibility and suitability of network coding for various applications. Second, our research results provide instrumental guidelines to the design of network coding from the perspective of complexity as well hardware implementation. Third, our research results lead to efficient decoders and hence reduce the area and power overheads of network coding.

In this paper, we focus on the generalized Gabidulin decoding algorithm [9] for the KK codes and decoding algorithm in [16] for Gabidulin codes for two reasons. First, compared with the decoding algorithm in [8], the generalized Gabidulin decoding [9] has a smaller complexity, especially for high-rate KK codes [9]. Second, components in the errors-only Gabidulin decoding algorithm in [16] can be easily adapted in the generalized Gabidulin decoding of KK codes. Thus, among the decoding algorithms for Gabidulin codes, we focus on the decoding algorithm in [16]. Finally, although we focus on random linear network coding with error control in this paper, our results can be easily applied to random linear network coding without error control. For random linear network coding without error control, the decoding complexity is primarily due to inverting of the global coding matrix via Gauss-Jordan elimination, which is considered in this paper.

One of our main contributions includes several algorithmic reformulations that reduce the computational complexities of decoders for both Gabidulin and KK codes. Our complexity-saving algorithmic reformulations are:

- We first adopt normal basis representations for all finite field elements, and then significantly reduce the complexity of bit-parallel normal basis multipliers by using our common subexpression elimination (CSE) algorithm.
- The decoding algorithms of both Gabidulin and KK codes involve solving key equations. Based on approaches in [23], [24], we reformulate the Berlekamp–Massey algorithm (BMA), and propose an inversionless BMA. Our inversionless BMA does not require any finite field inversions, and leads to reduced complexities as well as efficient architectures;
- The decoding algorithm of KK codes requires that the input be arranged in a row reduced echelon (RRE) form. We define a more generalized form called $n$-RRE form, and show that it is sufficient if the input is in the $n$-RRE form. This change not only reduces the complexity of reformulating the input, but also enables parallel processing of decoding KK codes based on Cartesian products.

The other main contribution of this paper is efficient decoder architectures for both Gabidulin and KK codes. Aiming to reduce the area and to improve the regularity of our decoder architectures, we have also reformulated other steps in the decoding algorithm. To evaluate the performance of our decoder architectures for Gabidulin and KK codes, we implement our decoder architecture for two rate-$1/2$ Gabidulin codes, an $(8, 4)$ code and a $(16, 8)$ code, and their corresponding KK codes. Our KK decoders can be used in network coding with various packet lengths by Cartesian product [9]. The synthesis results of our decoders show that our decoder architectures for Gabidulin and KK codes over small fields with limited error-correcting capabilities not only are affordable, but also achieve high throughput. Our decoder architectures and implementation results are novel to the best of our knowledge.

The rest of the paper is organized as follows. After briefly reviewing the background in Section III, we present our complexity-saving algorithmic reformulations and efficient decoder archi-
tectures in Sections III and IV respectively. In Section V the proposed architectures are implemented in Verilog and synthesized for area/performance evaluation. The conclusion is given in Section VI.

II. PRELIMINARIES

A. Notation

Let \( q \) denote a power of prime and \( \mathbb{F}_{q^m} \) denote a finite field of order \( q^m \). We use \( \mathbb{F}_{q}^{n \times m} \) to denote the set of all \( n \times m \) matrices over \( \mathbb{F}_{q} \) and use \( I_n \) to denote an \( n \times n \) identity matrix. For a set \( U \subseteq \{0, 1, \ldots, n-1\} \) and \( \ell^C \) denotes the complement subset \( \{0, 1, \ldots, n-1\} \setminus U \) and \( I_q \) denotes the columns of \( I_n \) in \( U \). In this paper, all vectors and matrices are in bold face.

The rank weight of a vector over \( \mathbb{F}_{q} \) is defined as the \textit{maximal} number of its coordinates that are linearly independent over the base field \( \mathbb{F}_{q} \). Rank metric is the weight of vector difference \( \mathbb{F}_{q}^{n \times m} \). For a column vector \( X \in \mathbb{F}_{q}^{n \times m} \), we can expand each of its component into a row vector over the base field \( \mathbb{F}_{q} \). Such a row expansion leads to an \( n \times m \) matrix over \( \mathbb{F}_{q} \). In this paper, we slightly abuse the notation so that \( X \) can represent a vector in \( \mathbb{F}_{q}^n \) or a matrix in \( \mathbb{F}_{q}^{n \times m} \), although the meaning is usually clear given the context.

Given a matrix \( X \), its row space, rank, and reduced row echelon (RRE) form \( [X] \) are denoted by \( \langle X \rangle \), rank \( X \), and RRE(\( X \)), respectively. For a subspace \( \langle X \rangle \), its dimension is denoted by \( \dim(X) \) and rank \( X = \dim(X) \). The rank distance of two vectors \( X \) and \( Y \) in \( \mathbb{F}_{q}^n \) is defined as \( d_R(X, Y) = \dim(X) + \dim(Y) - 2 \cdot \dim(X \cap Y) \).

Let the received matrix be \( Y = [A | y] \), where \( A \in \mathbb{F}_{q}^{N \times n} \) and \( y \in \mathbb{F}_{q}^{N \times m} \). Note that we always assume the received matrix is full-rank \( 9 \). The row and column rank deficiencies of \( A \) are \( \delta = N - \text{rank} A \) and \( \mu = n - \text{rank} A \), respectively. Then the RRE form of \( Y \) is expanded into \( Y = \begin{bmatrix} I_{\delta} & 0 \\ 0 & I_{\mu} \end{bmatrix} [X] \), where \( [X] \) is the RRE(\( Y \)). The tuple \( (r, L, E) \) is called a reduction of \( Y \) \( 9 \).

Polynomials over \( \mathbb{F}_{q^m} \) with only non-zero terms with degrees \( q^i \) are called \textit{linearized polynomials} \( 27, 28 \). For convenience, let \( [i] \) denote \( q^i \). In a linearized polynomial, the greatest value of \( i \) of non-zero terms is defined as its \( q \)-degree. The symbolic product of two linearized polynomials \( a(x) \) and \( b(x) \), denoted by \( \otimes \) (that is, \( a(x) \otimes b(x) = a(b(x)) \)), is also a linearized polynomial. The \( q \)-reverse of a linearized polynomial \( f(x) = \sum_{i=0}^{p} f_i x^i \) is given by the polynomial \( g(x) = \sum_{i=0}^{p} g_i x^i \), where \( g_i = f_{p-i} \) for \( i = 0, 1, \ldots, p \) and \( p \) is the \( q \)-degree of \( f(x) \). For a set \( \alpha \) of field elements, we use \text{minpoly}(\alpha) \) to denote its minimal linearized polynomial, which is the monic linearized polynomial of least degree such that all the elements of \( \alpha \) are its roots.

B. Gabidulin Codes and Their Decoding

A Gabidulin code \( 10 \) is a linear \( (n, k) \) code over \( \mathbb{F}_{q^m} \), whose parity-check matrix has a form as

\[
H = \begin{bmatrix}
    h_0^{[0]} & h_1^{[0]} & \cdots & h_{n-1}^{[0]} \\
h_0^{[1]} & h_1^{[1]} & \cdots & h_{n-1}^{[1]} \\
    \vdots & \vdots & \ddots & \vdots \\
h_0^{[n-k-1]} & h_1^{[n-k-1]} & \cdots & h_{n-1}^{[n-k-1]}
\end{bmatrix}
\] (1)

where \( h_0, h_1, \ldots, h_{n-1} \in \mathbb{F}_{q^m} \) are linearly independent over \( \mathbb{F}_{q} \). Since \( \mathbb{F}_{q^m} \) is an \( m \)-dimensional vector space over \( \mathbb{F}_{q} \), it is necessary that \( n \leq m \). Conversely, a generator matrix of this Gabidulin code is given by

\[
G = \begin{bmatrix}
g_0^{[0]} & g_1^{[0]} & \cdots & g_{n-1}^{[0]} \\
g_0^{[1]} & g_1^{[1]} & \cdots & g_{n-1}^{[1]} \\
    \vdots & \vdots & \ddots & \vdots \\
g_0^{[k-1]} & g_1^{[k-1]} & \cdots & g_{n-1}^{[k-1]}
\end{bmatrix}
\] (2)

where \( \sum_{i=0}^{n-1} g_i^{[k-1]} h^i = 0 \) for \( s = 0, 1, \ldots, n-1 \). The minimum rank distance of a Gabidulin code \( d = n - k + 1 \) and all Gabidulin codes are MRD codes.

The decoding process of Gabidulin codes includes five major steps: syndrome computation, key equation solver, finding the root space, finding the error locators by Gabidulin’s algorithm \( 10 \), and finding error locations. The data flow of Gabidulin decoding is shown in Figure 1.

Most key equation solvers are based on a modified Berlekamp–Massey algorithm (BMA) \( 16 \) or a modified Welch–Berlekamp algorithm (WBA) \( 27 \). Other decoding algorithms include \( 29, 30 \). The decoding algorithms in \( 31 \) are for cryptography applications and have very high complexity. In this paper, we focus on the modified BMA.

As in RS decoding, we can compute syndromes for Gabidulin codes as \( S = (S_0, S_1, \ldots, S_{d-2}) \) \( H(r) \) for any received vector \( r \). Then the syndrome polynomial \( S(x) = \sum_{j=0}^{d-2} S_j x^j \) can be used to solve the key equation

\[
\sigma(x) \otimes S(x) \equiv \omega(x) \mod x^{[d-1]} \] (3)

for the error span polynomial \( \sigma(x) \), using the BMA. Up to \( t = [(d - 1)/2] \) error values \( E_j \)’s can be obtained by finding a basis \( E_0, E_1, \ldots \) for the root space of \( \sigma(x) \) using the methods in \( 22, 33 \). Then we can find the error locators \( X_j \)’s corresponding to \( E_j \)’s by solving a system of equations

\[
S_l = \sum_{j=0}^{r-1} X_j^{[l]} E_j, \quad l = 0, 1, \ldots, d - 2 \] (4)

where \( \tau \) is the number of errors. Gabidulin’s algorithm \( 10 \) in Algorithm 1 can be used to solve \( 4 \). Finally, the error locations \( L_j \)’s are obtained from \( X_j \)’s by solving

\[
X_j = \sum_{i=0}^{n-1} L_j[i] h_i, \quad j = 0, 1, \ldots, \tau - 1. \] (5)

Algorithm 1 (Gabidulin’s Algorithm \( 10 \)).

\[
\text{Input: } S_0, S_1, \ldots, S_{d-2} \text{ and } E_0, E_1, \ldots, E_{\tau-1} \\
\text{Output: } X_0, X_1, \ldots, X_{\tau-1}
\]
Fig. 1. Data flow of Gabidulin decoding

1. Compute $\tau \times \tau$ matrices $A$ and $Q$ as

   $$A_{i,j} = \begin{cases} E_j & i = 0 \\ 0 & i \neq 0, j < i \\ A_{i-1,j} - A_{i-1,i-1} \left( \frac{A_{i-1,j}}{A_{i-1,i-1}} \right) & i \neq 0, j \geq i \end{cases}$$

   $$Q_{i,j} = \begin{cases} S_j & i = 0 \\ Q_{i-1,j} - A_{i-1,i-1} \left( \frac{Q_{i-1,j}}{A_{i-1,i-1}} \right) & i \neq 0, j \geq i \end{cases}$$

2. Compute $X_i$ recursively as $X_{\tau-1} = Q_{\tau-1,0}/A_{\tau-1,\tau-1}$ and $X_i = (Q_{i,0} - \sum_{j=i+1}^{\tau-1} A_{i,j} X_j)/A_{i,i}$, for $i = \tau - 2, \tau - 3, \ldots, 0$.

In total, the decoding complexity of Gabidulin codes is roughly $O(n^2(1 - R))$ operations over $\mathbb{F}_{q^m}$ [20], where $R$ is the code rate or $O(dm^3)$ operations over $\mathbb{F}_q$ [21]. Note that all polynomials involved in the decoding process are linearized polynomials.

Gabidulin codes are often viewed as the counterpart in rank metric codes of the well-known Reed–Solomon codes. As shown in Table I, an analogy between Reed–Solomon and Gabidulin codes can be established in many aspects. Such an analogy helps us understand the decoding of Gabidulin codes, and in some cases allows us to adapt innovations proposed for Reed–Solomon codes to Gabidulin code.

### Table I

**Analogies between Reed–Solomon and Gabidulin Codes**

<table>
<thead>
<tr>
<th>Metric</th>
<th>Reed–Solomon</th>
<th>Gabidulin</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ring of Polynomials</td>
<td>Linearized</td>
<td>Linearized Polynomials</td>
</tr>
<tr>
<td>Key Operation</td>
<td>Polynomial Multiplication</td>
<td>Symbolic Product</td>
</tr>
<tr>
<td>Parity Check Matrix</td>
<td>$[g_q]$</td>
<td>$[g_q]^{[\mu]}$</td>
</tr>
<tr>
<td>Key Equation Solver</td>
<td>BMA</td>
<td>Modified BMA</td>
</tr>
<tr>
<td>Error Value Solver</td>
<td>Forney’s Formula</td>
<td>Gabidulin’s Algorithm</td>
</tr>
</tbody>
</table>

C. KK Codes and Their Decoding

By the lifting operation [9], KK codes can be constructed from Gabidulin codes. Lifting can also be seen as a generalization of the standard approach to random linear network coding [7], which transmits matrices in the form $X = [I \mid x]$, where $X \in \mathbb{F}_q^{n\times M}$, $x \in \mathbb{F}_q^{n\times m}$, and $m = M - n$. Hence by adding the constraint that $x$ is the row expansion of codewords from a Gabidulin code $C$ over $\mathbb{F}_{q^m}$, error control is enabled in network coding.

In practice, the packet length could be very long. To accommodate long packets based on the KK codes, very large $m$ and $n$ are needed, which results in prohibitively high complexity due to the huge field size of $\mathbb{F}_{q^m}$. A low-complexity approach in [9] suggested that instead of using a single long Gabidulin code, a Cartesian product of many short Gabidulin codes with the same distance can be used to construct constant-dimension codes for long packets via the lifting operation.

In the decoding algorithm of [9], the matrix $Y$ is first turned into an RRE form. Based on an RRE form of $Y$, a three-tuple $(r, \tilde{L}, \tilde{E})$, referred to as a reduction of $Y$, is obtained. It was proved [9] that $d_S((X), (Y)) = 2 \text{ rank}[0_{r-x}] - \mu - \delta$, where $\mu = n - \text{rank} \tilde{L}$ and $\delta = N - \text{rank} \tilde{L}$. Now the decoding problem to minimize the subspace distance becomes a problem to minimize the rank distance.

The generalized rank decoding [9] finds an error word $\hat{e} = \arg \min_{e \in e - c} \text{ rank}[0_{L} \in \mathbb{E}]$. The error word $\hat{e}$ is expanded as a summation of products of column and row vectors [9] such that $\hat{e} = \sum_{j=0}^{\tau-1} L_j E_j$. Each term $L_j E_j$ is called either an erasure, if $L_j$ is known, or a deviation, if $E_j$ is known, or an error, if neither $L_j$ nor $E_j$ is known. In this general decoding problem, $L$ has $\mu$ columns from $\tilde{L}$ and $E$ has $\delta$ rows from $\tilde{E}$. Given a Gabidulin code of minimum distance $d$, the corresponding KK code is able to correct $\epsilon$ errors, $\mu$ erasures, and $\delta$ deviations as long as $2\epsilon + \mu + \delta < d$.

Algorithm 2 was proposed [9] for generalized decoding of the KK codes, and its data flow is shown in Figure 2. It requires $O(dm)$ operations in $\mathbb{F}_{q^m}$.

Algorithm 2 (General Rank Decoding [9]).

**Input:** received tuple $(r, \tilde{L}, \tilde{E})$

**Output:** error word $\hat{e}$

1. Compute $S = Hr$, $\tilde{X} = \tilde{L}^T h$, $\lambda_U(x) = \minpoly(\tilde{X})$, $\sigma_D(x) = \minpoly(\tilde{E})$, and $S_{DU}(x) = \sigma_D(x) \otimes S(x) \otimes \zeta_U(x)$, where $\zeta_U(x)$ is the $q$-reverse of $\lambda_U(x)$.

2. Compute the error span polynomial:

   a. Use the modified BMA [10] to solve the key equation $\sigma_F(x) \otimes S_{DU}(x) \equiv \omega(x) \bmod x^{[d-1]}$ such that $\deg \omega(x) < [\tau]$ where $\tau = \epsilon + \mu + \delta$.
   b. Compute $S_{FD}(x) = \sigma_F(x) \otimes \sigma_D(x) \otimes S(x)$.
   c. Use Gabidulin’s algorithm [10] to find $\beta$ that solves $S_{FD,I} = \sum_{j=0}^{\mu-1} X_j^{[\beta]} E_j$, $l = d - 2, d - 3, \ldots, d - 1 - \mu$.
   d. Compute $\sigma_U(x) = \minpoly(\beta)$ followed by $\sigma(x) = \sigma_U(x) \otimes \sigma_F(x) \otimes \sigma_D(x)$.

3. Find a basis $E$ for the root space of $\sigma(x)$.

4. Find the error locations:

   a. Solve $S_l = \sum_{j=0}^{\tau-1} X_j^{[\beta]} E_j$, $l = 0, 1, \ldots, d - 2$ using Gabidulin’s algorithm [10] to find the error locators $X_0, X_1, \ldots, X_{\tau-1} \in \mathbb{F}_{q^m}$.


b) Compute the error locations $L_j$’s by solving (5).
c) Compute the error word $\hat{e} = \sum_{j=1}^{\tau} L_j E_j$, where each $E_j$ is the row expansion of $E_j$.

III. COMPUTATIONAL COMPLEXITY REDUCTION

In general, RLNC is carried out over $\mathbb{F}_q$, where $q$ is any prime power. That is, packets are treated as vectors over $\mathbb{F}_q$.

Since our investigation of computational complexities is for both software and hardware implementations of RLNC, where data are stored and transmitted in bits, we focus on RLNC over characteristic-2 fields in our work, i.e., $q$ is a power of two.

In some cases, we further assume $q = 2$, as it leads to further complexity reductions.

A. Finite Field Representation

Finite field elements can be represented by vectors using different types of bases: polynomial basis, normal basis, and dual basis [34]. In rank metric decoders, most polynomials involved are linearized polynomials, and hence their evaluations and symbolic products require computing their $[i]$th powers. Suppose a field element is represented by a vector over $\mathbb{F}_q$, with respect to a normal basis, computing $[i]$th powers ($i$ is a positive or negative integer) of the element is simply cyclic shifts of the corresponding vector by $i$ positions, which significantly reduces computational complexities. For example, the computational complexity of Algorithm [1] is primarily due to the following updates in Step [1]:

$$A_{i,j} = A_{i-1,j} - (A_{i-1,j}A_{i-1,i-1})^{-1} A_{i-1,i-1}$$
$$Q_{i,j} = Q_{i-1,j} - (Q_{i-1,j+1}A_{i-1,i-1})^{-1} A_{i-1,i-1}$$

which require divisions and computing $[−1]$th powers. With normal basis representation, $[−1]$th powers are obtained by a single cyclic shift. Furthermore, when $q = 2$, they can be computed in an inversionless form

$$A_{i,j} = A_{i-1,j} - (A_{i-1,j}A_{i-1,i-1})^{-1}$$
$$Q_{i,j} = Q_{i-1,j} - (Q_{i-1,j+1}A_{i-1,i-1})^{-1},$$

which also avoids finite field divisions or inversions. Thus using normal basis representation also reduces the complexity of Gabidulin’s algorithm.

In addition to reduce complexities of finite field arithmetic operations, normal basis representation leads to reduced complexities in the decoding of Gabidulin and KK codes for several reasons. First, it was shown that using normal basis can facilitate the computation of symbolic product [20]. Second, it was also suggested [20] that solving (5) can be trivial using normal basis. When a normal basis is used as $h_i$’s and also used for representation, the matrix $H$, whose rows are vector representations of $h_i$’s with respect to the basis $h_i$’s, becomes an identity matrix with additional all-zero columns. Hence solving (5) requires no computation. These two complexity reductions were also observed in [21]. Third, if a normal basis of $\mathbb{F}_{2^n}$ is used as $h_i$’s and $n = m$, the parity check matrix $H$ in (1) becomes a cyclic matrix. Thus syndrome computation becomes part of a cyclic convolution of $(h_0, h_1, \ldots, h_{n-1})$ and $r$, for which fast algorithms are available [35], [36]. Using fast cyclic convolution algorithms are favorable when $m$ is large.

B. Normal Basis Arithmetic Operations

We also propose finite field arithmetic operations with reduced complexities, when normal basis representation is used. When represented by vectors, the addition and subtraction of two elements are simply component-wise addition, which is straightforward to implement. For characteristic-2 fields $\mathbb{F}_{2^m}$, inverses can be obtained efficiently by a sequence of squaring and multiplying, since $\beta^{-1} = \beta^{2^m-2} = \beta^2 \beta^4 \ldots \beta^{2^m-1}$ for $\beta \in \mathbb{F}_{2^m}$. Since computing $[i]$th powers requires no computation, the complexity of inversion in turn relies on that of multiplication. Division can be implemented by a concatenation of inversion and multiplication: $\alpha \beta^{-1}$, and hence the complexity of division also depends on that of multiplication in the end.

There are serial and parallel architectures for normal basis finite field multipliers. To achieve high throughput in our decoder, we consider only parallel architectures. Most normal basis multipliers are based on the Massey–Omura (MO) architecture [34], [37]. The complexity of a serial MO normal basis multiplier, $C_N$, is defined as the number of terms $a_i b_i$ in computing a bit of the product $c = ab$, where $a_i$’s and $b_i$’s are the bits of $a$ and $b$, respectively. It has been shown [38] that a parallel MO multiplier over $\mathbb{F}_{2^m}$ needs $m^2$ AND gates and at most $m(C_N + m - 2)/2$ XOR gates. For instance, for the fields $\mathbb{F}_{2^5}$ and $\mathbb{F}_{2^6}$, their $C_N$’s are minimized to 21 and 85, respectively [34]. Using a common subexpression elimination algorithm [39], we significantly reduce the number of XOR gates while maintaining the same critical path delays (CPDs) of one AND plus five XOR gates and one AND plus seven XOR gates as direct implementations, respectively. Our results are compared to those in [34], [38] in Table II, where we also provide the prime polynomial $P(x)$ for each field.

The reduced gate count for normal basis multiplication is particularly important for hardware implementations of RLNC. This improvement is transparent to the complexity of decoders, in terms of finite field operations. When decoders for RLNC are realized in hardware, the reduced gate count for normal basis multiplication will be reflected in reduced area and power consumption.

C. Inversionless BMA

The modified BMA for rank metric codes [16] is similar to the BMA for RS codes except that polynomial multiplications are replaced by symbolic products. The modified BMA [16] requires finite field divisions, which are more complex than other arithmetic operations. Following the idea of inversionless RS decoder [23], we propose an inversionless variant in Algorithm [3].

Algorithm 3. 1BMA
Input: Syndromes $S$
Output: $A(x)$
TABLE II

COMPLEXITIES OF BIT-PARALLEL NORMAL BASIS MULTIPLIERS OVER FINITE FIELDS (FOR THESE TWO FIELDS, ALL THREE IMPLEMENTATIONS HAVE THE SAME CPD.)

<table>
<thead>
<tr>
<th></th>
<th>(P(x))</th>
<th>AND</th>
<th>XOR</th>
</tr>
</thead>
<tbody>
<tr>
<td>(F_{2^m})</td>
<td>(-)</td>
<td>(m^2)</td>
<td>(144)</td>
</tr>
<tr>
<td>(F_{4^8})</td>
<td>((\sum_{i=0}^{15} x^i) - x^6 - x^3 - x^2)</td>
<td>64</td>
<td>160</td>
</tr>
<tr>
<td>(F_{2^{16}})</td>
<td>((\sum_{i=0}^{31} x^i) - x^{14} - x^9 - x^6 - x^4)</td>
<td>256</td>
<td>1344</td>
</tr>
</tbody>
</table>

5. Initialize: \(\Lambda(0)(x) = B^0(0)(x) = x^0\), \(\Gamma(0) = 1\), and \(L = 0\).

5.2 For \(r = 0, 1, \ldots, 2t - 1\),
   a) Compute the discrepancy \(\Delta_r = \sum_{j=0}^{L} \Lambda_{j}^{(r)} S_{r-j}^{[i]}\).
   b) If \(\Delta_r = 0\), then go to (e).
   c) Modify the connection polynomial: \(\Lambda^{(r+1)}(x) = (\Gamma^{(r)})[1] \Lambda^{(r)}(x) - \Delta_r x^{1} \otimes B^{(r)}(x)\).
   d) If \(2L > r\), go to (e). Otherwise, \(L = r + 1 - L\), \(\Gamma^{(r+1)} = \Delta_r\), and \(B^{(r+1)}(x) = \Lambda^{(r)}(x)\). Go to (a).
   e) Set \(\Gamma^{(r+1)} = (\Gamma^{(r)})[1]\) and \(B^{(r+1)}(x) = x^{1} \otimes B^{(r)}(x)\).

5.3 Set \(\Lambda(x) = \Lambda^{(2t)}(x)\).

Using a similar approach as in [23], we prove that the output \(\Lambda(x)\) of Algorithm 5 is the same as \(\sigma(x)\) produced by the modified BMA, except it is scaled by a constant \(C = \prod_{i=1}^{t-1} (\Gamma^{(2i)})[1]\). However, this scaling is inconsequential since the two polynomials have the same root space.

Using normal basis, the modified BMA in [16] requires at most \([ (d-2)/2 \] \) inversions, \( (d-1)(d-2) \) multiplications, and \( (d-1)(d-2) \) additions over \( F_{q^m} \) [20]. Our inversionless version, Algorithm 5 requires at most \(3/2(d-1) \) multiplications and \( (d-1)(d-2) \) additions. Since a normal basis inversion is obtained by \(m-1 \) normal basis multiplications, the complexity of normal basis inversion is roughly \(m-1 \) times that of normal basis multiplication. Hence, Algorithm 5 reduces the complexity considerably. Algorithm 5 is also more suitable for hardware implementation, as shown in Section IV.

D. Finding the Root Space

Instead of finding roots of polynomials in RS decoding, we need to find the root spaces of linearized polynomials in rank metric decoding. Hence the Chien search in RS decoding will have a high complexity for two reasons. First, it requires polynomial evaluations over the whole field, whose complexity is very high; Second, it cannot find a set of linearly independent roots.

A probabilistic algorithm to find the root space was proposed in [33]. For Gabidulin codes, it can be further simplified as suggested in [20]. But hardware implementations of probabilistic algorithms require random number generators. Furthermore, the algorithm in [33] requires symbolic long division, which is also not suitable for hardware implementations. According to [9], the average complexity of the probabilistic algorithm in [33] is \(O(dn)\) operations over \( F_{q^m} \), while that of Berlekamp’s deterministic method [32] is \(O(dn)\) operations in \( F_{q^m} \) plus \(O(m^3)\) operations in \( F_q \). Thus their complexity difference is small, and hence we focus on the deterministic method since it is much easier to implement.

Suppose we need to find the root space of a linearized polynomial \(r(x)\). Berlekamp’s deterministic method first evaluates the polynomial \(r(x)\) on a basis of the field \((a_0, a_1, \ldots, a_{m-1})\) such that \(v_i = r(a_i), i = 0, 1, \ldots, m - 1\). Then it expands \(v_i\)’s in the base field as columns of an \(m \times m\) matrix \(V\) and finds linearly independent roots \(z\) such that \(Vz = 0\). Using the representation based on \((a_0, a_1, \ldots, a_{m-1})\), the roots \(z\) are also the roots of the given polynomial. Finding \(z\) is to obtain the linear dependent combinations of the columns of \(V\), which can be done by Gaussian elimination.

E. n-RRE Form

Given a received subspace spanned by a set of received packets, the input of Algorithm 2 is a three-tuple, called a reduction of the received space represented by its generator matrix \(Y\); the three-tuple is obtained based on \(Y\) when it is in its RRE form [9]. Thus, before the decoding starts, preprocessing is performed on the received packets so as to obtain the RRE form of \(Y\). We show that \(Y\) needs to satisfy only a relaxed constraint. The relaxed constraint on \(Y\) does not affect the decoding outcome, while leading to two advantages. First, the relaxed constraint results in reduced complexities in the
preprocessing step. Second and more importantly, the relaxed constraint enables parallel processing of decoding KK codes based on Cartesian products.

We first define an $n$-RRE form for received matrices. Given a matrix $Y = [\hat{A} \mid y]$, where $\hat{A} \in \mathbb{F}_q^{N \times n}$ and $y \in \mathbb{F}_q^{N \times m}$, the matrix $Y$ is in its $n$-RRE form as long as $\hat{A}$ (its leftmost $n$ columns) is in its RRE form. Compared with the RRE form, the $n$-RRE form is more relaxed as it puts no constraints on the right part. We note that an $n$-RRE form of a matrix is not unique.

We first show that the relaxed constraint does not affect the decoding. Similar to [40, Proposition 7], we first show that a reduction based on $n$-RRE form is always possible. Given $Y = [\hat{A} \mid y]$ and $\text{RRE}(\hat{A}) = R\hat{A}$ where the matrix $R$ represents the reducing operations, the product $Y' = RY = [B' \mid Z']$ is in its $n$-RRE form. We note that $B' \in \mathbb{F}_q^{N \times n}$ and $Z \in \mathbb{F}_q^{N \times m}$, where the column and row rank deficiency of $B'$ are given by $\mu' = n - \text{rank} B'$ and $\delta' = N - \text{rank} B'$, respectively. We have the following result about the reduction based on $Y'$.

Lemma 1. Let $Y'$ and $\mu'$ and $\delta'$ be defined as above. There exists a tuple $(r', \hat{L}', \hat{E}') \in \mathbb{F}_q^{r' \times m} \times \mathbb{F}_q^{\mu' \times m} \times \mathbb{F}_q^{\delta' \times m}$ and a set $U'$ satisfying

$$\left[ \begin{array}{c}
[U'] \\
\hat{L}'
\end{array} \right] = \mu'^{}'$$

$$\left[ \begin{array}{c}
I_{\mu'} \hat{L}'
\end{array} \right] = \delta'$$

so that

$$\langle \left[ \begin{array}{c}
I_{\mu'} \hat{L}' \\
0
\end{array} \right] r' \rangle = \langle Y' \rangle = \langle Y \rangle.$$  (8)

Proof: See Appendix [A]

Lemma 2 shows that we can find an alternative reduction based on $n$-RRE form of $Y$, instead of an RRE form of $Y$. The key of our alternative reduction of $Y$ that the reduction is mostly determined by the first $n$ columns of $\text{RRE}(Y)$. Also, this alternative reduction does not come as a surprise. As shown in [40] Proposition 8, row operations on $\hat{E}$ can produce alternative reductions. Next, we show that decoding based on our alternative reduction is the same as in [40]. Similar to [40] Theorem 9, we have the following results.

Lemma 2. Let $(r', \hat{L}', \hat{E}')$ be a reduction of $Y$ determined by its $n$-RRE form, we have

$$d_S(\langle X \rangle, \langle Y \rangle) = 2 \text{rank} \left[ \begin{array}{c}
\hat{L}' \\
0
\end{array} \right] r' - x \hat{E}' - \mu' - \delta'.$$

Proof: See Appendix [B]

Lemma 2 shows that the subspace decoding problem is equivalent to the generalized Gabidulin decoding problem with the alternative reduction $(r', \hat{L}', \hat{E}')$, which is obtained from an $n$-RRE form of $Y$. We illustrate an example of KK decoding based on $n$-RRE approach in Appendix [C].

Our alternative reduction leads to two advantages. First, it results in reduced complexity in preprocessing. Given a matrix $Y$, the preprocessing needed to transform $Y$ into its $n$-RRE form is only part of the preprocessing to transform $Y$ into its RRE form. We can show that the maximum number of arithmetic operations in the former preprocessing is given by $(N - 1) \sum_{i=0}^{\text{rank} A - 1} (n + m - i)$, whereas that of the latter preprocessing is $(N - 1) \sum_{i=0}^{\text{rank} Y - 1} (n + m - i)$. Since $\text{rank} X \geq \text{rank} \hat{A}$, the relaxed constraint leads to a lower complexity, and the reduction depends on $\text{rank} Y$ and $\text{rank} \hat{A}$.

Second, the reduction for $n$-RRE forms is completely determined by the $n$ leftmost columns of $Y$. Instead of the whole matrix, which greatly simplifies hardware implementations. This advantage is particularly important for the decoding of constant-dimension codes that are lifted from Cartesian products of Gabidulin codes. First, since the row operations to obtain an $n$-RRE form depend on $\hat{A}$ only, decoding $[\hat{A} \mid y_{01} \cdots y_{1}]$ can be divided into parallel and smaller decoding problems whose inputs are $[\hat{A} \mid y_{0}], [\hat{A} \mid y_{1}], \ldots, [\hat{A} \mid y_{1}]$. Thus, for these constant-dimension codes, we can decode in a serial manner with only one small decoder, or in a partly parallel fashion with more decoders, or even in a fully parallel fashion. This flexibility allows tradeoffs between cost/area/power and throughput. Furthermore, since the erasures $\hat{L}$ is determined by $\hat{A}$ and is the same throughout all $[A \mid y]$, the computation of $\hat{X}$ and $\lambda(u)(x)$ in Algorithm [2] can be shared among these parallel decoding problems, thereby reducing overall complexity.

F: Finding Minimal Linearized Polynomials

Minimal linearized polynomials can be computed by solving systems of linear equations. Given roots $\beta_0, \beta_1, \ldots, \beta_{p-1}$, the minimal linearized polynomial $x[p] + \sum_{i=0}^{p-1} a_i x[i]$ satisfies

$$\left[ \begin{array}{c}
\beta_0 \beta_1 \ldots \beta_{p-1}
\beta_0^2 \beta_1^2 \ldots \beta_{p-1}^2
\vdots
\vdots
\beta_0^{p-1} \beta_1^{p-1} \ldots \beta_{p-1}^{p-1}
\end{array} \right] \left[ \begin{array}{c}
a_0 \\
a_1 \\
\vdots \\
\vdots \\
a_{p-1}
\end{array} \right] = \left[ \begin{array}{c}
\beta_0^p \\
\beta_1^p \\
\vdots \\
\vdots \\
\beta_{p-1}^p
\end{array} \right].$$  (9)

Thus it can be solved by Gaussian elimination over the extension field $\mathbb{F}_{q^m}$. Gabidulin’s algorithm is not applicable because the rows of the matrix are not the powers of the same element.

The complexity to solve (9) is very high. Instead, we reformulate the method from [27] Theorem 7]. The main idea of [27] Theorem 7 is to recursively construct the minimal linearized polynomial using symbolic products instead of polynomial multiplications in polynomial interpolation. Given linearly independent roots $w_0, w_1, \ldots, w_{p-1}$, we can construct a series of linearized polynomials as

$$F_0(x) = x[p] - w_0 x[0]$$

$$F_1(x) = (x[p] - (F_0(w_1))^{-1} x[0]) \otimes F_0(x)$$

$$\ldots$$

$$F_{p-1}(x) = (x[p] - (F_{p-2}(w_{p-1}))^{-1} x[0]) \otimes F_{p-2}(x).$$

In the method in [27] Theorem 7], the evaluation of $F_i(w_j)$ has an increasing complexity when the degree of $F_i(x)$ gets higher. To facilitate the implementation, we reformulate the algorithm to divide the evaluation into multiple steps. The idea is
based on the fact that $F_i(w_{i+1}) = (x_i^1 - (F_i-1(w_i))^y - x_i^0) \otimes F_i-1(w_{i+1})$. The reformulated algorithm is given in Algorithm 4. Thus the evaluation can also be done in a recursive way.

**Algorithm 4 (Algorithm for Minimal Linearized Polynomials).**

**Input:** Roots $w_0, w_1, \ldots, w_{p-1}$

**Output:** The minimal linearized polynomial $F(p)(x)$

1. Initialize $\gamma_j^{(0)} = w_j$, for $j = 0, 1, \ldots, p-1$ and $F(0)(x) = x^0$.

2. For $i = 0, 1, \ldots, p-1$, if $\gamma_i^{(i)} \neq 0$,
   a) $F(i+1)(x) = (F(i)(x)[1]) + (\gamma_i^{(i)} - 1 F(i)(x))$;
   b) For $j = i+1, i+2, \ldots, p-1$, $\gamma_j^{(i+1)} = (\gamma_j^{(i)})[1] + (\gamma_i^{(i)} - 1 \gamma_j^{(i)})$.

Since powers of $q$ require only cyclic shifting, the operations in Algorithm 4 are simple. Another advantage of Algorithm 4 is that it does not require the roots to be linearly independent. In Algorithm 4, $F(i)(w_j) = 0$ for $j = 0, 1, \ldots, i-1$ and $\gamma_i^{(i+1)} = F(i)(w_i)$. If $w_0, w_1, \ldots, w_j$ are linearly dependent, $\gamma_j^{(i)} = 0$ and hence $w_j$ is ignored. So Algorithm 4 integrates detection of linear dependency at no extra computational cost.

Essentially, Algorithm 4 breaks down evaluations of high $q$-degree polynomials into evaluations of polynomials with $q$-degree as one. It gets rid of complex operations while maintaining the *same* total complexity of the algorithm.

**IV. ARCHITECTURE DESIGN**

Aiming to reduce the storage requirement and total area as well as to improve the regularity of our decoder architectures, we further reformulate the steps in the decoding algorithms of both Gabidulin and KK codes. Again, we assume the decoder architectures are suitable for RLNC over $\mathbb{F}_q$, where $q$ is a power of two.

**A. High-Speed BMA Architecture**

To increase the throughput, regular BMA architectures with shorter CPD are necessary. Following the approaches in [24], we develop two architectures based on Algorithm 3 which are analogous to the riBM and RiBM algorithms in [24].

In Algorithm 3, the critical path is in step 3.2(a). Note that $\Delta_i$ is the $r$th coefficient of the discrepancy polynomial $\Delta_r(x) = \lambda_r(x) \otimes S(x)$. By using $\Theta^{(r)}(x) = B^{(r)}(x) \otimes S(x)$, $\Delta^{(r+1)}(x)$ can be computed as

$$\Delta^{(r+1)}(x) = \Delta^{(r+1)}(x) \otimes S(x)$$

$$= [(\Gamma^{(r)}[1] \Delta^{(r)}(x) - \Delta_r x^2] \otimes B^{(r)}(x)] \otimes S(x)$$

$$= [(\Gamma^{(r)}[1] \Delta^{(r)}(x) - \Delta_r x^2] \otimes B^{(r)}(x) + \Theta^{(r)}(x)] \otimes S(x)$$

which has the structure as step 3.2(c). Hence this reformulation is more conducive to a regular implementation.

Given the similarities between step 3.2(a) and (10), $\Delta(x)$ and $\Delta(x)$ can be combined together into one polynomial $\Delta(x)$. Similarly, $B(x)$ and $\Theta(x)$ can be combined into one polynomial $\Theta(x)$. These changes are incorporated in our RiBMA algorithm, shown in Algorithm 5.

**Algorithm 5. RiBMA**

**Input:** Syndromes $S$

**Output:** $\Lambda(x)$

1. Initialize $\Lambda(0) = \Lambda(0) = 0$ and $b = 0$.

2. For $r = 0, 1, \ldots, 2t - 1$,
   a) Modify the combined polynomial: $\Lambda^{(r+1)}(x) = \Gamma^{(r)}[r](x) - \Lambda^{(r)}(x) \otimes B^{(r)}(x)$;
   b) Set $b = b + 1$;
   c) If $\Lambda_0^{(r)} \neq 0$ and $b > 0$, set $b = \mu$, $\Lambda^{(r)} = \Lambda^{(r)}(x)$, and $\Theta^{(r)}(x) = \Theta^{(r)}(x)$;
   d) Set $\Lambda^{(r+1)}(x) = \sum_{i=0}^{2t-1} \Lambda^{(r+1)}[1] x^i$;
   e) Set $\Lambda^{(r+1)} = (\Gamma^{(r)})[1]$ and $\Theta^{(r+1)}(x) = x^1 \otimes \Theta^{(r)}(x)$.

3. Set $\Lambda(x) = \sum_{i=0}^{t} \Lambda^{(i+1)} x^i$.

Following Algorithm 5 we propose a systolic RiBMA architecture shown in Fig. 5 which consists of $3t + 1$ identical processing elements (BEs), whose circuitry is shown in Fig. 4. The central control unit BCTRL processes $b$ inside, generates the global control signals $ct^{(r)}$ and $\Gamma^{(r)}$, and passes along the coefficient $\Lambda_0^{(r)}$. The control signal $ct^{(r)}$ is set to 1 only if $\Lambda_0^{(r)} \neq 0$ and $k > 0$. In each processing element, there are two critical paths, both of which consist of one multiplier and one adder over $\mathbb{F}_{2^m}$. One starts from $\Lambda^{(r)}(x)$ and the other $\Lambda^{(r+1)}(x)$, while both end in $\Lambda^{(r)}(x)$.

![Fig. 4. The processing element BEi (xq is a cyclic shift, and requires no hardware but wiring)](image-url)

**B. Generalized BMA**

The key equation of KK decoding is essentially the same as [3] but $\omega(x)$ has $q$-degree less than $\tau$, instead of $[(d - 1)/2]$. Actually, in KK decoding, we do not know the exact value of $\tau$ before solving the key equation. But all we need is to determine the maximum number of correctable errors $t'$ given $\mu$ erasures and $\delta$ deviations, which is given by $t' = [(d - 1 - \mu - \delta)/2]$. Hence we need to generalize our previous
BMA architectures in Section III-C for KK decoding, as in Algorithm 6. To apply Algorithm 6 to Gabidulin decoding, we can simply use \( \theta = \mu + \delta = 0 \).

**Algorithm 6 (Generalized RiBMA).**

**Input:** \( S \) and \( \theta \)

**Output:** \( \Lambda(x) \)

1. Initialize as follows: \( t' = \lfloor (d - 1 - \theta)/2 \rfloor \), \( \tilde{\Delta}^{(0)}(x) = \Theta^{(0)}(x) = \sum_{i=0}^{\theta + 2t'-1} S_i x^i \), \( \Delta^{(0)}(2^t+x) = \Theta^{(0)} = 1 \), \( \Gamma^{(0)} = 1 \), and \( b = 0 \).
2. For \( r = 0, 1, \ldots, 2t' - 1 \),
   a. Modify the combined polynomial: \( \tilde{\Delta}^{(r+1)}(x) = \Gamma^{(r)} \Delta^{(r)}(x) - \tilde{\Delta}^{(r)}(x) \Theta^{(r)}(x) \);
   b. Set \( b = b + 1 \);
   c. If \( \tilde{\Delta}^{(r)} \neq 0 \) and \( b \geq 0 \), set \( b = b - \Gamma^{(r+1)} = \tilde{\Delta}^{(r)}(x) \) and \( \Theta^{(r)}(x) = \Delta^{(r)}(x) \);
   d. Set \( \tilde{\Delta}^{(r+1)}(x) = \sum_{i=0}^{2^{r+1} \cdot t'-1} \tilde{\Delta}^{(r)}(x^i) \), \( \tilde{\Delta}^{(r)}(x) = \sum_{i=0}^{2^{r+1} \cdot t'-1} \tilde{\Delta}^{(r)}(x^i) \Theta^{(r)}(x) \);
   e. Set \( \Gamma^{(r+1)} = (\Gamma^{(r)})[1] \) and \( \tilde{\Delta}^{(r)}(x) = x^1 \Theta^{(r)}(x) \).
3. Set \( \Lambda(x) = \sum_{i=0}^{2t'} \tilde{\Delta}^{(2t')} x^i \).

Compared with Algorithm 5, we replace \( t \) by \( t' \). The variable \( t' \) makes it difficult to design regular architectures. By carefully initializing \( \Delta^{(0)}(x) \) and \( \Theta^{(0)}(x) \), we ensure that the desired output \( \Lambda(x) \) is always at a fixed position of \( \Delta^{(2t')}(x) \), regardless of \( \mu + \delta \). Hence, the only irregular part is the initialization. The initialization of Algorithm 6 can be done by shifting in at most \( t' \) cycles. Hence we reuse the RiBMA architecture in Fig. 3 in the KK decoder and keep the same worse-case latency of \( 2t \) cycles.

**C. Gaussian Elimination**

We need Gaussian elimination to obtain \( n \)-RRE forms as well as to find root spaces. Furthermore, Gabidulin’s algorithm in Algorithm 7 is essentially a smart way of Gaussian elimination, which takes advantage of the properties of the matrix. The reduction (to obtain \( n \)-RRE forms) and finding the root space are Gaussian eliminations on matrices over \( \mathbb{F}_q \), while Gabidulin’s algorithm operates on matrices over \( \mathbb{F}_{q^m} \). In this section, we focus on Gaussian eliminations over \( \mathbb{F}_q \) and Gabidulin’s algorithm will be discussed in Section IV-D.

For high-throughput implementations, we adopt the pivoting architecture in \([41]\), which was developed for non-singular matrices over \( \mathbb{F}_2 \). It always keeps the pivot element on the top-left location of the matrix, by cyclically shifting the rows and columns. Our Gaussian elimination algorithm, shown in Algorithm 7, has three key differences from the pivoting architecture in \([41]\). First, Algorithm 7 is applicable to matrices over any field. Second and more importantly, Algorithm 7 detects singularity and can be used for singular matrices. This feature is necessary since singular matrices occur in the reduction for the RRE form and finding the root space. Third, Algorithm 7 is also flexible about matrix sizes, which are determined by the variable numbers of errors, erasures, and deviations.

**Algorithm 7 (Gaussian Elimination for Root Space).**

**Input:** \( m \times m \) matrices \( M \in \mathbb{F}_{q^m}^{m \times m} \), whose rows are evaluations of \( \sigma(x) \) over the normal basis, and \( B = I \)

**Output:** Linearly independent roots of \( \sigma(x) \)

1. Set \( i = 0 \).
2. For \( j = 0, 1, \ldots, m - 1 \),
   a. While \( M_{0,0} = 0 \) and \( j < m - i \), shiftup(\( M, i \)) and shiftup(\( B, i \))
   b. If \( M_{0,0} \) is not zero, eliminate(\( M \)), reduce(\( B, M \)), and \( i = i + 1 \).
3. The last \( m - i \) rows of \( M \) are all zeros and the last \( m - i \) rows of \( B \) are roots.

The eliminate and shiftup operations are quite similar to those in \([41]\) Algorithm 2. In eliminate(\( M \)), for all \( 0 \leq i < m \) and \( 0 \leq j < m \), \( M_{i,j} = M_{i,0} M_{i+1,j+1} \mod m - M_{i+1,0} M_{i,j+1} \mod m \) if \( i < m - 1 \), or \( M_{i,j} = M_{i,j+1} \mod m \). The procedure reduce(\( B, M \)) essentially repeats eliminate without column operations. For all \( 0 \leq i < m \) and \( 0 \leq j < m \), \( B_{i,j} = B_{i+1,j} - M_{i+1,0} B_{i,j} \) if \( i < m - 1 \), otherwise \( B_{i,j} = B_{i,j} \).
In the shiftup(\( M, \rho \)) operation, the first row is moved to the \((m - 1 - \rho)\)th row while the second to the \((m - 1 - \rho)\)th rows are moved up. That is, for \( 0 \leq j < m \), \( M_{i,j} = M_{i,j} \) if \( i = m - 1 - \rho \), and \( M_{i,j} = M_{i+1,j} \) for \( 0 \leq i < m - 2 - \rho \). In the shiftleft operation, all columns are cyclic shifted to the left except that the first column is moved to the last. In other words, for all \( 0 \leq i < m \) and \( 0 \leq j < m \), \( M_{i,j} = M_{i,(j+1)} \mod m \). By adding a shiftleft operation, Algorithm 7 handles both singular and non-singular matrices while \([41]\) Algorithm 2] only works on non-singular matrices. Since \( B \) is always full rank, the roots obtained are guaranteed to be linearly independent.

We can get the root space using Algorithm 7 and we can also use it in KK decoding to reduce the received vector to an \( n \)-RRE form. But Algorithm 7 only provides \( \hat{L} \), we need to extend it to obtain \( \hat{L}^* \), as in Algorithm 8.

**Algorithm 8 (Gaussian Elimination for \( n \)-RRE Forms).**

**Input:** \( N \times n \) matrix \( \hat{A} \) and \( N \times m \) matrix \( y \)

**Output:** \( \hat{L}^*, \hat{E}^*, \rho^*, \) and \( \mu^* \).
1. Set $i = 0$, $\mathcal{U}'$ and $\hat{L}$ as empty.
2. For each column $j = 0, 1, \ldots, n-1$
   a) While $A_{0,0} = 0$ and $j < n - i$, shiftup$(\hat{A}, i)$, shiftup$(y, i)$, shiftup$(\hat{L}', i)$.
   b) If $A_{0,0}$ is not zero, eliminate$(\hat{A})$, reduce$(y, \hat{A})$, shiftup$(\hat{L}', 0)$, $i = i + 1$.
   c) Otherwise, shiftleft$(\hat{A})$, append the first column of $\hat{A}$ to $\hat{L}'$, set the top-right element of $\hat{L}'$ to one, and add $j$ to $\mathcal{U}'$.
3. Set $\mu' = n - i$. The deviations $\hat{E}'$ are given by the first $\mu'$ rows of $y$.
4. For each column $j \in \mathcal{U}'$, shiftup$(\hat{L}', j)$ and shiftup$(y, j)$.
5. The received vector $r'$ is given by $y$.

In Algorithm 8 we incorporate the extraction of $\hat{L}'$, $\hat{E}'$, and $r'$ into Gaussian elimination. So our architecture has the same worst-case latency as Algorithm 7 and requires no extra cycles to extract $\hat{L}$ out of the $n$-RRE form. Hence the throughput also remains the same.

Algorithm 7 is implemented by the regular architecture shown in Fig. 5 which is a two-dimensional array of $m \times 2m$ processing elements (GE’s); The leftmost $m$ columns of processing elements correspond to $M$, and the rightmost $m$ columns $B$.

Algorithm 8 can be implemented with the same architecture with $N \times (n + m)$ GE’s; The leftmost $n$ columns of processing elements correspond to $\hat{A}$, and the rightmost $m$ columns $y$.

The elements for $\hat{L}'$ are omitted in the figure. The circuitry of the processing element GE is shown in Fig. 6. For row $i$, the control signal $ct_i$ for row $i$ chooses from five inputs based on the operation: keeping the value, shiftleft, eliminate (or reduce), and shiftup (using the first row or the next row).

In Gabidulin’s algorithm, there are two $\tau \times \tau$ matrices over $\mathbb{F}_q^m$, $A$ and $Q$. In step 12, it requires only $Q_{i,0}$’s to compute the coefficients. To compute $Q_{0,0}$ in (6), it requires only $Q_{1-1,0}$ and $Q_{i-1,1}$. And for $Q_{i+j}$ in (6), it requires only $Q_{i-1,j}$ and $Q_{i-1,j+1}$. Recursively only those $Q_{i,j}$’s where $i + j < \tau$ are necessary. Actually, given any $i$, entries $Q_{1,0}, Q_{1+1,0}, \ldots, Q_{\tau-1,0}$ can be computed with the entries $Q_{1-1,0}, Q_{1-1,1}, \ldots, Q_{1-1,\tau-1}$. With $Q_{0,0}, Q_{1,0}, \ldots, Q_{1-2,0}$, we need to store only $\tau$ values to keep track of $Q$. Hence we reduce the storage of $Q$ from $\tau \times \tau$ $m$-bit registers down to $\tau$. We cannot reduce the storage of $A$ to $\tau(\tau+1)/2$ because we have to use the pivoting scheme for short critical paths.

**D. Gabidulin’s Algorithm**

In Gabidulin’s algorithm, the matrix is first reduced to a triangular form. It takes advantage of the property of the matrix such that it requires no division in the first stage. In the first stage, we need to perform elimination on only one row. We use a similar pivoting scheme like Algorithm 7. When a row is reduced to have only one non-zero element, a division is used to obtain one coefficient of $X$. Then it performs a backward elimination after getting each coefficient. Hence we introduce a backward pivoting scheme, where the pivot element is always at the bottom-right corner.

---

**Fig. 8. The processing element $AE_{i,j}$**

**Fig. 9. The processing element $QE_{i}$**

In our decoder, Gabidulin’s Algorithm is implemented by the regular architecture shown in Fig. 7 which include a triangular array of $\tau \times \tau$ AE’s and a one-dimensional array of $\tau$ QE’s. The circuitry of the processing element $AE_{i,j}$ and $QE_{i}$ is shown in Fig. 8 and 9. The upper MUX in AE controls the output sending upward along the diagonal. Its control signal $ct_{ai}$ is 1 for the second row and 0 for other rows since we update $A$ one row in a cycle and we keep the pivot on the upper left corner in Step 11. The control of the lower MUX in AE is 0 for working on Step 11, and 1 for working on Step 12. Similarly the control of the MUX in $QE$ is 0 for working on Step 11, and 1 for working on Step 12. But in Step 11, only
part of QE’s need update and others should maintain their values and their control signals ctq’s are set to 2. Initially, \( A_{0,i} = E_i \) and \( q_i = S_i \) for \( i = 0, 1, \ldots, \tau - 1 \). Step 1 needs \( \tau \) substeps. In the first \( \tau - 1 \) substeps, \( \text{ctal}_{i+1} = 0 \), \( \text{ctau}_1 = 1 \), \( \text{ctq}_0 = \text{ctq}_1 = \cdots = \text{ctq}_{\tau} = 2 \), and \( \text{ctq}_{i+2} = \cdots = \text{ctq}_{\tau - 1} = 0 \) for substep \( i \). In the last substep, \( \text{ctau}_1 = 0 \) and all \( \text{ctq}_i \)’s are set to 2. This substep is to put the updated \( A \) into the original position. In Step 2, the pivot is in the right lower corner, where we compute \( X_i \)’s. Step 2 also needs \( \tau \) substeps, in which all \( \text{ctal}_i \)’s and \( \text{ctq}_i \)’s are set to 1. First \( X_{\tau - 1} \) is computed by \( A_{\tau - 1, \tau - 1}^{-1} q_{\tau - 1} - 1 \) where \( q_{\tau - 1} = Q_{\tau - 1, 0} \). Note that the inversion may need \( m - 2 \) clock cycles. In each substep, the matrix \( A \) is moving down the diagonal so the \( A_{1,i} \) to be inverted is always at the bottom right corner. At the same time, the \( q_i \)’s are also moving down. Basically, in substep \( p \), the architecture updates \( q_i \)’s to \( Q_{i-p,0} = \sum_{j=\tau-p}^{\tau-1} A_{i,j} X_j \) for \( i > p \) by doing one backward elimination at each substep.

**E. Low Complexity Linearized Interpolation**

Note that in each iteration of step 2(b), the \( q \)-degree of \( F(x) \) is no more than \( i+1 \) and we need only \( w_{i+1}, w_{i+2}, \ldots, w_{p-1} \). Hence the coefficients of \( F(x) \) and \( w_i \)’s can be packed into a register vector of length \( p + 1 \), where \( F(x) \) and \( w_i \)’s are updated currently, as in Algorithm 9. Along with the update,
more registers are used for $F(x)$ and less for $w_i$’s. In Algorithm 9 we keep $F_p = \gamma_p^{(i)} = 1$ during the interpolation.

Algorithm 9 (Reformulated Algorithm for Minimal Linearized Polynomials).

**Input:** Roots $w_0, w_1, \ldots, w_{p-1}$

**Output:** The minimal linearized polynomial $F(x)$

1. Initialize $\gamma_j^{(i)} = w_j$ for $j = 0, 1, \ldots, q-1$, $\gamma_p^{(i)} = 1$, and $\alpha = 0$.
2. For $i = 0, 1, \ldots, p-1$,
   a) If $\gamma_0^{(i)} \neq 0$,
      i) For $j = 0, 1, \ldots, p-1-\alpha$, $\gamma_j^{(i+1)} = (\gamma_j^{(i)})^{[1]} + (\gamma_0^{(i)})^{q-1}\gamma_j^{(i)}$.
      ii) For $j = p-\alpha, p-\alpha+1, \ldots, p-1$, $\gamma_j^{(i+1)} = (\gamma_j^{(i)})^{[1]} + (\gamma_0^{(i)})^{q-1}\gamma_j^{(i)}$.
      iii) $\alpha = \alpha + 1$.
   b) Otherwise, for $j = 0, 1, \ldots, p-1-\alpha$, $\gamma_j^{(i+1)} = \gamma_j^{(i+1)}$, $\gamma_p^{(i)} = 0$, and $\gamma_j^{(i+1)} = \gamma_j^{(i)}$ for $j = p-\alpha+1, \ldots, p-1$.
   c) $\gamma_p^{(i+1)} = \gamma_p^{(i)}$.
3. For $i = p, p+1, \ldots, 2p-\alpha-1$,
   a) For $j = 0, 1, \ldots, p-1-\alpha$, $\gamma_j^{(i+1)} = \gamma_j^{(i)}$ and $\gamma_p^{(i)} = 0$.
4. $F(x) = \sum_{i=0}^{p} \gamma_i^{2p-\alpha-1} x^i$.

The interpolation is actually done after Step 2. If the roots are not linearly independent, $F(x)$ may not be aligned with $\gamma_0$. Therefore, in Step 3, we shift the registers until $\gamma_0$ is non-zero. In other words, it is shifted to the left for $p-\alpha$ times.

Algorithm 9 is implemented by the systolic architecture shown in Fig. 10 which consists of $p$ processing elements (ME’s). We note that we need only $p$ processing elements since we assume that the highest coefficient is fixed to 1. The circuitry of the processing element ME$_j$ is shown in Fig. 11. The cr signal is 1 only when $\gamma_0 \neq 0$. The ct$_j$ signal for each cell is 1 only if $j < p-\alpha$. Basically, ct$_j$ controls if the update is for $F(x)$ or $w_i$’s.

![Fig. 10. Architecture of linearized polynomial interpolation](image)

**F. Decoding Failure**

A complete decoder declares decoding failure when no valid codeword is found within the decoding radius of the received word. To the best of our knowledge, decoding failures of Gabidulin and KK codes were not discussed in previous works. Similar to RS decoding algorithms, a rank decoder can return decoding failure when the roots of the error span polynomial $\lambda(x)$ are not unique. That is, the root space of $\lambda(x)$ has a dimension smaller than the $q$-degree of $\lambda(x)$. Note that this applies to both Gabidulin and KK decoders. For KK decoders, another condition of decoding failure is when the total number of erasures and deviations exceeds the decoding bound $d-1$.

**G. Latency and Throughput**

We analyze the worst-case decoding latencies of our decoder architectures, in terms of clock cycles, in Table III.

<table>
<thead>
<tr>
<th>TABLE III</th>
<th>WORST-CASE DECODING LATENCY (IN TERMS OF CLOCK CYCLES)</th>
</tr>
</thead>
<tbody>
<tr>
<td>n-RRE</td>
<td>Gabidulin</td>
</tr>
<tr>
<td>Syndrome</td>
<td>$S$</td>
</tr>
<tr>
<td>$\lambda_D(x)$</td>
<td>$n-d$</td>
</tr>
<tr>
<td>$\sigma_D(x)$</td>
<td>$d-1$</td>
</tr>
<tr>
<td>$\sigma_U(x)$</td>
<td>$(m+2d-1)$</td>
</tr>
<tr>
<td>$\sigma(x)$</td>
<td>$(m+2d-1)$</td>
</tr>
<tr>
<td>Root space base $E$</td>
<td>$(m+1)/2$</td>
</tr>
<tr>
<td>Error locator $L$</td>
<td>$2t + m2$</td>
</tr>
<tr>
<td>Error word $e$</td>
<td>$t$</td>
</tr>
</tbody>
</table>

As in [41], the latency of Gaussian elimination for the n-RRE form is at most $n(N + 1)/2$ cycles. Similarly, the latency of finding the root space is at most $m(m + 1)/2$. For Gabidulin’s algorithm, it needs one cycle per row for forward elimination and the same for backward elimination. For each coefficient, it takes $m$ cycles to perform a division. Hence it needs at most $2(d-1) + m(d-1)$ and $2(d-1) + m(d-1)$ for $\beta$ and $L$ respectively. The latencies of finding the minimal linearized polynomials are determined by the number of registers, which is $2t$ to accommodate $\lambda_D(x)$, $\sigma_D(x)$, and $\sigma_U(x)$, whose degrees are $\mu$, $\delta$, $\mu$, respectively. The $2t$ syndromes can be computed by $2t$ sets of multiply-and-accumulators in $n$ cycles. Note that the computations of $S(x)$, $\lambda_U(x)$, and $\sigma_D(x)$ can be done concurrently. The latency of RiBMA is $2t$ for $2t$ iterations. The latency of a symbolic product $a(x) \otimes b(x)$ is determined by the $q$-degree of $a(x)$. When computing $S_{DU}(x)$, we are concerned about only the terms of $q$-degree less than $d-1$ because only those are meaningful for the key equation. For computing $S_{FD}(x)$, the result of $\sigma_D(x) \otimes S(x)$ in $S_{DU}(x)$ can be reused, so it needs only one symbolic product. In total, assuming $n = m$, the decoding latencies of our
Gabidulin and KK decoders are \( n(n+3)/2 + (n+5)t \) and \( n(N+n+4)/2 + (4n+24)t \) cycles, respectively.

One assumption in our analysis is that the unit that computes \( x^{q-1} \) in Figs. 9 and 11 is implemented with pure combinational logic, which leads to a long CPD for large \( q \)'s. For large \( q \)'s, to achieve a short CPD, it is necessary to pipeline the unit that computes \( x^{q-1} \). There are two ways to pipeline it: \( x^{q-1} = x \cdot x^2 \cdot x^{q/2} \) that requires \( \log_2 q - 1 \) multiplications, or \( x^{q-1} = x^q/x \) that requires \( m \) multiplications for division.

To maintain a short CPD, \( x^{q-1} \) needs to be implemented sequentially with one clock cycle for each multiplication. Let \( c_{qm} = \min \{ \log_2 q - 1, m \} \) and it requires at most \( 2(c_{qm}+2)t \) clock cycles for getting minimal linearized polynomials \( \lambda_i(x) \), \( \sigma_D(x) \), and \( \sigma_U(x) \). Similarly, it requires at most \( c_{qm}(d-1) \) more cycles to perform forward elimination in Gabidulin’s algorithm for the error locator, and the latency of this step will be \( (m + c_{qm} + 2)(d-1) \) cycles.

In our architectures, we use a block-level pipeline scheme for high throughput. Data transfers between modules are buffered into multiple stages so the throughput is determined by only the longest latency of a single module. For brevity, we present only the data flow of our pipelined Gabidulin decoder in Fig. 12. The data in different pipeline stage are for different decoding sessions. Hence these five units can work on five different sessions currently for higher throughput. If some block finishes before others, it cannot start another session until all are finished. So the throughput of our block-level pipeline decoders, the throughput is determined by the block with the longest latency. For both Gabidulin and KK decoders, the block of finding root space is the bottleneck that requires the longest latency in the worst case scenario. Both decoders have the same bottleneck latency despite different areas.

### V. Implementation Results and Discussions

To evaluate the performance of our decoder architectures, we implement our architectures for Gabidulin and KK codes for RLCN over \( \mathbb{F}_2 \). Note that although the random linear combinations are carried out over \( \mathbb{F}_2 \), decoding of Gabidulin and KK codes are performed over extension fields of \( \mathbb{F}_2 \).

Due to hardware limitation in Fig. 5. We need to restrict \( N \). In our implementation of KK decoders, we assume \( N \) is no more than \( m \). When \( N > m \), the number of deviations \( N - n \) is at least \( N - m \) since \( n \leq m \). If \( N \) is greater than \( m \), we simply choose \( m \) linearly independent packets randomly. In some cases, it may lose useful information of deviations and hence gets higher error rates. In other cases, the extra deviations are not introduced into the received codeword and they reduce error correcting capability. So choosing \( N = m \) is a trade-off to avoid false deviations at the cost of possibly useful deviations. If \( N \) is smaller than \( m \), it implies \( \hat{E} \) is not linearly independent. But since Algorithm 4 can handle linearly dependent roots, the whole decoder works the same.

#### A. Implementation Results

We implement our decoder architecture in Verilog for an (8, 4) Gabidulin code over \( \mathbb{F}_{2^8} \) and a (16, 8) one over \( \mathbb{F}_{2^{16}} \), which can correct errors of rank up to two and four, respectively. We also implement our decoder architecture for their corresponding KK codes, which can correct \( e \) errors, \( \mu \) erasures, and \( \delta \) deviations as long as \( 2e + \mu + \delta \) is no more than five or nine, respectively. Our designs are synthesized using Cadence RTL Compiler 7.1 and MOSIS SCmos TSMC 0.18\( \mu \)m standard cell library [22]. The synthesis results are given in Tables IV and V. The total area in Tables IV and V include both cell area and estimated net area, and the total power in Tables IV and V includes both leakage and estimated dynamic power. All estimation are made by the synthesis tool. To provide a reference for comparison, the gate count of our (8, 4) KK decoder is only 63% to that of the (255, 239) RS decoder over the same field \( \mathbb{F}_{2^8} \) in [23], which is 115,500. So for Gabidulin and KK codes over small fields, which have limited error-correcting capabilities, their hardware implementations are feasible. The area and power of decoder architectures in Tables IV and V appear affordable except for applications with very stringent area and power requirements.

### Table IV

<table>
<thead>
<tr>
<th>Code</th>
<th>Gates</th>
<th>Area (mm²)</th>
<th>CPD (ns)</th>
<th>Estimated Power (mW)</th>
<th>Throughput (Mbit/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gabidulin</td>
<td>19420</td>
<td>0.466</td>
<td>5.572</td>
<td>0.001</td>
<td>498</td>
</tr>
<tr>
<td>KK</td>
<td>72527</td>
<td>1.741</td>
<td>3.696</td>
<td>0.003</td>
<td>962</td>
</tr>
</tbody>
</table>

### Table V

<table>
<thead>
<tr>
<th>Code</th>
<th>Gates</th>
<th>Area (mm²)</th>
<th>CPD (ns)</th>
<th>Estimated Power (mW)</th>
<th>Throughput (Mbit/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gabidulin</td>
<td>129218</td>
<td>3.101</td>
<td>4.427</td>
<td>7.38171</td>
<td>425</td>
</tr>
<tr>
<td>KK</td>
<td>437362</td>
<td>10.497</td>
<td>4.651</td>
<td>2732.035</td>
<td>809</td>
</tr>
</tbody>
</table>

#### B. Implementation Results of Long Codes

Although the area and power shown in Tables IV and V are affordable and high throughputs are achieved, the Gabidulin and KK codes have very limited block lengths 8 and 16. For practical network applications, the packet size may be large [22]. One approach to increase the block length of a constant-dimension code is to lift a Cartesian product of Gabidulin codes [40]. We also consider the hardware implementations for this case. We assume a packet size of 512 bytes,
and use a KK code that is based on Cartesian product of 511 length-8 Gabidulin codes. As observed in Section III-E, the n-RRE form allows us to either decode this long KK code in serial, partly parallel, or fully parallel fashion. For example, more decoder modules can be used to decode in parallel for higher throughput. We list the gate counts and throughput of the serial and factor-7 parallel schemes based on the (8, 4) KK decoder in Table VI and those of the serial and factor-5 parallel schemes based on the (16, 8) KK decoder in Table VII.

### Table VI
**Performance of (8, 4) KK Decoders for 512-byte packets**

<table>
<thead>
<tr>
<th></th>
<th>Serial</th>
<th>7-Parallel</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gates (Gates)</td>
<td>72527</td>
<td>507689</td>
</tr>
<tr>
<td>Area (mm²)</td>
<td>2.366</td>
<td>16.562</td>
</tr>
<tr>
<td>Estimated Power (mW)</td>
<td>150.291</td>
<td>2452.037</td>
</tr>
<tr>
<td>Latency (cycles)</td>
<td>18552</td>
<td>2784</td>
</tr>
<tr>
<td>Throughput (Mbit/s)</td>
<td>482</td>
<td>3374</td>
</tr>
</tbody>
</table>

### Table VII
**Performance of (16, 8) KK Decoders for 512-byte packets**

<table>
<thead>
<tr>
<th></th>
<th>Serial</th>
<th>5-Parallel</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gates (Gates)</td>
<td>437362</td>
<td>2186810</td>
</tr>
<tr>
<td>Area (mm²)</td>
<td>15.249</td>
<td>76.245</td>
</tr>
<tr>
<td>Estimated Power (mW)</td>
<td>2725.035</td>
<td>13625.175</td>
</tr>
<tr>
<td>Latency (cycles)</td>
<td>33184</td>
<td>7440</td>
</tr>
<tr>
<td>Throughput (Mbit/s)</td>
<td>406</td>
<td>2030</td>
</tr>
</tbody>
</table>

### C. Discussions

Our implementation results above show that the hardware implementations of RLNC over small fields and with limited error control are quite feasible, unless there are very stringent area and power requirements. However, small field sizes imply limited block length and limited error control. As shown above, the block length of a constant-dimension code can be increased by lifting a Cartesian product of Gabidulin codes. While this easily provides arbitrarily long block length, it does not address the limited error control associated with small field sizes.

For example, a Cartesian product of (8, 4) Gabidulin codes has the same error correction capability as the (8, 4) KK decoder, and their corresponding constant-dimension codes also have the same error correction capability. If we want to increase the error correction capabilities of both Gabidulin and KK codes, longer codes are needed and in turn larger fields are required. A larger field size implies a higher complexity for finite field arithmetic, and longer codes with greater error correction capability also lead to higher complexity. It remains to be seen whether the decoder architectures continue to be affordable for longer codes over larger fields, and this will be the subject of our future work.

### VI. Conclusion

This paper presents novel hardware architectures for Gabidulin and KK decoders. It not only reduces the algorithm complexity but also fits the decoders into regular architectures suitable for circuit implementation. Synthesis results using a standard cell library confirm that our designs achieve high speed and high throughput.

### Acknowledgment

The authors would like to thank Dr. D. Silva and Prof. F. R. Kschischang for valuable discussions, and thank H. Xie for pointing out errors in the decoding example.

### References


APPENDIX A

PROOF OF LEMMA 1

Proof: Let the RRE and an n-RRE forms of $Y$ be given by

$$\text{RRE}(Y) = \begin{bmatrix} W & \tilde{r} \\ 0 & \tilde{E} \end{bmatrix}$$

$$Y' = \begin{bmatrix} W' & \tilde{r}' \\ 0 & \tilde{E}' \end{bmatrix}$$

Since the RRE form of $\tilde{A}$ is unique, $W = W'$. Thus, $\mu = \mu'$ and $\delta = \delta'$. In the proof of Proposition 7, $U$ is chosen based on $W$. Thus, we choose $U = U'$. Since $\hat{L}$ is uniquely determined by $W$ and $L'$ is by $W'$, we also have $\hat{L} = L'$. Finally, choosing $r' = I_n r$, we can show that holds by following the same steps as in the proof of Proposition 7.
APPENDIX B
PROOF OF LEMMA

Proof: This follows a similar approach as in [9, Appendix C]. We have

\[
\text{rank } \begin{bmatrix} X \\ Y \end{bmatrix} = \text{rank} \begin{bmatrix} I + \hat{L}'I_{\hat{b}'} + \hat{r}' \\ 0 \end{bmatrix} = \text{rank} \begin{bmatrix} \hat{L}'I_{\hat{b}'} + \hat{r}' - x \\ 0 \end{bmatrix}
\]

such that the erasures are given by \( \hat{L}' = (0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1) \) and \( \hat{r}' = (219, 227, 55, 199, 27, 47, 0, 0) \).

The decoding process is as follows:

1. Computing the auxiliary syndrome polynomial
   a) syndromes \( S = \mathbf{H} \hat{r}' = (185, 169, 45, 130) \)
   b) erasures \( \hat{X} = \mathbf{L}' \hat{h} = (205, 130) \)
   c) erasure polynomial \( \lambda_U(x) = \text{minpoly}(\hat{X}) = x^2 + 146x^{10} + 49x^{20} \)
   d) deviation polynomial \( \sigma_D(x) = \text{minpoly}(\hat{E}') = x^2 + 142x^2 + 143x^{10} \)
   e) \( \lambda_U(x) = 49x^2 + 201x^1 + x^0 \) and \( S_{DU}(x) = \sigma_D(x) \otimes S(x) \otimes \lambda_U(x) = 142x^7 + 198x^6 + 141x^5 + 44x^4 + 124x^3 + 251x^2 + 53x^1 + 241x^0 \).

2. Computing the error span polynomial
   a) solving \( \sigma_F(x) \otimes S_{DU}(x) \equiv \omega(x) \mod x^{d-1} \), in which \( \deg \omega(x) < \tau \) and \( \mu + \delta \leq \tau \leq \delta - d - 1 \), gives \( \sigma_F(x) = x^{d-1} \)
   b) \( S_{FD}(x) = \sigma_F(x) \otimes \sigma_{DU}(x) \otimes S(x) = 4x^5 + 225x^4 + 140x^3 + 115x^2 + 216x + 241x^0 \)
   c) solving \( S_{FD}(x) = \sum_{j=0}^{d-1} X_j \beta_j \), for \( d = 2, 3 \), gives \( \beta = (48, 186) \)
   d) \( \sigma_U(x) = \text{minpoly}(\beta) = x^2 + 150x^1 + 69x^0 \)
   e) \( \sigma(x) = \sigma_U(x) \otimes \sigma_F(x) \otimes \sigma_{PU}(x) = x^4 + 238x^3 + 187x^2 + 141x + 217x^0 \).

3. Finding the roots of the error span polynomial \( \sigma(x) = \mathbf{E} = (254, 157, 4, 251) \)

\(^1\) \( X_i = X_i, i = 0, 1, \ldots, \mu - 1 \).
4) Finding the error locations
   a) solving $S_l = \sum_{j=0}^{\tau-1} X_j^{[l]} E_j, l = 0, 1, \ldots, d - 2$
      where the rank of error word $\tau \leq d - 1$, gives
      $X = (205, 130, 204, 1)$
   b) the error locations $L = (64, 128, 191, 255)$ and
      
      $L^T = \begin{bmatrix}
      0 & 0 & 0 & 0 & 0 & 1 & 0 \\
      0 & 0 & 0 & 0 & 0 & 0 & 1 \\
      1 & 1 & 1 & 1 & 1 & 0 & 1 \\
      1 & 1 & 1 & 1 & 1 & 1 & 1 
      \end{bmatrix}$
      such that $X = L^T \mathbf{h}$
   c) the error word is given by
      
      $e = \sum_{j=0}^{\tau-1} L_j E_j$
      
      $= (255, 255, 255, 255, 255, 5, 98)$
   d) the decoding output is
      
      $\hat{x} = x = (36, 28, 200, 56, 228, 208, 5, 98)^T$. 