Letter to the Editor on the Paper ‘Variable Horizon Robust Predictive Control via Adjustable Controllability Sets’ by M.N. Demenkov and N.B. Filimonov

1. Letter by B. Kouvaritakis

This letter has been prepared in response to an invitation by the European Journal of Control for an item which was to appear in a Discussion Session concerning on the paper ‘Variable Horizon Robust Predictive Control via Adjustable Controllability Sets’ by Demenkov and Filimonov. I found this paper particularly hard to read because its English is poor, its style convoluted and unhelpful, the language ambiguous, and its contents technically unsound. The criticisms about the English, style and language are quite apparent and need no further elaboration, hence this letter will concentrate on the question of technical correctness with particular reference to the claim of closed-loop stability.

The starting point of the work is the steepest descent control (SDC) proposed in the Bulletin of Bauman, Moscow State Technical University (in Russian) in which a positive definite function is minimized over a range of input horizons, and the first element of the input sequence that achieves the minimum of the infima in the shortest time is implemented. It is claimed that under some undisclosed conditions (discussed in an unspecified publication), the minimum of the infima behaves like a Lyapunov function and therefore guarantees closed-loop stability. SDC is further claimed to have a convergence property (as yet unproved but verified by simulation) according to which the state–space trajectories converge to the origin faster than other approaches. The paper claims to extend SDC to systems which are subject to polytopic uncertainty in a manner that guarantees closed-loop stability.

It is unfortunately the case that SDC does not guarantee closed-stability for the case with no uncertainty and thus cannot guarantee the stability of uncertain systems. Furthermore, the lack of stability also invalidates the claim about fast convergence which in any case concentrates on the behaviour of a “terminal” state and ignores transient behaviour. To prove that SDC cannot guarantee stability consider the simplest possible case of a two-state linear unconstrained system with no uncertainty and choose $F(x) = \|x\|_2^2$ with an overall control horizon $T = 1$, for which SDC would simply choose the current input $u$ to minimize the norm of the next predicted state:

$$x_{k+1} = Ax_k + Bu_k$$

thereby prescribing the state feedback law

$$u = -\frac{B^TA}{B^TB}x$$

yielding the closed-loop dynamics

$$x_{k+1} = A_{cl}x_k, \quad A_{cl} = A - \frac{BB^TA}{B^TB}.$$

It is trivial to show that the eigenvalues of $A_{cl}$ are at

$$\lambda_1 = 0, \quad \lambda_2 = \text{trace}(A) - \frac{BB^TA}{B^TB},$$

the second of which can very easily lie outside the unit circle thereby indicating instability, i.e. divergence rather than fast convergence. Similar arguments could be constructed for system of any order $n$. It is of course true that for $T \geq n$, SDC will be identical to deadbeat control and as such will be closed stable but this would not be true in the constrained case. It is very easy to construct examples which illustrate instability,
and the model below is selected so that instability can be proved explicitly for given initial conditions:

$$A = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & 2 \\ -1 & 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

for which the first eigenvector \( w = [2 \ -1 \ -1]^T \) is chosen to be orthogonal to both \( B \) and \( AB \). Then, it is trivial to show (analytically) that for \( T=2 \) in the unconstrained case SDC will lead to instability for \( x_0 = w + A^{-1}B\gamma \) for any scalar \( \gamma \); it also leads to instability for a range of different initial conditions. Of course in the unconstrained case, setting \( T=3 \) removes this problem, but this does not hold true when the input is limited: for example, SDC goes unstable for a control horizon \( T=3 \) and an input limit \( u_{\text{max}} = 5, \gamma = 1 \), as it does for a wide range of initial conditions, input horizons and input limits.

2. Answer of the Author M.N. Demenkov

I accept the criticism from Basil Kouvaritakis concerning the quality of our paper in terms of the English, style and language. These are definitely our “Achilles’ heel”. But I am absolutely disagree with Kouvaritakis in terms of the technical correctness. The main argument of Basil Kouvaritakis formulated in the letter (“... SDC does not guarantee closed stability for the case with no uncertainty and thus cannot guarantee the stability of uncertain system”) is not correct. In our paper the stability guarantee for the uncertain case has been proofed totally independently on the case with no uncertainty. Moreover, the numerical counterexamples presented in the letter are technically inconsistent with formulated problems. Two given examples under selected control constraints are not stabilizable at all with taken initial conditions by any possible control law. The system from the last example is treated by Kouvaritakis as unstabilizable by our control strategy, although it is not a case. The system is actually stabilizable via SDC with the horizon and control constraints considered in the counterexample and this is shown hereafter.

2.1. Stability of Uncertain Systems

Kouvaritakis argues that our results are not applicable to uncertain systems, because the results are not applicable to the case with no uncertainty. This is the main argument against our paper and it lies in the way of common sense. However, our proof is in principle based on the properties of uncertain systems and it fails in the case of systems with no uncertainty. As a result we could not proof in our paper the closed-loop system stability without uncertainty.

In our design method, the available information about system parameters always affect the parameters of the designed controller. It looks rather paradoxical, but our proof of stability can guarantee that the trajectories of an uncertain closed-loop system converge to origin, while this proof cannot guarantee asymptotic stability when information about uncertainty is not available. I mean here the case when a system with no uncertainty is closed by controller, which is designed taking into account the information only about this (“certain”) system.

The closed-loop stability of an uncertain system is proved only inside some constrained region in state-space and only if this system satisfies the condition \( V(x) > 0, \forall x \neq 0 \) in this region. Let me define an uncertain system in the polytopic form (I shall use the notation \( x^{(i)} \) for a vector \( x \) components):

$$x_{k+1} = \sum_{i=1}^{q} \lambda^{(i)}_k (A_i x_k + B_i u_k),$$

$$\sum_{i=1}^{q} \lambda^{(i)}_k = 1, \quad \lambda^{(i)}_k \geq 0,$$

then (in rather informal style) the cost function is defined as

$$V(x_k) = \min_{j=1,T} \min_u \max_{\lambda} F(x_{k+1}),$$

where \( F(x) \) is a positive definite terminal weighting function (actually, we use a polyhedral Minkowski function).

For a “certain” system

$$x_{k+1} = Ax_k + Bu_k,$$

$$V(x_k) = \min_{j=1,T} \min_u F(x_{k+1}),$$

in this case \( V(x) = 0 \) in some nonzero region containing origin, and our stability proof is not applicable.

Given the current state \( x_k \), the first element of the shortest input sequence that determines the value of \( V(x_k) \) is implemented as the current input \( u_k \). The fact is that \( V(x) \) for the “certain” case is different from \( V(x) \) for the uncertain case.

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For an uncertain system, the nonzero states for which \( V(x) = 0 \) may not exist at all. This situation is possible because computation of the cost function is performed using only available information about plant parameters, but not the actual parameters of the plant. In general, you cannot predict exactly where the next plant state will be due to unknown disturbances in the plant parameters, even if you know exactly the current state and input. That is why you may be unable to minimize the terminal weighting function up to zero value – the algorithm described in our paper tries to achieve this minimum for the worst case.

The stability in the uncertain case is based on the following assumptions:

1. Some region in state–space exists, which is positively invariant for the closed-loop system.
2. In this region, \( V(x) > 0 \), \( \forall x \neq 0 \).
3. In this region, \( V(x_{k+1}) < V(x_k) \).

Given a polyhedral function \( F(x) \), the Algorithm 2 described in our paper can be used to get a region, which satisfies the first assumption. Also, this region is constructed so as to satisfy the third assumption if the second is satisfied. The second assumption must be checked independently. It is satisfied if you cannot steer the current state to origin in one step despite parameters variation – see Lemma 8 in our paper. If this region has nonzero volume and the second assumption is fulfilled, then the stability is guaranteed for any initial condition taken in this region.

The second assumption can be verified considering only unconstrained case (we can always scale the control and the state so as to satisfy constraints). It is trivial to show that an uncertain system states, which can be steered to origin in one step by means of control input \( u \), must satisfy the following equalities:

\[
A_i x + B_i u = 0, \quad i = 1, q.
\]

Hence, \( V(x) \) is a positive definite function if the following matrix has empty null space:

\[
M = \begin{bmatrix}
A_1 & B_1 \\
\ldots & \ldots \\
A_q & B_q
\end{bmatrix}
\]

i.e. its rank is greater than or equal to \( n + m \), where \( n \) is the system order and \( m \) is the number of inputs. It is clear that for a “certain” system, the rank of \( M \) is always less than \( n + m \).

Note that not all uncertain systems satisfy this condition. Let consider the following system:

\[
A_1 = \begin{bmatrix}
0.8 & 1 \\
0 & 1
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
1.2 & 1 \\
0 & 1
\end{bmatrix},
\]

\[
B_1 = B_2 = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}.
\]

In this case,

\[
\text{rank} \left( \begin{bmatrix}
A_1 & B_1 \\
A_2 & B_2
\end{bmatrix} \right) = 3
\]

while \( n + m = 5 \). For example, the state \( x = [0 \ 0.5774]^T \) can be steered to origin using the control \( u = [-0.5774 -0.5774]^T \) despite uncertainty. So for this system, the stability cannot be guaranteed.

Setting

\[
B_1 = B_2 = \begin{bmatrix}
0 \\
1
\end{bmatrix}
\]

in the system definition given above permits to prove that \( V(x) \) is positive definite, in this case the rank of \( M \) is equal to \( n + m \). This system is used as the example in our paper.

The fact that the stability cannot be guaranteed in this case for the way without uncertainty is not our secret, discovered by Kouvaritakis, but it directly follows from our Lemma 8 and Theorem 5 stated in the paper. It must be admitted that we missed the explanation of this fact in our paper, and that is why the questions arise about applicability in the case with no uncertainty. We have discussed in our paper just the case when \( V(x) \) is positive definite. I did not intend to say that the stability when \( V(x) = 0 \) cannot be granted, but if it is possible, it must be proved in another way not considered in our paper.

Let me take note of the case when controller was designed for some uncertain system, but the controlled system contains no uncertainty. If the parameters of this “certain” system lie inside the limits for the uncertain system parameters, the closed-loop system is guaranteed to be asymptotically stable. In order to construct controller for a “certain” system, you have to put some uncertainty around nominal plant so as to satisfy the condition \( V(x) > 0, \forall x \neq 0 \) in some region containing origin, and all will be fine.

### 2.2. Time-optimality and Stability of Systems with no Uncertainty

Kouvaritakis remarks that we claim SDC to be faster (in term of convergence) than other approaches for “certain” systems. In contrary, we do not claim in the paper that SDC being applied to systems with no uncertainty has a convergence property according to which the
state-space trajectories always converge to origin faster than other approaches. Of course, for these systems the time-optimal control method is known (see e.g. [1]), and our method is not the time-optimal while \( V(x) > 0 \).

However, for any “certain” system a null-controllable region exists for which \( V(x) = 0 \), and in this region SDC is guaranteed to be stable. When \( V(x) = 0 \), SDC would pick the shortest input sequence that steers the current state to origin, so the stability follows. SDC operates inside this region in the same way as the time-optimal law. As the horizon length increases, the volume of null-controllable set should increase too. So, for a “certain” system, the method is always applicable in this region where it can be regarded as a variant of the time-optimal one.

Consider, e.g. the following second-order system:

\[
A = \begin{bmatrix} 1.2 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

It is stated by Kouvaritakis that in unconstrained case for \( T = 1 \) and \( F(x) = ||x||^2 \),

\[
u_k = Kx_k, \quad K = \frac{B^T A}{B^T B},
\]

and for the given system,

\[
K = \begin{bmatrix} 0 & -1 \end{bmatrix}.
\]

The eigenvalues of \( A + BK \) are \( \lambda_1 = 0 \) and \( \lambda_2 = 1.2 \), so the closed-loop system is unstable. Kouvaritakis agrees to the fact that for \( T \geq 2 \) the closed-loop system will be stable, but he has doubt about constrained case.

In contrary, one can compute the set of states for which \( V(x) = 0 \) using \( T = 3 \) and \( u_{\text{max}} = 1 \). Please make sure that this set is nonzero. In Fig. 1, this set is shown together with some trajectories of the closed-loop system. Using adjustable controllability sets considered in our paper, SDC operates in this null-controllable region exactly as the time-optimal law described in [1].

In the case of uncertainty, in the paper we do not claim, but guess that convergence is fast enough. It is not a theorem. It may be wrong, and our scope of the available control design methods may be too limited.

### 2.3. Refutation of Counterexamples

Kouvaritakis has three numerical examples proposed in order to prove instability of SDC in the case of “certainty”. As is shown above, these examples are not the counterexamples for our paper. Nevertheless, it is interesting to look inside these examples, in order to better understand our stability results.

Our stability results are always connected with some positively invariant constrained region containing origin. Our Algorithm 2 can help to find this region. If this region just not exists, the stability is not proved. If this region exists, the stability guaranteed only for initial conditions in this region, and not for any point in state-space. In the “certain” case, the trajectories emanated in the region, which is constructed by Algorithm 2, never leave this region, but

![Fig. 1. Null-controllable set (where stability is guaranteed) for SDC with \( T = 3 \) and \( u_{\text{max}} = 1 \). Here, \( x^{(1)} \) represents the first, and \( x^{(2)} \) the second component of state vector.](image-url)
this property by itself cannot guarantee the convergence to origin. The invariance property in this case can be established using our proof for uncertain systems.

We do not claim in the paper that such a region always exists. We do not claim in the paper that the stability results hold for any possible horizons, control constraints, etc. Kouvaritakis cannot prove that in unconstrained case with $T \geq n$ (here $n$ is system order) SDC goes unstable. His unconstrained examples with a too short horizons proves nothing – of course, region of the closed-loop stability depends on the control horizon length.

It is much more interesting to look inside the constrained case. The way how Kouvaritakis tries to disprove our results (by examples) in the presence of control constraints can be used even in the uncertain system case with the polyhedral function $F(x)$. That is why I consider this way here, while our paper is exactly about the last case, not about the case of systems with no uncertainty and the quadratic functions.

The “disproving” arguments are like these: if some initial conditions exist, for which method does not work, then this method is not applicable at all. It is not true. Closed-loop stability and even controllability of an open-loop system can be local only. Consider, e.g. the following first-order open-loop unstable continuous system:

$$\dot{x} = x + u,$$

where $x \in \mathbb{R}^1$. The stability condition is $x + u < 0$ for $x > 0$ and $x + u > 0$ for $x < 0$. If $|u| \leq 1$, it is trivial to show that the controllability region of origin is defined by $|x| < 1$. For $|x| > 1$, it is absolutely impossible to drive the current system state towards origin – this state goes to infinity irrespective of the current input. But, it is possible to provide the system with constrained stability region, which is equal to controllability one, using the following control law:

$$u = kx, \quad k < -1.$$

This is the simple example given here for illustrating purposes. The same way of reasoning is applicable to a discrete system (see e.g. [2]). An open-loop unstable discrete system of any order with the constrained control has limited controllability region for the origin. This region is defined as

$$C = \{ x_0 \in \mathbb{R}^n \mid \exists u_k \in U, \lim_{k \to \infty} x_k = 0 \}.$$

Here, $U$ represents control constraints.

Note that all three third-order systems proposed by Kouvaritakis are unstable ones. Therefore, in the presence of control constraints they are not stabilizable from initial states those are outside of controllability region, by any possible controller in open- or closed-loop fashion. The fact that there is a range of initial conditions for which constrained system goes to infinity cannot disprove the local stability.

Let me examine the first Kouvaritakis counter-example:

$$A = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & 2 \\ -1 & 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$  \(^\text{3}\)

The matrix $A$ has exactly two unstable eigenvalues, and one stable. Using the Schur decomposition [3], one can obtain the following second-order unstable subsystem:

$$z_{k+1} = A_z z_k + B_z u_k, \quad z_k = Z x_k,$$

where $x_k$ and $u_k$ are the state and the control of the “full” system. The following numerical results were obtained using Robust Control Toolbox [4] with MATLAB\(^\text{3}\) 5.3 (namely the function blkrsch):

$$\begin{bmatrix} A_z | B_z \end{bmatrix} = \begin{bmatrix} 1.6348 & 1.0397 & 1.6528 \\ 0.4634 & 4.5103 & 0.5126 \end{bmatrix},$$

$$Z = \begin{bmatrix} 0.7719 & 0.5304 & 0.3506 \\ -0.5762 & 0.3506 & 0.7383 \end{bmatrix}.$$

In order to stabilize the “full” system, one may stabilize only this unstable subsystem [5].

The predicted state of the subsystem can be expressed as

$$z_N = A_z^N z_0 + \sum_{k=0}^{N-1} A_z^{N-k-1} B_z u_k.$$  \(^\text{3}\)

Let me suppose that $z_N = 0$. All stabilizable states $z_0$ can be expressed as

$$z_0 = -\sum_{k=0}^{N-1} A_z^{-k-1} B_z u_k,$$

where $N \to \infty$.

Note that $A_z$ has only unstable eigenvalues, hence it is not necessary to consider $N = \infty$. In this example, all elements of the $A_z^{-k}$ become negligible (less than $10^{-10}$) for $k \geq 50$. For all practical purposes, $N = 50$ is enough.

\(^3\text{MATLAB is a trademark of The MathWorks Inc.}\)
How one can determine the controllability region, viz. the set of all $z_0$ in the presence of control constraints? This controllability region can be computed as the Minkowski sum \([6]\) of a set of line segments defined by $\pm A^{-k}B_zu_{\max}$ for $k = 1, N$. Some aspects of controllability region construction via set operations, such as Minkowski set addition and subtraction, are described in \([7]\).

More precisely, the Minkowski sum of two sets $X$ and $Y$ in $R^d$ is

$$X + Y = \{x + y| x \in X, y \in Y\}.$$ 

The controllability set

$$C = \sum_{k=1}^{N} LS(k),$$

where line segment $LS(k)$ is defined as

$$LS(k) = \{\lambda A^{-k}B_zu_{\max} \mid -1 \leq \lambda \leq 1\}.$$ 

The controllability region computed as the Minkowski sum with $N = 50$ and $u_{\max} = 5$ is shown in Fig. 2. Let me now graphically compare this region with the line $x_0 = w + A^{-1}B\gamma$ (where $w = [2 -1 -1]^T$) for which the point having $\gamma = 1$ is claimed by Kouvaritakis as unstabilizable via SDC with $u_{\max} = 5$ and $T = 3$. In the state–space of unstable subsystem, this line is expressed as

$$z(\gamma) = Zx_0 = Zw + ZA^{-1}B\gamma.$$ 

One can see in Fig. 2 that this line has no points of intersection with the controllability region. Hence, it is impossible to steer any point of this line to origin using $u_{\max} = 5$. For any $\gamma$, these states are not stabilizable neither using SDC, nor via any other possible control law. So it is incorrect to claim that if SDC cannot stabilize these states, then SDC does not work at all.

I found that the same situation holds in the second counterexample:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & 1.4857 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$ 

In this case, the $A$ matrix still has two unstable eigenvalues and one stable eigenvalue. It is not difficult to isolate the following second-order unstable subsystem:

$$Az = \begin{bmatrix} 2 & 0 \\ 0 & 1.4857 \end{bmatrix}, \quad B_z = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$ 

$$Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$ 

In Fig. 3, the controllability region computed for $u_{\max} = 5$ is shown and the line $z(\gamma)$ is drawn in parallel to region boundary. Although this line is very close with the boundary, it is still has no points of intersection. The point having $\gamma = 1$ is clearly outside of the controllability region, hence this point cannot be stabilized at all.

Fig. 2. The first counterexample of Kouvaritakis: the states $x_0 = w + A^{-1}B\gamma$ are not stabilizable for any $\gamma$ by any possible control law utilizing $u_{\max} = 5$. The point having $\gamma = 1$ is marked by circle.
Fig. 3. The second counterexample of Kouvaritakis: the states \( x_0 = w + A^{-1}B\gamma \) are still not controllable to origin using \( u_{\text{max}} = 5 \). The point having \( \gamma = 1 \) is marked by circle.

Let me consider the third counterexample:

\[
A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & -5 & 3 \\ 1 & 4 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]

It is the different case from the first and the second, because the matrix \( A \) has only one unstable and two stable eigenvalues. In the same way, the following first-order open-loop unstable subsystem can be isolated:

\[
z_{k+1} = a_z z_k + b_z u_k, \quad z_k = Zx_k,
\]

where \( a_z = -6.8917, \quad b_z = -0.5359, \quad Z = [-0.1704 \ldots -0.8551 \ldots 0.4896] \). The controllability region is in interval form

\[-z_{\text{max}} \leq z_0 \leq z_{\text{max}},\]

and its computation is quite simple:

\[
z_{\text{max}} = \max_{\{u_k\}_{k=0}^{N-1}} z_0 = \sum_{k=1}^{N} a_z^{-k} b_z \text{sign}(a_z^{-k} b_z) u_{\text{max}}.
\]

For \( N = 50 \), I found \( z_{\text{max}} = 0.4548 \), and as in the first and the second examples, \( a_z^{-k} \) is negligible for \( k \geq 50 \). In the state–space of the “full” system, this interval corresponds to the stratum bounded by two planes (see Fig. 4), which is defined as

\[-z_{\text{max}} \leq Zx \leq z_{\text{max}}.
\]

I found that the point \( x_0 = w + A^{-1}B = [3 \ 1 \ 3]^T \) belongs to the controllability region. Since \( V(x_0) = 0 \), the SDC with \( T = 3 \) and \( u_{\text{max}} = 5 \) can stabilize this point in a minimum time fashion using the following input sequence:

\[
\{u_k\}_{k=0}^{2} = \{-1.375, \ -0.375, 0.125\}.
\]

Not all points \( x_0 = w + A^{-1}B\gamma \) are controllable to origin. By substituting \( x_0 \) to the inequalities, which define controllability region, one can find the range of \( \gamma \) for which stabilization is possible:

\[
\frac{-z_{\text{max}} - Zw}{ZA^{-1}B} \leq \gamma \leq \frac{-z_{\text{max}} - Zw}{ZA^{-1}B}.
\]

In conclusion, I wish to sincerely thank Basil Kouvaritakis for the opportunity of discussing here some interesting properties of uncertain and unstable constrained linear systems. The original software for MATLAB 5.3 used here for computations is available from the author by request.

3. Answer of the Author N.B. Filimonov$^4$

I undoubtedly agree with Prof. Kouvaritakis concerning the not quite perfect translation of our paper.

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into English as it makes difficult to read it. However, it is not possible to agree to the remarks concerning the essence and the scientific value of the results presented in the paper. We have concentrated our attention only on the algorithmic features of predictive control strategy applied to dynamic systems with parametric indeterminacy in the discussed paper. In this connection, the most important questions of validity of the idea itself of the given control strategy and in particular its efficiency for considered class of discrete systems were left out. I thank the editorial staff for opportunity to give the absolute answer and once more to draw the readers attention to the very perspective approach of our developing.

The key argument of the criticism of our paper is the thesis of Kouvaritakis concerning non-efficiency of the considered predictive control strategy not only for the class of linear systems with parametric indeterminacy but even for the class of linear stationary discrete systems with complete parametric determination.

The control strategy for linear discrete dynamic systems with multistep prediction forms the base of the paper. This strategy was suggested for the first time in the paper [8] and got its development in the papers by N.B. Filimonov (see e.g. [9]). In particular, in [10] this strategy was extended to the class of nonlinear discrete dynamic systems.

Let us give account of the essence of this control strategy in short and let us give its necessary ground.

Let the discrete model of a control system dynamics be represented by the vector linear difference equation

\[ x(t + 1) = Ax(t) + Bu(t), \]  
\[ x(0) = x_0, \]  

where \( t \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\} \) is the discrete time; \( x \in \mathbf{X} = \mathbb{R}^n \) is the state vector; \( u \in \mathbf{U} \subseteq \mathbb{R}^r \) is the control vector; \( \mathbf{U} \) is \( r \)-dimensional hyperparallelepiped; \( A \in \mathbb{R}^{nxn} \) and \( B \in \mathbb{R}^{nxr} \) are the coefficient matrices.

Let us consider the problem of constrained stabilization. Let us suppose that the goal state is \( x^* = 0 \). In this connection, we will assume that the controllability region of origin \( G \subseteq \mathbf{X} \) exists: any point in \( G \) is controllable to origin using admissible control sequence.

The idea of integration of the Lyapunov’s functions method, multistep prediction of controlled state moves and extreme control with the method of steepest descent turns out to be very perspective for the class of discrete dynamic systems with constrained control. Discrete control strategy for this class on the base of multistep prediction of system dynamics is the development of this idea.

Let us realize the control of the system via prediction of its state moves for some finite period of time \( T \in \mathbb{Z}_+, T > 0 \) called in the sequel as horizon length. If \( x(t) = x \) is the current state of the system, then its possible controlled state moves \( \xi(t) \) in consequent \( T \) time instants may be described by the following linear
equation:

\[ \xi(t + \tau) = A^\top x + \sum_{\theta=0}^{\tau-1} A^{\theta-1} B v(t + \theta), \]

\[ \tau = 1, T, \]

where \( v \in \mathbb{U}. \)

Let us denote by \( x[t, x_0] \) the trajectory of the system (1) with the initial condition (2), and we will denote by \( \Omega(x, T) \) reachability set of a state \( x \) in \( T \) steps, which is the set of points in \( X \), into which it is possible to steer this state \( x \) in no more than \( T \) steps.

Let us assume that the distance of the current system state \( x \) from the goal state \( x^* \) is being estimated via some convex positive definite function \( Q(x) \):

\[ Q(0) = 0, \quad Q(x) > 0, \quad x \neq 0. \]

We will use it, in the first place, as the goal function for the controller with the extremum at the point \( x^* = 0 \) and, secondly, as the performance criterion for a local controlled state moves.

The essence of strategy of discrete predictive control with multistep prediction reduces to the following.

At every current time instant, the system state moves are being directed towards \( T \)-reachable state having the minimum value of the criterion \( Q \); this value is determined as the solution to the following constrained optimization problem:

\[ V(x) = \min_{\xi \in \Omega(x, T)} Q(\xi) \]

\[ = \min_{v[t, t+T-1]} \min_{\tau \in [1, T]} Q(\xi(t + \tau)), \quad (3) \]

where \( V(x) \) is the cost function induced by the goal one and \( v[t, t+T-1] = \{ v(t + \theta) \in \mathbb{U} | \theta = 0, T-1 \} \) are the predicted control moves.

Then, aiming point \( \bar{x} \) situated on the extreme trajectory is being determined:

\[ Q(\bar{x}) = V(x) \quad \text{for} \quad V(x) > 0, \]

\[ \bar{x} = 0 \quad \text{for} \quad V(x) = 0. \]

The desired state on the next step is being directed towards aiming point \( \bar{x} \):

\[ u(t) = v(t). \]

It is possible that the solution to the problem is not unique, i.e. there are different trajectories, which steer system state to the same aiming point or there are a number of aiming points with equal values of the goal function. In this case, among alternative extreme trajectories the shortest one is implemented, which is the trajectory driving system to the aiming point quicker than others.

As far as the movement to the minimum value of the goal function in limits of prediction horizon is being formed in controller then the described control strategy actually realizes the method of steepest descent, which is well-known among numerical methods of optimization.

Let us introduce the following notations: \( D_c \) (where \( c \geq 0 \)) is a level set of the function \( V(x) \):

\[ D_c = \{ x \in X | V(x) \leq c \}, \]

\[ D_H \setminus D_0 = \{ x | x \in D_H, x \notin D_0 \}, \]

\( \Phi^\tau : X \rightarrow X \) is the state transformation that transforms an initial state in the closed-loop system to its future state in \( \tau \) steps:

\[ x(t + \tau) = \Phi^\tau x(t). \]

The theoretical justification of the efficiency of the given control strategy is the following theorem.

**Theorem.** Let us consider the closed-loop system that is defined as the system (1) being controlled by means of chosen control strategy, i.e. the first element of the shortest input sequence that minimizes goal function according to (3) is implemented as the current input. Let us suppose \( T \geq 2 \) and let the following sets exist:

\[ D_h \subset D_H \subset X, \quad 0 \leq h \leq H, \]

so that

\[ \sup_{x \in D_H \setminus D_0} (V(x) - Q(x)) < 0, \quad (4) \]

and all trajectories emanated from \( D_h \) does not leave \( D_H \) in \( T \) steps:

\[ \Phi^\tau D_h \subset D_H, \quad \tau = 1, T. \]

Then, for any initial state, \( x(0) \in D_h \):

(a) the set \( D_h \) is positively invariant for the closed-loop system:

\[ x(t) \in D_h \quad \text{for} \quad t > 0, \quad (5) \]

and the sequence of the cost function values is non-increasing along any trajectory of the closed-loop system:

\[ V(x(t)) \leq V(x(t-1)); \quad (6) \]

(b) while system state is in \( D_H \setminus D_0 \), the value of the cost function decreases in no more than \( T \) steps:

\[ V(x(t)) < V(x(t-T)) \quad \text{for} \quad t \geq T \quad \text{and} \quad x(t) \in D_H \setminus D_0. \quad (7) \]
Proof. Let us come to proof of the proposition (a) of the theorem. According to (3), the equality

$$Q(\hat{x}) = \min_{\xi \in \Omega(x, T)} Q(\xi) = V(x),$$

is valid, moreover using the hypothesis of the theorem, we have

$$V(x) < Q(x) \quad \text{for } x \in D_H \setminus D_0.$$

So far as

$$V(x) = 0 \quad \text{for } x \in D_0,$$

and $Q(x) \geq 0$, it is clear that

$$V(x) \leq Q(x) \quad \text{for } x \in D_H. \quad (8)$$

Consider the time instant $t = t_0$ and let $x(t_0) \in D_H$. Two cases of the closed-loop system behaviour for $t \geq t_0$ are possible.

The first case. At the time instant $t = t_0 + 1$, the system state coincides with the aiming point:

$$x(t_0 + 1) = \hat{x}(t_0). \quad (9)$$

Then, from the control strategy (3), we have

$$V(x(t_0)) = Q(\hat{x}(t_0)) = Q(x(t_0 + 1)). \quad (10)$$

According to (8),

$$V(x(t_0 + 1)) \leq Q(x(t_0 + 1)).$$

From the last inequality and the equality (10), it follows that

$$V(x(t_0 + 1)) \leq V(x(t_0)) \leq h. \quad (11)$$

The second case. At the time instant $t = t_0 + 1$, the system state does not coincide with the aiming point, i.e.

$$x(t_0 + 1) \neq \hat{x}(t_0). \quad (12)$$

However, the state $x(t_0 + 1)$ belongs to a system trajectory that is connected with $\hat{x}(t_0)$. It means that the state $x(t_0 + 1)$ can be steered to $\hat{x}(t_0)$. Hence,

$$\hat{x}(t_0) \in \Omega(x(t_0 + 1), T)$$

and

$$V(x(t_0 + 1)) = \min_{x \in \Omega(x(t_0 + 1), T)} Q(x) \leq Q(\hat{x}(t_0)).$$

Taking into account the following equality:

$$V(x(t_0)) = Q(\hat{x}(t_0)),$$

we again derive the inequality (11), i.e.

$$x(t_0 + 1) \in D_h.$$

Therefore, for any time instant $t \geq 0$,

$$V(x(t)) \leq h,$$

i.e. $x(t) \in D_h$, and the inequality (6) holds.

This completes the proof of the proposition (a).

Let us come to proof of the proposition (b) of the theorem.

Let us take into account the condition (5). Pick any time instant $t_1 \geq T$ and suppose $x(t_1) \notin D_0$. Then, from (6), $x(t) \notin D_0$ at $t = t_0, t_0 + 1, \ldots, t_1$, where $t_0 = t_1 - T$.

In the same way, here two cases of the closed-loop system behaviour for $t \geq t_0$ are possible.

The first case. At the next time instant $t = t_0 + 1$, the system state coincides with the aiming point, i.e. the condition (9) is satisfied.

From (9) and (4), we have

$$V(x(t_0 + 1)) - V(x(t_0)) = V(x(t_0 + 1)) - Q(x(t_0 + 1)) < 0.$$

But according to (6),

$$V(x(t_1)) \leq V(x(t_1 - 1)) \leq \cdots \leq V(x(t_0 + 1)).$$

Hence,

$$V(x(t_1)) < V(x(t_0)). \quad (13)$$

If we substitute $t$ in place of $t_1$, we obtain (7). The second case. Suppose that at the time instant $t = t_0 + 1$, the system state does not coincide with the aiming point, i.e. (12) holds.

Two situations are possible here.

First situation. The value of the cost function does not change over $T - 1$ next steps:

$$V(x(t)) = V(x(t_0)),$$

where $t = t_0 + 1, \ldots, t_0 + T - 1$.

Then, at the time instant $t_1 = t_0 + T$, the state coincides with the aiming point, i.e. the first case is realized and therefore inequality (13) follows.

Second situation. At the time instant $t = t_0 + \tau + 1$, where $\tau \in \{0, \ldots, T - 1\}$, the value of the cost function is decreased:

$$V(x(t_0 + \tau + 1)) < V(x(t_0)). \quad (14)$$

But according to (6),

$$V(x(t_1)) \leq V(x(t_1 - 1)) \leq \cdots \leq V(x(t_0 + \tau + 1)).$$
From these inequalities and from (14) we obtain (13). Again, substituting $t$ in place of $t_1$ we have obtained (7). This completes the proof of the proposition (b) and hence of the theorem. \hfill $\Box$

**Remark.** Note that in the case

$$x(t) \in D_0,$$

the aiming point will coincide with the origin

$$\hat{x} = x^* = 0.$$

It is clear that at every time instant, the length of system trajectory is decreased per unit. As a result, the state can be steered to origin in no more than $T$ steps.

As far as $V(x(t))$ is monotonically decreasing function then outside of the set $D_0$ it serves as the Lyapunov’s function for the closed-loop system.

It is possible to show that if the system is stabilizable then for any choice of the goal function $Q$ and for a horizon length $T \geq n$, the hypothesis of theorem will be fulfilled. And what is more for $T$ that is large enough the stability region of the closed-loop system will approximate the controllability region $G$. In deciding on horizon length $T$ it is well to bear in mind that for every control problem in accordance with the systems specific characteristic its own “sufficient horizon length” exists so that for its further increase the discovered prognostic control practically is not changed.

Such an efficiency of the described control strategy with multistep prediction is guaranteed at least for $T \geq n$, in doing so, the required efficiency of control process is provided by the proper choice of a horizon length $T$.

It should be noted that so far as set $D_0 = \{x \in X \mid V(x) \leq 0\}$ includes all those states of system which it is possible to steer to the origin $x^* = 0$ not more than in $T$ steps, then for $x(0) \in D_0$ the described control strategy is time-optimal.

In conclusion, let me point to the important advantage of the algorithmic realization of the described strategy of predictive control: in deciding on the polyhedral goal function $Q(x)$, the optimization problem, which is being solved by controller at every time instant, reduces to the linear programming problems. It should be separately emphasize that linear structure of the given optimization problem is being reserved while introducing the polyhedral constraints on the trajectories of controlled system state moves.

**References**