Independence and Domination Separation on Chessboard Graphs

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Abstract

A *legal placement* of Queens is any placement of Queens on an order N chessboard in which any two attacking Queens can be separated by a Pawn. The *Queens independence separation number* is the minimum number of Pawns which can be placed on an $N \times N$ board to result in a separated board on which a maximum of m independent Queens can be placed. We prove that $N + k$ Queens can be separated by k Pawns for large enough N and provide some results on the number of fundamental solutions to this problem. We also introduce separation relative to other domination-related parameters for Queens, Rooks, and Bishops.

1 Introduction

The well-known 8-Queens problem involves placing 8 Queens on an 8×8 chessboard such that no two Queens attack each other. A collection of some references to the N-Queens problem can be found in [15], [18] includes a variety of other interesting chessboard problems, and a storage scheme for parallel memory systems based on solutions to the N-Queens problem is given in [10]. Various algorithms for solving the N-Queens problem are given in [1, 8, 9]. We primarily use the notation and terminology of [12, 13].

In January 2004, the *Chess Variant Pages* [2] proposed a variation of the traditional 8-Queens problem. The new problem is to place nine Queens on an 8×8 board by using Pawns to block all Queens that would otherwise attack each other. An example in which three Pawns were needed to separate the nine Queens was provided. A contest problem was to determine the *least* number of

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Pawns needed to separate these nine Queens. We say the chessboard (graph) resulting from the addition of one or more Pawns is a *separated* board (graph).

A *legal placement* of Queens is defined in [5] to be any placement in which any two attacking Queens can be separated by a Pawn. The *Queens independence separation number*, $s_Q(\beta, m, N)$ is the minimum number of Pawns which can be placed on an $N \times N$ board to result in a separated board on which a maximum of m independent Queens can be placed. More generally, for any set of natural numbers A and graph parameter π defined on the Queens graph, we define $s_Q(\pi, A, b)$ to be the minimum number of Pawns on a $b \times b$ board which causes the value of π to be an element of the set A. The *upper separation number* relative to a given parameter, $S_Q(\pi, A, b)$, is the maximum cardinality of a minimal set of Pawns that results in π taking a value in A.

We consider $s_Q(\pi, A, b)$ for $\pi = \gamma, \beta, i$, where γ is the domination number, β is the independence number, and i is the independent domination number. If $A = \{a\}$ and no confusion will arise, we denote $s_Q(\pi, A, b)$ by $s_Q(\pi, a, b)$ rather than $s_Q(\pi, \{a\}, b)$. Similarly, $s_Q(\pi, [x, y], b)$ will be used when $A = \{[x, y] : x \in \mathbb{N} \cup \{0\}, y \in \mathbb{N}\}$.

It is proved in [5] that $s_Q(\beta, N + 1, N) = 1$ for $N > 5$ and conjectured that $s_Q(\beta, N + k, N) = k$ for large enough N. We prove this conjecture, namely that if $N > \max\{87 + k, 25k\}$ then $N + k$ Queens can be safely placed on the board by using k Pawns to separate the Queens. We also consider these values for other chess pieces such as Bishops ($s_B(\pi, A, b)$) and Rooks ($s_R(\pi, A, b)$).

2 Domination separation

A set S of vertices in a graph is a *dominating* set if every vertex in the graph is either in the set S or is adjacent to a vertex in S. For a graph G, the minimum number of vertices in a dominating set is denoted $\gamma(G)$. A set of vertices S is *independent* if no two vertices in S are adjacent. The maximum cardinality of a maximal independent set is denoted $\beta(G)$ while the minimum cardinality of a maximal independent set is denoted $i(G)$. It is well known [12] that an independent set is maximally independent if and only if it is dominating, so $i(G)$ is also called the *independent domination number*.

We consider domination separation for Queens, Bishops, and Rooks. We begin with a useful lower bound for the number of Pawns needed to reduce the domination number of a given chessboard graph. For any chess piece Π, let Π_N denote the order N graph associated with that piece.

Lemma 1 *For any chess piece Π and $\gamma(\Pi_N) > k$, $s_\Pi(\gamma, \gamma(\Pi_N) - k, N) \geq k$.*

Proof. It is shown in [4] that removal of a single vertex from a graph cannot decrease the domination number by more than 1, and removal of an edge alone never decreases the domination number.

Since placing a Pawn has the same effect as removing one vertex and some edges, the result follows inductively. ■

2.1 The Queens graph

The domination number $\gamma(Q_N)$ is the minimum number of Queens needed to ensure that every square of an $N \times N$ board is either occupied or under attack. It can be shown that five Queens are necessary in order to dominate an 8×8 board. While the value of $\gamma(Q_N)$ is unknown for all but small values of N, the current best upper bound for the domination number of the Queens graph is $\gamma(Q_N) \leq (101N/195) + O(1)$ and is due to Burger and Mynhardt [3].

A computer search of all possible cases for $n \leq 9$ gives the following proposition.

Proposition 2 *For $N \leq 9$, one Pawn never decreases γ on a separated Queens graph, and one Pawn increases γ on a separated Queens graph only if $N = 3$ or 6.*

It can be deduced from an example on page 15 of [18] that it is possible to decrease the domination number of a separated Queens graph by using two Pawns. This gives rise to the following two open questions.

Open Question 1 *For which size boards will two Pawns decrease γ?*

Open Question 2 *Can two Pawns decrease γ by two?*

2.2 The Bishops graph

Values for independence in the Bishops graph were given in [5].

Proposition 3 ([5]) *Let $N > 0$ be given.*

1. *For $N \geq 3$, $s_B(\beta, 2N-1, N) = 1$.*
2. *For odd $N \geq 3$, $s_B(\beta, 2N, N) = 1$.*

Since it is shown in [7] that $\gamma(B_N) = i(B_N) = N$, it follows that $s_B(\gamma, N, N) = s_B(i, N, N) = 0$.

From [12], the *open neighborhood* $N(v)$ of a vertex v is the set of all vertices adjacent to v while the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. Let S be a set of vertices with $u \in S$. A vertex v is said to be a *private neighbor of u with respect to S* if $N[v] \cap S = \{u\}$.

The proof of the following lemma is modeled after the proof of Theorem 1 in [7], which also appears as Theorem 6 in [6].

Lemma 4 *Let S be a minimum (independent) dominating set of an even order N Bishops graph. If one Bishop attacks no other Bishop, then that Bishop has a private neighbor other than the square it occupies.*

Proof. We prove this by contradiction. Suppose we have a minimum (independent) dominating set with a Bishop b that attacks no other Bishop and has no private neighbors other than its own square. Suppose, without loss of generality, that b is on a white square.

Consider either of the diagonals occupied by b. Since b has no private neighbor other than its own square and since b does not attack other Bishops, every square in the open neighborhood of b must be attacked by another Bishop not in the closed neighborhood of b. Such a Bishop that attacks one square of our chosen diagonal cannot attack another square on that same diagonal. So the white squares must contain at least as many Bishops as there are squares on our diagonal.

If the diagonal has more than $N/2$ squares we have a contradiction, since in a minimum (independent) dominating set on an even order Bishops graph $N/2$ are on black squares and $N/2$ are on white squares. (The Bishops graph has domination number N and is a disjoint union of two isomorphic halves, so each half has domination number $N/2$.) So each diagonal occupied by b has at most $N/2$ squares. So the closed neighborhood of b has at most $2(N/2) - 1 = N - 1$ squares. But this contradicts the fact that every square on the Bishops graph has at least N squares in its closed neigborhood. ■

Proposition 5 *For even N, $s_B(\gamma, N-1, N) \geq 2$ and $s_B(i, N-1, N) \geq 2$.*

Proof. Suppose to the contrary that 1 Pawn is sufficient to reduce the domination number. That is, suppose there is a placement of $N-1$ Bishops that cover all but one square on the board, which contains a Pawn. The square containing the Pawn cannot be attacked by any of the Bishops, else removal of the Pawn would yield a covering of the entire board with only $N-1$ Bishops.

Replacing the Pawn with a Bishop b yields a minimum dominating set with the property that no other Bishop attacks b. By Lemma 4, it must be the case that b has a private neighbor not attacked by any other Bishop. This contradicts the assumption that $N-1$ Bishops can cover all but one square. ■

Proposition 6 *For odd $N \geq 3$, $s_B(\gamma, N-1, N) = s_B(i, N-1, N) = 1$.*

Proof. First note from Lemma 1 that the domination number cannot decrease by more than 1.

Label the columns from $-k$ to k on a $2k+1$ board, and label the rows from 0 to $2k+1$. To reduce the domination number, place the Bishops on column 0 then replace the Bishop on row 0 with a Pawn. The resulting set of $N-1$ Bishops is a minimum independent dominating set. ■

Note that placing a single Pawn at the center of a 3×3 Bishops graph increases the domination number by three to 6.

Proposition 7 *For $N > 3$, the addition of a single Pawn increases the (independent) domination number on a separated Bishops graph by no more than 2.*

Proof. We consider two cases, one in which the Pawn is not at the exact center of the board, and one with the Pawn at the exact center. Note in the second

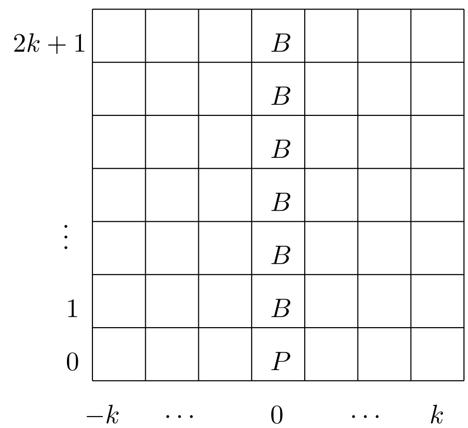

Figure 1: For odd $N \geq 3$, $s_B(\gamma, N-1, N) = 1$.

case that the board must be of odd order.

Case I: Suppose the Pawn is not at the exact center of the board. If the Pawn is not on the column closest to the center, place Bishops on that column. If the Pawn is on the column closest to the center, then rotate the board prior to placing the Bishops. Suppose, without loss of generality, that the Pawn is to the left of the Bishops. The Bishops cover everything to their right while the Pawn blocks at most two diagonals, both of which are to the left of the Pawn. Place Bishops on those diagonals to the left of the Pawn. This yields an independent dominating set with at most $N+2$ Bishops.

Case II: Suppose the Pawn is at the exact center of the board. Label this square as the origin of a rectangular coordinate system. Place Bishops on squares $(0, \pm 1)$, $(-1, \pm 1)$, and $(1, \pm 1)$. For $5 < j \leq N$, place Bishops on $(0, \pm(j-1)/2)$. It is easy to see this yields an independent dominating set of order $N+1$.

Thus, addition of a single Pawn can never increase the (independent) domination number by more than 2. ■

Note that in certain cases the maximum increase of 2 is possible. Number the rows and columns of the Bishops board with row 0 at the bottom and column 0 to the left. Computer searches for odd $N \leq 11$ show that the domination number can be increased to $N+2$ by placing a Pawn in row $\lfloor (N+1)/4 \rfloor$ and column $\lfloor (N+4)/4 \rfloor$.

Conjecture 3 *On an odd order Bishops board with $N \geq 3$, the domination number is increased by 2 if a single Pawn is placed in row $\lfloor (N+1)/4 \rfloor$ and column $\lfloor (N+4)/4 \rfloor$.*

2.3 The Rooks graph

It is shown in [5] that for $N \geq k+2$, k Pawns suffice to increase the independence number of the Rooks graph to $N+k$. We consider the number of Pawns needed to change the domination number. Note that since N Rooks are necessary to dominate an $N \times N$ board [13], it follows that $s_R(\gamma, N, N) = 0$.

Proposition 8 *For $N \geq 2$, $s_R(\gamma, N-1, N) = s_R(i, N-1, N) = 1$.*

Proof. Label the columns from 1 to N and the rows from 1 to N beginning in the upper left of the board. Suppose that a single Pawn is placed in column r, where $1 \leq r \leq N$. Place Rooks in the remaining columns, with the row determined by any permutation of the columns which contain Rooks. ■

Proposition 9 *For $N \geq k$, $s_R(\gamma, N-k, N) = s_R(i, N-k, N) = k^2$.*

Proof. Let $m = s_R(\gamma, N-k, N)$. Suppose we have an order N chessboard with m Pawns and $N-k$ Rooks such that the Rooks dominate those squares not occupied by Pawns. First we note that no Rook attacks any of the Pawns since otherwise we could remove the attacked Pawns, which contradicts the minimality of m.

Next we note that if a row (column) contains p (> 0) Pawns, then there must be at least $N-p$ Rooks on the board, since we need a distinct Rook to cover each of the empty squares in the row (column). Since we want the (independent) domination number to be $N-k$, each nonempty row or column must have at least k Pawns. If $k=0$, we need no Pawns. If $k>0$, we need at least one Pawn so at least one row has a Pawn. That row must have k Pawns, and each of those Pawns must occupy a column with at least k Pawns. So the board must have at least k^2 Pawns.

Finally, we note that k^2 Pawns suffice: Place the k^2 Pawns in the corner formed by the first k rows and k columns, and place the Rooks on the unoccupied squares of the main diagonal. (See Figure 2). ■

Corollary 10 *Placing two or three Pawns on an order N board reduces the Rooks (independent) domination number by at most* 1.

It is possible to increase the domination number on the Rooks graph by adding Pawns. For example, by placing Pawns on all the squares of the second and fourth columns and rows of a 5×5 board, we can increase the domination number from 5 to 9.

Open Question 4 *Under what conditions can adding Pawns increase the Rooks domination number?*

3 Queens independence separation

For an $N \times N$ chessboard, we wish to place $N+k$ independent Queens.

P	P	P						
P	P	P						
P	P	P						
			R					
				R				
					R			
						R		
							R	
								R

Figure 2: For $N = 9$, $s_R(\gamma, N-3, N) = 9$.

Theorem 11 *For each positive integer k and $N > \max\{87+k, 25k\}$, $s_Q(\beta, N+k, N) = k$.*

Proof Sketch. There are seven patterns to consider. Each begins with a known solution [9] to a smaller N-Queens problem. The proof that each pattern holds involves elementary but extremely tedious computations. Let $n = N - k$.

Pattern I. Let $n \equiv 0 \mod 6$, $n \geq 30$, and $k < n/8$.
Solution: Number the rows $-k, \cdots, -1, 0, \cdots, n-1$ and the columns $0, 1, \cdots, n-1+k$. Place the Queens at $(2i+1, i)$ for $i = 0, \cdots n/2-1$, $(2i-n, i)$ for $i = n/2, \cdots, n-1$, $(-i, n-1-2i)$ for $i = 1, \cdots, k$, and $((n-n \mod 12)/4-3+2i, n+i-1)$ for $i = 1, \cdots, k$. Place the Pawns at $((n-n \mod 12)/4-3+2i, n-1-2i)$ for $i = 1, \cdots, k$.

Pattern II. Let $n \equiv 4 \mod 6$, $n \geq 22$, and $k < n/8$.
Solution: Number the rows $-k, \cdots, -1, 0, \cdots, n-1$ and the columns $0, 1, \cdots, n-1+k$. Place the Queens at $(2i+1, i)$ for $i = 0, \cdots n/2-1$, $(2i-n, i)$ for $i = n/2, \cdots, n-1$, $(-i, n-1-2i)$ for $i = 1, \cdots, k$, and $((n-12+n \mod 12)/4-2+2i, n+i-1)$ for $i = 1, \cdots, k$. Place the Pawns at $((n-12+n \mod 12)/4-2+2i, n-1-2i)$ for $i = 1, \cdots, k$.

Pattern III. Let $n \equiv 1 \mod 6$, $n \geq 23$, and $k < (n-1)/8$.
Solution: Number the rows $-k-1, \cdots, n-2$ and the columns $-1, \cdots, n-2+k$. Place the Queens and Pawns as in Pattern I with $n := n-1$. Place an extra Queen at $(-k-1, n-2+k)$.

Pattern IV. Let $n \equiv 5 \mod 6$, $n \geq 23$, and $k < (n-1)/8$.
Solution: Number the rows $-k-1, \cdots, n-2$ and the columns $-1, \cdots, n-2+k$. Place the Queens and Pawns as in Pattern I with $n := n-1$. Place an extra Queen at $(-k-1, n-2+k)$.

Pattern V. Let $n \equiv 2 \mod 6$, $n \geq 56$, and $k \leq n/10$.
Solution: Number the rows $-k, \cdots, -1, 0, \cdots, n-1$ and the columns $0, 1, \cdots, n-1+k$. Place the Queens at $((n-6)/4 + ((n-2) \mod 12)/4 + 2i, n-1+i)$ for $i = 1, \cdots, k$, $(-i, n-8-2i)$ for $i = 1, \cdots, k$, $((2i+4) \mod n, i)$ for $i = 0, \cdots, n/2-1$, and $((2i+7) \mod n, i)$ for $i = n/2, \cdots, n-1$. Place the Pawns at $((n-6)/4 + ((n-2) \mod 12)/4 + 2i, n-8-2i)$ for $i = 1, \cdots, k$.

Pattern VI. Let $n \equiv 3 \mod 12$, $n \geq 87$, and $k \leq n/24$.
Solution: Number the rows $-k, \cdots, -1, 0, \cdots, n-1$ and the columns $0, 1, \cdots, n-1+k$. Place the Queens at $(-i, n-11-3i)$ and $(n/3 - 3 + 3i + ((n-3) \mod 12)/6, n-1+i)$ for $i = 1, \cdots, k$, $((3i+5) \mod n, i)$ for $i = 0, 1, \cdots, n/3-1$, $((3i+9) \mod n, i)$ for $i = n/3, \cdots, 2n/3-1$, and $((3i+13) \mod n, i)$ for $i = 2n/3, \cdots, n-1$. Place the Pawns at $(n/3-3+3i+((n-3) \mod 12)/6, n-11-3i)$.

Pattern VII. Let $n \equiv 9 \mod 12$, $n \geq 87$, and $k \leq n/24$.
Solution: Number the rows $-k, \cdots, -1, 0, \cdots, n-1$ and the columns $0, 1, \cdots, n-1+k$. Place the Queens at $(-i, n-11-3i)$ for $i = 1, \cdots, k$, $(n/3+1, n)$, $(n/3+3i-1, n-1+i)$ for $i = 2, \cdots, k$, $((3i+5) \mod n, i)$ for $i = 0, 1, \cdots, n/3-1$, $((3i+9) \mod n, i)$ for $i = n/3, \cdots, 2n/3-1$, and $((3i+13) \mod n, i)$ for $i = 2n/3, \cdots, n-1$. Place the Pawns at $(n/3+1, n-14)$ and $(n/3+3i-1, n-11-3i)$ for $i = 2, \cdots, k$. ■

The bounds on N given in Theorem 11 are not best possible for all k. By computer searches with $N < 91$, we have been able to show that $N \geq 7$ and $N \geq 8$ are sufficiently large for $k = 2$ and $k = 3$ respectively. From Theorem 11, we see that $N > 25k$ is large enough for $k \geq 4$. However, this bound can most likely be decreased as well.

The following proposition provides some examples for which more than k additional Pawns are needed to increase the independence number by k.

Proposition 12 *Some values of* $s_Q(\beta, m, N)$.

1. (Zhao [19]) $s_Q(\beta, 6, 5) = 3$.
2. $s_Q(\beta, 8, 6) = 3$.
3. $s_Q(\beta, 9, 5) = 16$.

4 An inverse problem

Given a board of order N, consider the maximum number of independent Queens that can be placed on the board if k Pawns have already been placed. Note that the maximum number of Queens that can be placed is no more than $N+k$. In many cases, the maximum may be considerably less.

The following proposition answers a question of Hammons [11] and describes possible locations of Pawns if $N+k$ independent Queens and k Pawns are to be placed.

Proposition 13 *If $N+k$ Queens and k Pawns are placed on an $N \times N$ board so that no two Queens attack each other, then no Pawn can be on the first or last row, first or last column, or any square adjacent to a corner.*

Proof. First we note that a Pawn divides a row or column into at most two independent parts, so p Pawns divide a row or column into at most $p+1$ parts.

Suppose there is a Pawn on the first row. If there are p Pawns on that row, there are at most p Queens in the first row. With $k-p$ Pawns in other rows, we can place at most $k-p+N-1$ Queens in those rows. So we can place at most $p+k-p+N-1 = N+k-1$ independent Queens, which contradicts the fact that there are $N+k$ independent Queens on the board. So no Pawn can be on the first row. Symmetric arguments show there are no Pawns on the last row, or the first or last column.

Now consider the squares diagonally adjacent to a corner. Without loss of generality, suppose a Pawn is in the second row and second column. There must be a Queen in the first column of the second row, else there are at most $N+k-1$ parts of rows to place Queens. Similarly, there must be a Queen in the first row of the second column. However, these two Queens attack each other. Thus, there can be no Pawn in the second row and second column. Symmetric arguments complete the proof. ■

Based on computer searches, we conjecture that for $N \geq 10$ a Pawn can be placed in any square not included in the above proposition in order to allow placement of $N+1$ independent Queens on an $N \times N$ board.

5 Counting fundamental solutions

Our primary focus so far has been to determine whether even a single solution exists for a given N and k. A more interesting and difficult problem arises when we consider the total number of solutions to separation problems. A *fundamental solution* of a chessboard problem is a class of solutions such that all the members of the class are rotations or reflections of one another. Knuth [16] mentions the existence of a method for counting the number of solutions to the N-Queens problem without actually generating the solutions. Note, however, this method is still exponential. We discuss an implementation of a method for generating fundamental solutions of the Queens independence separation problem which is an improvement of traditional backtracking techniques.

5.1 Exact covers and dancing links

Given a set, S, an *exact cover* of S is a set $\{S^1, S^2, \ldots, S^k\}$ of subsets of S where all elements of S are in one and only one subset S^j. A typical problem is to find all exact covers for a given set S under some additional constraints for the subsets. Solutions to this problem are found using backtracking and recursion. Backtracking and recursion require potential solutions to have elements 'appended' and 'removed' when either a solution is found or a constraint violated.

The Dancing Links (DLX) algorithm was first presented in 1979 by Hitotumatu and Noshita [14]; over twenty years later, Knuth [16] named and publicized the technique as an efficient method for implementing backtracking algorithms. In particular, [16] provides applications of DLX for solving various forms of exact cover problems. DLX is a more efficient way to implement backtracking algorithms due to a data structure which, by design, provides faster 'append' (cover) and 'removal' (uncover). The primary goal in using this method is to save computation time rather than space.

The generalized cover problem does not require that all elements of S be in the set of subsets $\{S^1, S^2, ..., S^k\}$. For example, consider the N-Queens problem. The rows of the chessboard require an exact cover with one Queen in each row, as do the columns. The diagonals, however, can have at most one Queen to have a valid solution. Therefore, the N-Queens problem is a generalized exact cover problem.

As defined in [16], a *universe* is a multi-dimensional data structure composed of intersecting double, circularly linked lists. The components of the universe are either a header node, Column Header, or Column Object. The universe is an expanded representation of the $N \times N$ chessboard.

Knuth defines each row of the chessboard as a rank R, the column of the chessboard as a file F, negative diagonal from lower right to upper left as A, and positive diagonal from lower left to upper right as B. As a result, each block of the chessboard is defined by a combination of specific, horizontally linked, Column Objects R, F, A, and B. The rank and file columns are the primary columns and must be covered. The two diagonals are secondary columns and may each contain zero or one Queen.

The header node links together the Column Headers of the universe. There is one Column for each R, F, A, and B. The Column Headers contain the name of the universe Column (e.g., $R1$) and link vertically the number of chessblocks Column Objects.

The Column Objects link horizontally to the other three Columns to define the chessboard block.

5.1.1 N-Queens universe

The N-Queens universe consists of a header node, Column Headers, and Column Objects. The header node is created and linked to itself in the constructor of its class. Then, $2N$ Column Headers are added to represent the N ranks and N

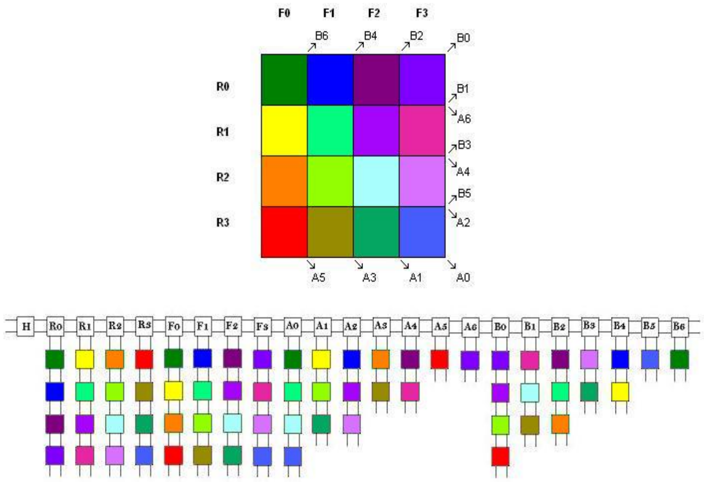

Figure 3: Chessboard and Corresponding Universe

columns. Then $2(2N-1)$ Column Headers are added for the negative diagonal A and positive diagonal B.

The Column Objects are the final object types added. Each rank and file have N Column Objects. The number of Column Objects for A and B is determined by the number of chessblocks contained in the diagonal. For example, if A_0 is defined to be the main negative diagonal, it has N Column Objects. If A_N is the corner diagonal, it contains 1 Column Object.

Figure 3 shows a mapping of a 4×4 chessboard to the Dancing Links universe for solving the N-Queens problem.

0	0	0	0
1	P	2	2
3	3	3	3
4	4	4	4

Figure 4: Recomputed Rows with 1 Pawn

5.1.2 N+k Queens Universe

When one Pawn is placed on the chess board, it divides the chessboard blocks for its row, column, negative, and positive diagonal. In the Dancing Links universe, this added Pawn splits the corresponding four Columns (R, F, A, B), resulting in four additional and unique Columns to be covered.

To analyze $N + k$ we solve the exact cover problem used in the N-Queens problem with the addition of a preprocessor to determine all valid Pawn placements, iterate through the Pawn placements, and determine unique row and column ids.

Given a valid Pawn placement, the total rows and columns of the chessboard are recomputed. If a Pawn splits a row, k, then up to Pawn is row k, and after the Pawn is row $k + 1$. The next column row is then row $k + 2$. Figure 4 shows the row identifiers when a Pawn is placed in location $(1, 1)$.

6 Conclusions and Future Work

Computer counts of fundamental solutions to the independence separation problem have provided some results for small N as shown in Tables 1, 2, and 3. An improvement of the algorithm used in [5] for $N + 2$ Queens resulted in the discovery of additional fundamental solutions for the cases $N = 9, 10$, and 11 that were not included in [5].

Using C++, we implemented sequential and parallel algorithms with backtracking as well as the previously described parallel algorithm with Dancing Links on the Midas cluster located in the Department of Mathematics and Computer Science at Morehead State University. The Midas cluster consists of a head node and eight compute nodes running OSCAR [17] under the Fedora Core 2 Linux distribution. The compute nodes are recycled Gateway E-4200 computers with a combined 3072MB RAM, while the head node is an IBM NetVista with 512MB RAM. The nodes communicate through a switched Gigabit Ethernet private network.

The times given in Table 4 are the average elapsed seconds over five to ten trials for each value of N for the $N + 1$ independence separation problem. Our current version counts the total number of solutions and at its conclusion, compresses the solutions into a set of fundamental solutions. It is obviously faster to derive an algorithm that computes the fundamental solutions first and determines the total solutions later.

Currently, the $N+k$-Queens universe is prepended to the N-Queens universe solver, which then uses DLX. In the future we hope to reduce runtimes even more by utilizing DLX to add and remove Pawns from the solution set.

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N	Solutions	Fundamental solutions
5	0	0
6	16	2
7	20	3
8	128	16
9	396	52
10	2288	286
11	11152	1403
12	65172	8214
13	437848	54756
14	3118664	389833
15	23387448	2923757
16	183463680	22932960

Table 1: $N + 1$ Queens and 1 Pawn on $N \times N$ chessboard

N	Solutions	Fundamental solutions
6	0	0
7	4	1
8	44	6
9	280	37
10	1304	164
11	12452	1572
12	105012	13133

Table 2: $N + 2$ Queens and 2 Pawns on $N \times N$ chessboard

N	Solutions	Fundamental solutions
7	0	0
8	8	1
9	44	6
10	528	66
11	5976	751

Table 3: $N + 3$ Queens and 3 Pawns on $N \times N$ chessboard

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N	Sequential	Parallel with backtracking	Parallel with DLX
6	0.002	0.002	0.0065
7	0.014	0.005	0.0198
8	0.064	0.020	0.0366
9	0.572	0.142	0.1011
10	3.051	0.635	0.2213
11	28.555	4.521	1.4471
12	173.007	27.018	6.0479
13	1667.834	261.106	42.3711
14		1812.864	256.9734
15		17353.400	1987.509
16			14536.260

Table 4: Elapsed time in seconds of sequential versus parallel implementations

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