

# Harmonic Bergman projectors and Calderón-Zygmund theory on homogeneous trees

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August 4, 2022

## Abstract

In this paper we investigate some properties of the harmonic Bergman spaces  $\mathcal{A}^p(\sigma)$  on a  $q$ -homogeneous tree, where  $q \geq 2$ ,  $1 \leq p < \infty$ , and  $\sigma$  is a finite measure on the tree with radial decreasing density, hence nondoubling. These spaces were introduced by J. Cohen, F. Colonna, M. Picardello and D. Singman. When  $p = 2$  they are reproducing kernel Hilbert spaces and we compute explicitly their reproducing kernel. We then study the boundedness properties of the Bergman projector on  $L^p(\sigma)$  for  $1 < p < \infty$  and their weak type (1,1) boundedness for suitable exponentially decreasing measures on the tree. The weak type (1,1) boundedness is a consequence of a Calderón-Zygmund decomposition adapted to this setting and the fact that the Bergman kernel satisfies an appropriate integral Hörmander type condition.

## Introduction

The main focus of this paper is on the projectors associated to the harmonic Bergman spaces on homogeneous trees introduced in [7]. The Bergman spaces  $\mathcal{A}^p(\sigma)$ ,  $1 \leq p < \infty$ , are in some ways the harmonic analogues of the classical holomorphic Bergman spaces on the hyperbolic disk, whereby  $p$ -integrability is relative to the reference measure  $\sigma$  on the tree, that is a finite measure with radial density with respect to the counting measure, and where harmonicity is relative to the so-called combinatorial Laplacian. The analogy between the hyperbolic disk and the homogeneous tree inspires many ideas behind our constructions (see [4], [6]).

The space  $\mathcal{A}^2(\sigma)$  is, as expected, a reproducing kernel Hilbert space (RKHS) and the problem of understanding the associated projectors hinges on the explicit knowledge of the kernel, an information that we derive by introducing a somewhat canonical basis for  $\mathcal{A}^2(\sigma)$ . The core of this contribution is devoted to proving

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This work is partially supported by the project "Harmonic analysis on continuous and discrete structures" funded by Compagnia di San Paolo (Cup E13C21000270007). Furthermore, they are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

Keywords: Bergman spaces; homogeneous trees; Bergman projectors; Calderón-Zygmund theory.

Mathematics Subject Classification (2010): 05C05; 46E22; 43A85.

that, for a prototypical class of measures, the extension of the Bergman projector is bounded on  $L^p(\sigma)$  if and only if  $p > 1$ , and is of weak type (1,1). The results are thus almost faithful reformulations of those that hold true for the holomorphic Bergman spaces on the hyperbolic disk ([3], [12], [14], [16], and [18]), but many of the key ingredients, first and foremost the explicit determination of the reproducing kernel, call for a rather different approach.

Let  $X$  be a homogeneous tree. A function on the tree is said to be harmonic if the mean of its values on the neighbors of any vertex coincides with the value at that vertex. J. Cohen, F. Colonna, M. Picardello, and D. Singman introduce harmonic Bergman spaces on homogeneous trees in [7]. They consider a family of reference measures which consists of finite measures which are absolutely continuous with respect to the counting measure and whose Radon-Nikodym derivative  $\sigma$  is a radial strictly positive decreasing function on  $X$ . For every  $1 \leq p < \infty$ , the harmonic Bergman space  $\mathcal{A}^p(\sigma)$  is the closed subspace of  $L^p(\sigma)$  consisting of harmonic functions. The requirement for the measure  $\sigma$  to be finite is suggested by the fact that the only harmonic function which is  $p$ -integrable with respect to the counting measure is the null function.

In the context of the hyperbolic disk, when  $p = 2$ , the weighted Bergman spaces are RKHS, and the holomorphic Bergman kernel is known as well as the properties of the associated projector. Indeed, the extension of the holomorphic Bergman projector to the weighted  $L^p$ -spaces is bounded if and only if  $p > 1$ , see [14], [16] and [18]. Furthermore, it is of weak type (1,1), see [3] and [12]. In our work, first of all, we show that  $\mathcal{A}^2(\sigma)$  is a RKHS for every reference measure  $\sigma$  and we provide an explicit formula for the reproducing kernel  $K_\sigma$  in Theorem 11. Since  $\mathcal{A}^2(\sigma)$  is closed in  $L^2(\sigma)$ , there exists an orthogonal projection  $P_\sigma: L^2(\sigma) \rightarrow \mathcal{A}^2(\sigma)$ . We prove that, for a particular class of reference measures,  $P_\sigma$  extends to a bounded operator from  $L^p(\sigma)$  to  $\mathcal{A}^p(\sigma)$  if and only if  $p > 1$ . Moreover, we show that  $P_\sigma$  is of weak type (1,1): to do so we construct a Calderón-Zygmund decomposition of integrable functions adapted to the measure metric space  $(X, \sigma)$ . The measure  $\sigma$  is not doubling, hence the construction of the Calderón-Zygmund decomposition is not obvious. We show that the Bergman kernel satisfies an integral condition that we call of Hörmander type. Then we show that all the integral operators bounded on  $L^2(\sigma)$  whose kernel satisfies the Hörmander type condition are of weak type (1,1).

The measures we focus on are exponentially decreasing radial measures, i.e. they are exponentially decreasing with respect to the distance from  $o$  and can be viewed as natural counterparts of the measures involved in the definition of the standard weighted holomorphic Bergman spaces on the hyperbolic disk. The fact that the extension of the projector to the weighted  $L^p$ -spaces is bounded if and only if  $p > 1$  follows from the fact that the projector coincides with a particular Toeplitz-type operator (see Section 3.4 in [18]).

In the spirit of the results of [1] and [2] on the disk, one could investigate the boundedness of the Bergman projectors for general reference measures. In [7], [8], [9], the authors introduce and study the optimal measures, a subclass of the reference measures which, roughly speaking, decrease fast as the distance from the origin increases. The exponentially decreasing radial measures are optimal in this sense. The study of the boundedness of the Bergman projector for optimal measures is still work in progress. Another related question is whether the Calderón-Zygmund theory that we develop here could be applied to other operators.

The paper is organized as follows. In the first section we recall the definition of the harmonic Bergman spaces and, for every reference measure, we provide an orthonormal basis of the Hilbert space  $\mathcal{A}^2(\sigma)$ . The basis plays a fundamental role in Section 2 in the proof of the two formulae for the kernel of the RKHS  $\mathcal{A}^2(\sigma)$ : the first is a recursive formula, while the second is the explicit formula of the kernel given in Theorem 11. In Section 3 we focus on the exponentially decreasing radial measures and state two results characterizing the boundedness of the extension of a class of Toeplitz-type operators inspired by the operators considered in [18] (see Theorem 14 and Theorem 15). As a consequence, in Theorem 17 we show that the extension of the harmonic Bergman projector to the weighted  $L^p$  spaces is bounded if and only if  $p > 1$ . The last section is devoted to the Calderón-Zygmund decomposition of integrable functions (presented in Proposition 29), the formulation of the Hörmander type condition (see Definition 30), and the weak type (1,1) boundedness of the integral operators with kernel satisfying the Hörmander type condition. The weak type (1,1) boundedness of the Bergman projectors is obtained as byproduct.

In what follows, we shall use the symbol  $\simeq$  ( $\lesssim$ , or  $\gtrsim$ ) between two quantities when the left hand side is equal (smaller than or equal to, or greater than or equal to, respectively) to the right hand side up to the multiplication by a (fixed) positive constant. Furthermore we assume the following convention on sums: the sum is null whenever the starting index is greater than the final index. If  $Y \subseteq X$ , we denote by  $\mathbb{1}_Y$  the characteristic function of  $Y$ . Finally, we adopt the symbol  $\sqcup$  for disjoint unions.

## 1 Harmonic Bergman spaces

### 1.1 Preliminaries on homogeneous trees

We present some preliminary notions and results on homogeneous trees; for more details we refer to [5], [10], [11], [13].

A *graph* is a pair  $(X, \mathfrak{E})$ , where  $X$  is the set of *vertices* and  $\mathfrak{E}$  is the family of *unoriented edges*, where an edge is a two-element subset of  $X$ . If two vertices  $u, v$  in  $X$  are joined by an edge, they are called *adjacent* and this is denoted by  $u \sim v$ . A *tree* is an undirected, connected, cycle-free graph. A *q-homogeneous tree* is a tree in which each vertex has exactly  $q + 1$  adjacent vertices. With slight abuse, we refer to the set of vertices  $X$  as the tree itself. We fix an origin  $o \in X$ .

From now on we consider a  $q$ -homogeneous tree  $X$  with  $q \geq 2$ . Given  $u, v \in X$ , with  $u \neq v$ , we denote by  $[u, v]$  the unique ordered  $t$ -uple  $(x_0 = u, x_1, \dots, x_{t-1} = v) \in X^t$ , where  $\{x_i, x_{i+1}\} \in \mathfrak{E}$  and all the  $x_i$  are distinct. We say that  $[u, v]$  is the path starting at  $u$  and ending at  $v$ . With slight abuse of notation, if  $[u, v] = (x_0, \dots, x_{t-1})$  we write  $x_i \in [u, v]$ ,  $i \in \{0, \dots, t-1\}$ . In particular, if  $u$  and  $v$  are adjacent, both  $[u, v], [v, u] \in X^2$  are oriented, unlike the edge  $\{u, v\} \in \mathfrak{E}$  which is not. A homogeneous tree  $X$  carries a natural distance  $d: X \times X \rightarrow \mathbb{N}$ , where for every  $u, v \in X$  the distance  $d(u, v)$  is the minimal length of a *path* joining  $u$  and  $v$ . If  $v \in X$ , then we denote by  $S(v, n)$  and  $B(v, n)$  the sphere and the ball centered at  $v$  with radius  $n \in \mathbb{N}$ , respectively, i.e.,

$$S(v, n) = \{x \in X : d(v, x) = n\} \quad \text{and} \quad B(v, n) = \{x \in X : d(v, x) \leq n\}.$$

It is straightforward to check that

$$\#S(v, n) = \begin{cases} 1, & n = 0, \\ (q+1)q^{n-1}, & n \neq 0. \end{cases} \quad (1)$$

We call *norm* of a vertex  $v$  in  $X$  its distance from  $o$ , i.e.  $|v| = d(o, v)$ . We say that a function  $f$  on  $X$  is *radial* (with respect to  $o$ ) if its value at a vertex  $x \in X$  depends only on  $|x|$ . If  $v \neq o$ , then we define the *sector of  $v$*  as the subset

$$T_v := \{x \in X : [o, v] \subseteq [o, x]\},$$

and we adopt the convention  $T_o = X$ . Moreover, we call *successors* of  $v$  the elements of the set  $s(v) = \{u \in X : u \sim v, |u| = |v| + 1\}$ . Evidently,

$$\#s(v) = \begin{cases} q, & \text{if } v \neq o; \\ q+1, & \text{if } v = o. \end{cases}$$

For every  $v \neq o$  we call predecessor of  $v$  and denote by  $p(v)$  the only neighbor of  $v$  which is not a successor of  $v$ ; it follows that  $|p(v)| = |v| - 1$ . The vertex  $o$  is the only one having no predecessors, and  $s(o) = S(o, 1)$ . This defines the predecessor function  $p: X \setminus \{o\} \rightarrow X$ , and, for every positive integer  $\ell$ , its  $\ell$ -fold composition  $p^\ell: X \setminus B(o, \ell - 1) \rightarrow X$  is the  $\ell$ -th predecessor function.

## 1.2 Harmonic functions and harmonic Bergman spaces

**Definition 1.** Let  $f$  be a complex valued function on  $X$ . The *combinatorial Laplacian* of  $f$  is defined by

$$Lf(v) := \frac{1}{q+1} \sum_{u \sim v} f(u) - f(v), \quad v \in X.$$

We say that  $f$  is *harmonic* on  $Y \subseteq X$  if  $Lf = 0$  on  $Y$ . Equivalently,  $f$  is harmonic on  $Y$  if

$$f(v) = \frac{1}{q+1} \sum_{u \sim v} f(u), \quad v \in Y. \quad (2)$$

We shall call a function harmonic if it is harmonic on  $X$ .

It is easy to prove that a function is harmonic if and only if for every  $v \in X$  and  $n \in \mathbb{N}$ , we have

$$f(v) = \frac{1}{\#S(v, n)} \sum_{d(v, x) = n} f(x). \quad (3)$$

The harmonicity property (2) implies a certain rigidity for the function. In particular, the value of a harmonic function at a vertex  $y \in X$  “propagates” to every layer of the sector  $T_y$ , as showed in the following proposition, which is a modified version of [7, Lemma 4.1]. In that lemma, the authors show that a function which is harmonic and radial on a sector  $T_y$ ,  $y \in X \setminus \{o\}$ , is completely described by its values at  $y$  and  $p(y)$ . We consider a harmonic function on the sector  $T_y$ , removing the radially assumption, and we formulate a result for its average on  $S(o, n) \cap T_y$ ,  $n \geq |y|$ . We omit the proof since it is an easy adaptation of the proof of [7, Lemma 4.1].

**Proposition 2.** *Let  $y \in X \setminus \{o\}$ . If  $f: X \rightarrow \mathbb{C}$  is harmonic on  $T_y$ , then for every  $n \in \mathbb{N}$ ,  $n \geq |y|$ , we have*

$$\sum_{\substack{|x|=n \\ x \in T_y}} f(x) = \left( \sum_{j=0}^{n-|y|} q^j \right) f(y) - \left( \sum_{j=0}^{n-|y|-1} q^j \right) f(p(y)). \quad (4)$$

Furthermore, if  $f: X \rightarrow \mathbb{C}$  is radial on  $T_y$  and satisfies (4) for every  $n \geq |y|$ , then  $f$  is harmonic on  $T_y$ .

We introduce a technique which allows to extend a function which is harmonic on a ball centered in  $o$  to a function harmonic on  $X$ . Let  $n \in \mathbb{N}$  and  $g$  be a function on  $X$  which is harmonic on  $B(o, n)$ . It is easy to see that there are infinitely many ways to extend  $g$  to a harmonic function on  $X$  which coincides with  $g$  on  $B(o, n+1)$ . As we see next, there is however a unique harmonic function  $g_n^H$  on  $X$  which is radial when restricted on  $T_y$  for every  $y \in S(o, n+1)$ .

Let  $x \in X \setminus B(o, n)$ . There exists a unique  $y \in S(o, n+1)$  such that  $x \in T_y$ , and  $y = p^{|x|-n-1}(x)$  (where  $p^0 = \text{id}_X$ ). Since we aim to construct  $g_n^H$  radial and harmonic on  $T_y$ , by Proposition 2 we have that

$$\begin{aligned} g_n^H(x) &= \frac{1}{\#S(o, |x|) \cap T_y} \sum_{\substack{|z|=|x|, \\ z \in T_y}} g_n^H(z) \\ &= q^{|y|-|x|} \left[ \left( \sum_{j=0}^{|x|-|y|} q^j \right) g(y) - \left( \sum_{j=0}^{|x|-|y|-1} q^j \right) g(p(y)) \right] \\ &= q^{n+1-|x|} \left[ \left( \sum_{j=0}^{|x|-n-1} q^j \right) g(p^{|x|-n-1}(x)) - \left( \sum_{j=0}^{|x|-n-2} q^j \right) g(p^{|x|-n}(x)) \right] \\ &= \left( \sum_{j=0}^{|x|-n-1} q^{-j} \right) g(p^{|x|-n-1}(x)) - \left( \sum_{j=1}^{|x|-n-1} q^{-j} \right) g(p^{|x|-n}(x)). \end{aligned}$$

For simplicity we introduce the notation

$$a_n = \sum_{j=0}^n q^{-j} = \frac{q - q^{-n}}{q - 1}, \quad n \in \mathbb{N},$$

and we set  $a_{-1} = 0$ . Hence

$$g_n^H(x) = \begin{cases} g(x), & |x| \leq n, \\ a_{|x|-n-1} g(p^{|x|-n-1}(x)) - (a_{|x|-n-1} - 1) g(p^{|x|-n}(x)), & |x| > n. \end{cases} \quad (5)$$

The function  $g_n^H$  defined above is harmonic on  $X$  by Proposition 2 and because

$$X = B(o, n) \cup \bigcup_{y \in S(o, n+1)} T_y.$$

Observe that  $g_n^H$  is indeed harmonic on  $B(o, n)$  because  $a_0 = 1$  and  $a_{-1} = 0$ , yield  $g_n^H = g$  on  $B(o, n+1)$ , and not only on  $B(o, n)$ . Furthermore, the extension  $g_n^H$  is radial on every sector “starting” from a point in  $S(o, n+1)$  by construction.

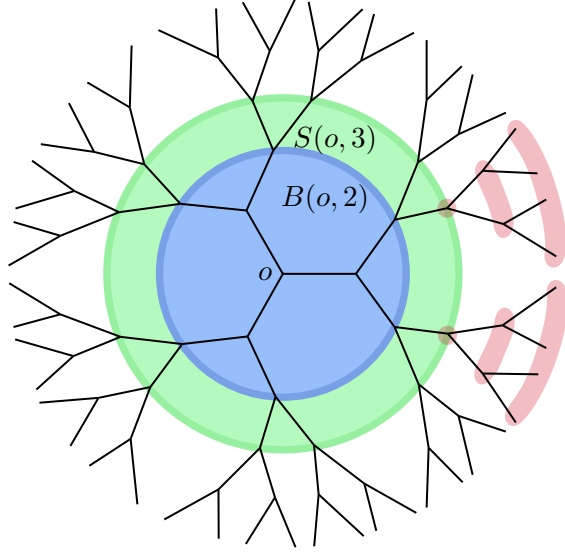


Figure 1: The function  $g$  is harmonic on  $B(o, 2)$ , that is the set of vertices in the blue area. The function  $g_2^H$  is obtained by extending the values of  $g$  in  $S(o, 3)$  (the green area) along sectors in such a way that  $g_2^H$  is harmonic on  $X$  and constant on the vertices lying on the same red arc, that is on the “layers” of the sectors.

### 1.3 Harmonic Bergman spaces

Homogeneous trees are classically endowed with the counting measure. The main feature of such measure is the invariance under the group of isometries of the tree. When studying spaces of harmonic functions, this measure is however inadequate because the only harmonic function that is  $p$ -summable,  $1 \leq p < \infty$ , with respect to the counting measure is the null function, as we show in the following statement.

**Proposition 3.** *If  $f$  is a harmonic function in  $L^p(X)$ ,  $1 \leq p < \infty$ , then  $f$  is the null function.*

*Proof.* Suppose that  $f$  is harmonic. We have that

$$\begin{aligned}
 \sum_{x \in X} |f(x)|^p &= \sum_{n=0}^{+\infty} \sum_{|x|=n} |f(x)|^p \\
 &= \frac{1}{(q+1)^p} \sum_{n=0}^{+\infty} \sum_{|x|=n} \left| \sum_{y \sim x} f(y) \right|^p \\
 &\leq \frac{(q+1)^{p-1}}{(q+1)^p} \sum_{n=0}^{+\infty} \sum_{|x|=n} \sum_{y \sim x} |f(y)|^p \\
 &= \frac{1}{q+1} (q+1) \|f\|_{L^p(X)}^p = \|f\|_{L^p(X)}^p < +\infty,
 \end{aligned}$$

since every vertex is neighbor of exactly  $q+1$  other vertices. Hence the unique inequality in the computation above is an equality, so that

$$(q+1)^{p-1} \sum_{y \sim x} |f(y)|^p = \left| \sum_{y \sim x} f(y) \right|^p = (q+1)^p |f(x)|^p,$$

which means that  $|f|^p$  is harmonic, too. If  $f$  is not the null function, then there exists  $v \in X$  such that  $f(v) \neq 0$ . Hence by (3), we have

$$\sum_{x \in X} |f(x)|^p = \sum_{n=0}^{+\infty} \sum_{d(v,x)=n} |f(x)|^p = |f(v)|^p \sum_{n=0}^{+\infty} \#S(v, n) = +\infty,$$

which is a contradiction. Hence  $f = 0$ .  $\square$

Since we are interested in Bergman spaces of harmonic functions, the previous proposition leads to consider finite measures on  $X$ . In [7], the authors introduce harmonic Bergman spaces with respect to the following class of measures.

**Definition 4.** A *reference measure* on  $X$  is a finite measure that is absolutely continuous with respect to the counting measure and whose Radon-Nikodym derivative  $\sigma$  is a radial strictly positive decreasing function on  $X$ . With slight abuse of notation we denote by  $\sigma$  the reference measure, too. Given a reference measure  $\sigma$  on  $X$  for every  $p \in [1, \infty)$  the *Bergman space*  $\mathcal{A}^p(\sigma)$  is the space of harmonic functions on  $X$  such that

$$\|f\|_{\mathcal{A}^p(\sigma)}^p := \sum_{x \in X} |f(x)|^p \sigma(x) < +\infty.$$

If  $\sigma$  is a reference measure on  $X$ , and if we denote by  $\sigma_n$  the value of  $\sigma$  on the sphere  $S(0, n)$ , then by (1) the total mass of  $\sigma$  is denoted by  $B_\sigma$  and its value is

$$B_\sigma = \sigma_0 + \frac{q+1}{q} \sum_{n=1}^{+\infty} \sigma_n q^n < +\infty.$$

**Example 5.** Let  $\alpha > 1$ . Interesting examples of reference measures are the exponentially decreasing radial measures, consisting of the measures having density

$$\mu_\alpha(x) = q^{-\alpha|x|}, \quad x \in X.$$

Indeed,  $\mu_\alpha$  is radial, positive and decreasing. Furthermore,

$$B_{\mu_\alpha} = 1 + \frac{q+1}{q} \sum_{n=1}^{+\infty} q^{(1-\alpha)n} = 1 + \frac{q+1}{q} \frac{q^{1-\alpha}}{1-q^{1-\alpha}} = \frac{q^\alpha + 1}{q^\alpha - q} < +\infty.$$

**Proposition 6.** *The reference measures are not doubling.*

*Proof.* Let  $\sigma$  be a reference measure. For every  $n \in \mathbb{N}$ , let  $v_n \in X$  be such that  $|v_n| = 2n$ . Then  $\max\{\sigma(x) : x \in B(v_n, n)\} = \sigma_n$  and so

$$\sigma(B(v_n, n)) = \sum_{\{x \in B(v_n, n)\}} \sigma(x) \leq \sigma_n |B(v_n, n)| \lesssim q^n \sigma_n.$$

On the other side, since  $o \in B(v_n, 2n)$ , we have

$$\frac{\sigma(B(v_n, 2n))}{\sigma(B(v_n, n))} \gtrsim \frac{\sigma(o)}{q^n \sigma_n} \xrightarrow{n \rightarrow \infty} \infty,$$

by the finiteness of  $\sigma$ . This proves that  $\sigma$  is not doubling.  $\square$

Given a reference measure  $\sigma$ , we introduce the decreasing sequence  $(b_n)_{n \in \mathbb{N}}$  which collects some important information on  $\sigma$ . For every  $n \in \mathbb{N}$ , we define

$$b_n = b_n(\sigma) = \sum_{m=n+1}^{+\infty} \left[ \sigma_m a_{m-n-1} \left( \sum_{k=0}^{m-n-1} q^k \right) \right]. \quad (6)$$

The sums are finite because  $\sigma$  is a finite measure on  $X$ . We shall use the notation  $b_n$  instead of  $b_n(\sigma)$  whenever the measure is clear from the context.

The next lemma is a technical result that is very useful in what follows.

**Lemma 7.** *Let  $n \in \mathbb{N}$  and  $g$  be a function on  $X$  which is harmonic and vanishes on  $B(o, n)$ . Then there exists a constant  $b'_n > 0$  such that for every  $f \in \mathcal{A}^2(\sigma)$*

$$\langle f, g_n^H \rangle_{\mathcal{A}^2(\sigma)} = \sum_{|y|=n+1} (b_n f(y) - b'_n f(p(y))) \overline{g(y)},$$

where  $b_n$  is defined in (6).

**Remark 8.** The constant  $b'_n$  has a structure similar to that of  $b_n$ , as can be seen in the proof below, but we are not interested in it.

*Proof.* Observe that, from the fact that  $g|_{B(o, n)} = 0$  and (5), for every  $x \in X$  with  $|x| > n$  we have  $g_n^H(x) = a_{|x|-n-1} g(p^{|x|-n-1}(x))$ . Take  $f \in \mathcal{A}^2(\sigma)$ . Then, by applying Proposition 2 to  $f$ , we have

$$\begin{aligned} \langle f, g_n^H \rangle_{\mathcal{A}^2(\sigma)} &= \sum_{m=n+1}^{+\infty} \sigma_m \sum_{|x|=m} f(x) \overline{g_n^H(x)} \\ &= \sum_{m=n+1}^{+\infty} \sigma_m \sum_{|y|=n+1} \sum_{\substack{|x|=m \\ x \in T_y}} f(x) a_{|x|-n-1} \overline{g(p^{|x|-n-1}(x))} \\ &= \sum_{|y|=n+1} \overline{g(y)} \sum_{m=n+1}^{+\infty} \sigma_m a_{m-n-1} \sum_{\substack{|x|=m \\ x \in T_y}} f(x) \\ &= \sum_{|y|=n+1} \overline{g(y)} \sum_{m=n+1}^{+\infty} \sigma_m a_{m-n-1} \left[ \left( \sum_{k=0}^{m-n-1} q^k \right) f(y) - \left( \sum_{k=0}^{m-n-2} q^k \right) f(p(y)) \right] \\ &= \sum_{|y|=n+1} (b_n f(y) - b'_n f(p(y))) \overline{g(y)}, \end{aligned}$$

as required. □

#### 1.4 A canonical orthonormal basis of $\mathcal{A}^2(\sigma)$

The goal of this section is the construction of an orthonormal basis of the space  $\mathcal{A}^2(\sigma)$ .

Let us consider the linear spaces

$$W_v := \left\{ \varphi : s(v) \rightarrow \mathbb{C} : \sum_{z \in s(v)} \varphi(z) = 0 \right\} \simeq \mathbb{C}^{|s(v)|-1} = \begin{cases} \mathbb{C}^q, & v = o, \\ \mathbb{C}^{q-1}, & v \in X \setminus \{o\}. \end{cases}$$



For every  $v \in X$  we set  $I_v = \{1, \dots, |s(v)|-1\}$ . For every  $v \in X$  we fix an orthonormal basis  $\{e_{v,j}\}_{j \in I_v}$  of  $W_v$  with respect to the scalar product

$$\langle \varphi, \psi \rangle_{W_v} = \sum_{y \in s(v)} \varphi(y) \overline{\psi(y)}.$$

Let  $v \in X$  and  $j \in I_v$ . We consider the extension by zero to all of  $X$  of  $e_{v,j}$ , namely,

$$E_{v,j}(x) = \begin{cases} e_{v,j}(x), & x \in s(v); \\ 0, & x \notin s(v). \end{cases}$$

It is immediate to see that  $E_{v,j}$  is harmonic on  $B(o, |v|)$  and vanishes on  $B(o, |v|)$ . We denote the harmonic extension of  $E_{v,j}$  by  $f_{v,j} = (E_{v,j})_{|v|}^H$ , namely

$$f_{v,j}(x) = \begin{cases} 0, & \text{if } x \notin T_v \setminus \{v\}, \\ a_{|x|-|v|-1} E_{v,j}(p^{|x|-|v|-1}(x)), & \text{otherwise.} \end{cases} \quad (7)$$

Hence  $f_{v,j}$  is harmonic for every  $v \in X$  and  $j \in I_v$ . Furthermore  $f_{v,j}$  is bounded, since for every  $x \in X$

$$|f_{v,j}(x)| \leq (1 - q^{-1})^{-1} \|e_{v,j}\|_\infty.$$

Hence  $f_{v,j} \in \mathcal{A}^2(\sigma)$  for every reference measure  $\sigma$ . Notice that, upon setting  $f_0(x) \equiv 1$ , the family

$$\mathcal{B} = \{f_0\} \cup \{f_{v,j} : v \in X, j \in I_v\} \subseteq \mathcal{A}^2(\sigma)$$

is independent of the choice of the reference measure  $\sigma$ .

**Proposition 9.** *The family  $\mathcal{B}$  is a complete orthogonal system in  $\mathcal{A}^2(\sigma)$  for every reference measure  $\sigma$ .*

*Proof.* Fix a reference measure  $\sigma$ . The fact that  $f_0$  is orthogonal to every other function of the family follows from the harmonicity of  $f_{v,j}$  and (3). Indeed

$$\langle f_{v,j}, f_0 \rangle_{\mathcal{A}^2(\sigma)} = \sum_{x \in X} f_{v,j}(x) \sigma(x) = \sum_{n=0}^{+\infty} \sigma_n \sum_{|x|=n} f_{v,j}(x) = 0.$$

Let us consider  $v, w \in X$  with  $v \neq w$ . Without loss of generality we may consider two situations: either  $T_v \cap T_w = \emptyset$  or  $T_v \subsetneq T_w$ . In the first case  $f_{v,j} \perp f_{w,k}$  for every  $j \in I_v$  and  $k \in I_w$ , because their supports are disjoint. If  $T_v \subsetneq T_w$ , then we can suppose that  $|w| < |v|$ . Since  $f_{v,j}|_{B(o, |w|+1)} = 0$ , from Lemma 7 we have

$$\langle f_{v,j}, f_{w,k} \rangle_{\mathcal{A}^2(\sigma)} = \sum_{|y|=|w|+1} (b_{|w|} f_{v,j}(y) - b'_{|w|} f_{v,j}(p(y))) \overline{E_{w,k}(y)} = 0$$

It remains to prove orthogonality in the case  $v = w$ . Let  $j, k \in I_v$  be such that  $j \neq k$ . We know that  $f_{v,k}|_{B(o, |v|)} = 0$ , so that by Lemma 7

$$\langle f_{v,j}, f_{v,k} \rangle_{\mathcal{A}^2(\sigma)} = b_{|v|} \sum_{|y|=|v|+1} E_{v,j}(y) \overline{E_{v,k}(y)} = b_{|v|} \sum_{y \in s(v)} e_{v,j}(y) \overline{e_{v,k}(y)} = 0,$$

where we used the fact that  $\text{supp}(E_{v,k}), \text{supp}(E_{v,j}) \subseteq s(v)$  and the orthogonality of  $e_{v,j}$  and  $e_{v,k}$  in  $W_v$ .

We now show that  $\mathcal{B}$  is complete. Take  $g \in \mathcal{A}^2(\sigma)$  such that  $\langle g, f \rangle_{\mathcal{A}^2(\sigma)} = 0$  for every  $f \in \mathcal{B}$ . We show that  $g$  is the null function in  $\mathcal{A}^2(\sigma)$ . In particular we prove by induction that  $g = 0$  on every  $B(o, m)$ ,  $m \in \mathbb{N}$ .

We start by observing that  $\langle g, f_0 \rangle_{\mathcal{A}^2(\sigma)} = 0$  implies  $g(o) = 0$ . Indeed by (3)

$$0 = \langle g, f_0 \rangle_{\mathcal{A}^2(\sigma)} = \sum_{n=0}^{+\infty} \sigma_n \sum_{|x|=n} g(x) = \left(1 + \frac{q+1}{q} \sum_{n=1}^{+\infty} q^n \sigma_n\right) g(o) = B_\sigma g(o). \quad (8)$$

We assume now  $g = 0$  on  $B(o, m)$  for some  $m \in \mathbb{N}$ . Let  $v \in S(o, m)$ . Observe that since  $g$  is harmonic and  $g(v) = 0$ , we have  $g|_{s(v)} \in W_v$ . Hence for every  $j \in I_v$

$$0 = \langle g, f_{v,j} \rangle_{\mathcal{A}^2(\sigma)} = b_m \sum_{y \in s(v)} \overline{e_{v,j}(y)} g(y) \quad (9)$$

and this implies that  $g(y) = 0$  for every  $y \in s(v)$  and so for every  $y \in S(o, m+1)$ , namely  $g$  vanishes on  $B(o, m+1)$ . The fact that  $g \equiv 0$  follows by induction.  $\square$

We now fix a measure  $\sigma$  and compute the norm of functions of the family  $\mathcal{B}$  in  $\mathcal{A}^2(\sigma)$ . Evidently,  $\|f_0\|_{\mathcal{A}^2(\sigma)}^2 = B_\sigma$ . Let  $v \in X$  and  $j \in I_v$ . By (9), we have

$$\|f_{v,j}\|_{\mathcal{A}^2(\sigma)}^2 = \langle f_{v,j}, f_{v,j} \rangle_{\mathcal{A}^2(\sigma)} = b_{|v|} \sum_{y \in s(v)} \overline{e_{v,j}(y)} e_{v,j}(y) = b_{|v|}. \quad (10)$$

Hence the norm of  $f_{v,j}$  does not depend on  $j$  and coincides with the constant in (6). Hence

$$\mathcal{B}_\sigma = \{B_\sigma^{-\frac{1}{2}} f_0\} \cup \{b_{|v|}^{-\frac{1}{2}} f_{v,j} : v \in X, j \in I_v\} \quad (11)$$

is an orthonormal basis of  $\mathcal{A}^2(\sigma)$ .

## 2 The reproducing kernel of $\mathcal{A}^2(\sigma)$

In this section we fix a reference measure  $\sigma$ . We show that the Bergman space  $\mathcal{A}^2(\sigma)$  is a RKHS and we obtain first a recursive formula for the kernel and we then derive a formula in closed form. Observe that the main ingredient used in the proofs are the harmonic extension and the orthonormal basis defined in the previous section together with the fact that  $W_v$  are reproducing kernel Hilbert spaces, too.

Let  $z \in X$ . We consider the evaluation functional  $\Phi_z : \mathcal{A}^2(\sigma) \rightarrow \mathbb{C}$  defined by  $\Phi_z g = g(z)$ . Observe that  $\Phi_z$  is a bounded operator. Indeed by the Cauchy-Schwarz inequality

$$|g(z)| = \frac{1}{q+1} \left| \sum_{x \sim z} g(x) \right| \leq \frac{1}{q+1} \sum_{x \sim z} |g(x)| \leq \frac{1}{q+1} \|g\|_{\mathcal{A}^2(\sigma)} \left\| \frac{\mathbb{1}_{S(z,1)}}{\sigma} \right\|_{L^2(\sigma)},$$

where  $\mathbb{1}_{S(z,1)}$  is the characteristic function of the sphere  $S(z,1)$ . Since for every  $x \in S(z,1)$ ,  $\sigma(x) \geq \sigma_{|z|+1}$ , we have that

$$\left\| \frac{\mathbb{1}_{S(z,1)}}{\sigma} \right\|_{L^2(\sigma)}^2 = \sum_{d(z,x)=1} \frac{1}{\sigma(x)^2} \leq \sum_{d(z,x)=1} \frac{1}{\sigma_{|z|+1}^2} = \frac{q+1}{\sigma_{|z|+1}^2}.$$

Hence

$$|g(z)| \leq (q+1)^{-\frac{1}{2}} \sigma_{|z|+1}^{-1} \|g\|_{\mathcal{A}^2(\sigma)}.$$

Thus  $\mathcal{A}^2(\sigma)$  is a RKHS, that is for every  $z \in X$  there exists  $K_z \in \mathcal{A}^2(\sigma)$  such that

$$\langle g, K_z \rangle_{\mathcal{A}^2(\sigma)} = g(z), \quad g \in \mathcal{A}^2(\sigma).$$

Let  $K: X \times X \rightarrow \mathbb{C}$  be the kernel defined by  $K(z, x) := K_z(x)$ .

Since  $\mathcal{B}_\sigma$  defined in (11) is an orthonormal basis of  $\mathcal{A}_\sigma^2$ , for every  $z \in X$  we can write

$$K_z = \sum_{f \in \mathcal{B}_\sigma} \langle K_z, f \rangle_{\mathcal{A}^2(\sigma)} f = \sum_{f \in \mathcal{B}_\sigma} \overline{f(z)} f = \frac{1}{B_\sigma} + \sum_{v \in X} \sum_{j \in I_v} \frac{\overline{f_{v,j}(z)} f_{v,j}}{b_{|v|}}. \quad (12)$$

We recall that by (7), for every  $z \in X$

$$\{v \in X : f_{v,j}(z) \neq 0 \text{ for some } j \in I_v\} = \begin{cases} \emptyset, & \text{if } v = o; \\ [o, p(z)], & \text{if } v \neq o. \end{cases}$$

Hence for every  $z \in X$  the sum in (12) is finite and the decomposition of  $K_z$  holds true pointwise.

Our goal is to compute  $K_z$ . To this end, we introduce the auxiliary function  $\Gamma: X \times X \times X \rightarrow \mathbb{R}$  which is a parametrization of the family of reproducing kernels for the spaces  $\{W_v\}_{v \in X}$ . For every  $(v, z, x) \in X \times X \times X$  we set

$$\Gamma(v, z, x) = \begin{cases} 0, & \text{if } \{z, x\} \not\subseteq T_v \setminus \{v\}; \\ \frac{\#s(v) - 1}{\#s(v)}, & \text{if } \{z, x\} \subseteq T_y \text{ for some } y \in s(v); \\ -\frac{1}{\#s(v)}, & \text{otherwise.} \end{cases}$$

Observe that  $\Gamma$  is symmetric in the second and third variables. Furthermore,  $\Gamma(v, z, \cdot)$  is the null function if  $z \notin T_v \setminus \{v\}$  and whenever  $z \in T_v \setminus \{v\}$  we have  $\text{supp}(\Gamma(v, z, \cdot)) = T_v \setminus \{v\}$ . Moreover, the values of  $\Gamma(v, z, \cdot)$  on  $T_v \setminus \{v\}$  are completely determined by the values on  $s(v)$ , as the value of  $\Gamma(v, z, \cdot)$  at  $x \in T_v \setminus \{v\}$  is equal to the value at  $p^{|x|-|v|-1}(x) \in s(v)$ .

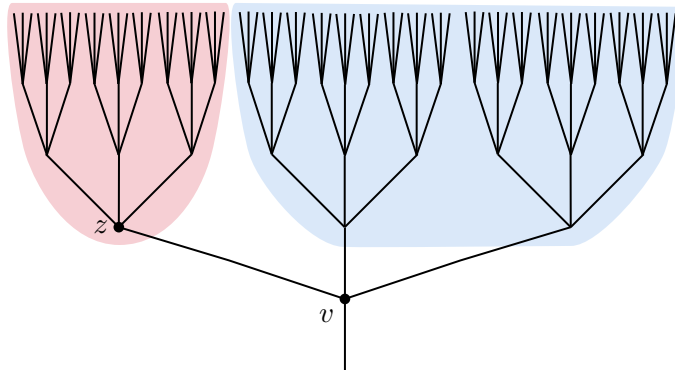


Figure 2: Partial representation of the function  $\Gamma(v, z, \cdot)$  on  $T_v$ . The value of  $\Gamma(v, z, \cdot)$  at the vertices in the red area is  $\frac{\#s(v)-1}{\#s(v)}$ , while in the blue area is  $-\frac{1}{\#s(v)}$ . Clearly,  $\Gamma(v, z, v) = 0$ .

We now show that  $\Gamma(v, \cdot, \cdot)$  is the reproducing kernel of  $W_v$ , namely that for  $z \in s(v)$  we have

$$\varphi(z) = \langle \varphi, \Gamma(v, z, \cdot) \rangle_{W_v}, \quad \varphi \in W_v.$$

First of all  $\Gamma(v, z, \cdot) \in W_v$  because

$$\sum_{y \in s(v)} \Gamma(v, z, y) = -(\#s(v) - 1) \frac{1}{\#s(v)} + \frac{\#s(v) - 1}{\#s(v)} = 0.$$

Furthermore,

$$\begin{aligned} \langle \varphi, \Gamma(v, z, \cdot) \rangle_{W_v} &= \frac{\#s(v) - 1}{\#s(v)} \varphi(z) - \frac{1}{\#s(v)} \sum_{\substack{y \in s(v) \\ y \neq z}} \varphi(y) \\ &= \frac{\#s(v) - 1}{\#s(v)} \varphi(z) + \frac{1}{\#s(v)} \varphi(z) = \varphi(z), \end{aligned}$$

because  $\varphi \in W_v$ .

It is easy to see that  $\Gamma(v, z, \cdot)$  is harmonic on  $B(o, |v|)$  so that we can consider the harmonic extension  $(\Gamma(v, z, \cdot))_{|v|}^H$ , which is bounded by construction. Indeed from the definition of harmonic extension we have for every  $x \in T_v \setminus \{v\}$

$$(\Gamma(v, z, \cdot))_{|v|}^H(x) = \left( \sum_{j=0}^{|x|-|v|-1} q^{-j} \right) \Gamma(v, z, p^{|x|-|v|-1}(x)) = a_{|x|-|v|-1} \Gamma(v, z, x), \quad (13)$$

and it vanishes elsewhere. We recall that if  $z \notin T_v$ , then  $\Gamma(v, z, \cdot) = (\Gamma(v, z, \cdot))_{|v|}^H$  is the null function.

**Proposition 10.** *Let  $z \in X$  and  $[o, z] = \{v_t\}_{t=0}^{|z|}$ . The kernel  $K_z$  is*

$$K_z = \begin{cases} \frac{1}{B_\sigma}, & \text{if } z = o, \\ K_o + \frac{1}{b_0} (\Gamma(o, z, \cdot))_0^H, & \text{if } |z| = 1, \\ -\frac{1}{q} K_{v_{m-2}} + \frac{q+1}{q} K_{v_{m-1}} + \frac{1}{b_{m-1}} (\Gamma(v_{m-1}, z, \cdot))_{m-1}^H, & \text{if } |z| = m > 1. \end{cases}$$

*Proof.* Since the measure  $\sigma$  is finite and the constant functions are harmonic,  $K_o = \frac{1}{B_\sigma} \in \mathcal{A}^2(\sigma)$ . The reproducing property follows from the computations in (8).

Now we observe that for every  $v, z \in X$  such that  $z \in T_v$  and  $g \in \mathcal{A}^2(\sigma)$ , by Lemma 7 and  $\text{supp}(\Gamma(v, z, \cdot)) = T_v \setminus \{v\}$ , we have

$$\begin{aligned} \langle g, (\Gamma(v, z, \cdot))_{|v|}^H \rangle_{\mathcal{A}^2(\sigma)} &= \sum_{|y|=|v|+1} (b_{|v|} g(y) - b'_{|v|} g(p(y))) \Gamma(v, z, y) \\ &= b_{|v|} \sum_{y \in s(v)} g(y) \Gamma(v, z, y) - b'_{|v|} g(v) \sum_{y \in s(v)} \Gamma(v, z, y) \\ &= b_{|v|} \sum_{y \in s(v)} g(y) \Gamma(v, z, y), \end{aligned} \quad (14)$$

where we used  $\Gamma(v, z, \cdot)|_{s(v)} \in W_v$ .

We now consider the case when  $|z| = 1$ . The function  $K_z \in \mathcal{A}^2(\sigma)$  because it is sum

of functions in  $\mathcal{A}^2(\sigma)$ . We prove the reproducing property. For  $g \in \mathcal{A}^2(\sigma)$ , by the reproducing formula of  $K_o$  and (14) with  $v = o$ ,

$$\begin{aligned} \langle g, K_z \rangle_{\mathcal{A}^2(\sigma)} &= g(o) + \frac{1}{b_0} \langle g, (\Gamma(o, z, \cdot))_0^H \rangle_{\mathcal{A}^2(\sigma)} \\ &= g(o) + \sum_{|y|=1} g(y) \Gamma(o, z, y) \\ &= g(o) + \frac{q}{q+1} g(z) - \frac{1}{q+1} \sum_{\substack{|y|=1 \\ y \neq z}} g(y) = g(z), \end{aligned}$$

where we used that  $g$  is harmonic at  $o$ .

It remains to consider the case when  $|z| = m > 1$ . We have  $K_z \in \mathcal{A}^2(\sigma)$  since it is the sum of bounded and harmonic functions. For  $g \in \mathcal{A}^2(\sigma)$  by induction on  $m$  and (14) with  $v = v_{m-1}$  we have

$$\begin{aligned} \langle g, K_z \rangle_{\mathcal{A}^2(\sigma)} &= -\frac{1}{q} g(v_{m-2}) + \frac{q+1}{q} g(v_{m-1}) + \frac{1}{b_{m-1}} \langle g, (\Gamma(v_{m-1}, z, \cdot))_{m-1}^H \rangle_{\mathcal{A}^2(\sigma)} \\ &= -\frac{1}{q} g(v_{m-2}) + \frac{q+1}{q} g(v_{m-1}) + \sum_{y \in s(v_{m-1})} \Gamma(v_{m-1}, z, y) g(y) \\ &= -\frac{1}{q} g(v_{m-2}) + \frac{1}{q} \sum_{y \sim v_{m-1}} g(y) + \frac{q-1}{q} g(z) - \frac{1}{q} \sum_{\substack{y \in s(v_{m-1}) \\ y \neq z}} g(y) = g(z), \end{aligned}$$

where we used the fact that  $g$  is harmonic at  $v_{m-1}$ . □

In Proposition 10 the kernel  $K_z$  is expressed through a two-step recursive formula. We want to find an explicit formula for  $K_z$ .

**Theorem 11.** *For every  $(z, x) \in X \times X$*

$$K(z, x) = \frac{1}{B_\sigma} + \frac{q^2}{(q-1)^2} \sum_{v \in X} \frac{1}{b_{|v|}} \Gamma(v, z, x) (1 - q^{|v|-|z|}) (1 - q^{|v|-|x|}). \quad (15)$$

*Proof.* Let  $z \in X$  and  $[o, z] = \{v_t\}_{t=0}^{|z|}$ . We start by proving that

$$K_z(x) = \frac{1}{B_\sigma} + \sum_{t=0}^{|z|-1} a_{|z|-t-1} \frac{1}{b_t} (\Gamma(v_t, v_{t+1}, x))_t^H, \quad x \in X. \quad (16)$$

The case  $z = o$  follows trivially from Proposition 10 and the convention on sums stated in the Introduction. We prove (16) by induction on  $m = |z| \geq 1$ . The case  $m = 1$  directly follows from Proposition 10, too. Let  $m \in \mathbb{N}$ ,  $m > 1$  and  $z \in X$ , with  $|z| = m$ , and suppose that (16) holds for every vertex in  $B(o, m-1)$ . Hence

by Proposition 10 we have

$$\begin{aligned}
K_z &= -\frac{1}{q}K_{v_{m-2}} + \frac{q+1}{q}K_{v_{m-1}} + \frac{1}{b_{m-1}}(\Gamma(v_{m-1}, z, \cdot))_{m-1}^H \\
&= -\frac{1}{q} \left[ \frac{1}{B_\sigma} + \sum_{t=0}^{m-3} a_{m-t-3} \frac{1}{b_t} (\Gamma(v_t, v_{t+1}, \cdot))_t^H \right] \\
&\quad + \frac{q+1}{q} \left[ \frac{1}{B_\sigma} + \sum_{t=0}^{m-2} a_{m-t-2} \frac{1}{b_t} (\Gamma(v_t, v_{t+1}, \cdot))_t^H \right] + \frac{1}{b_{m-1}} (\Gamma(v_{m-1}, z, \cdot))_{m-1}^H \\
&= \frac{1}{B_\sigma} + \sum_{t=0}^{m-2} \left( \frac{q+1}{q} a_{m-t-2} - \frac{1}{q} a_{m-t-3} \right) \frac{1}{b_t} (\Gamma(v_t, v_{t+1}, \cdot))_t^H \\
&\quad + \frac{1}{b_{m-1}} (\Gamma(v_{m-1}, z, \cdot))_{m-1}^H \\
&= \frac{1}{B_\sigma} + \sum_{t=0}^{m-1} a_{m-t-1} \frac{1}{b_t} (\Gamma(v_t, v_{t+1}, \cdot))_t^H,
\end{aligned}$$

where we used  $(q+1)a_{n-1} - a_{n-2} = qa_n$ . Hence we proved (16) by induction. Since  $\text{supp}((\Gamma(v_t, v_{t+1}, \cdot))_t^H) = T_{v_t} \setminus \{v_t\}$ , we have that the  $t$ -th term of the sum in (16) does not vanish if and only if  $x \in T_{v_t} \setminus \{v_t\}$ , that is when  $v_t \in [o, x]$ , and hence by (13), we have

$$\begin{aligned}
K(z, x) &= \frac{1}{B_\sigma} + \sum_{v \in X} \frac{1}{b_{|v|}} a_{|z|-|v|-1} a_{|x|-|v|-1} \Gamma(v, z, x) \\
&= \frac{1}{B_\sigma} + \frac{q^2}{(q-1)^2} \sum_{v \in X} \frac{1}{b_{|v|}} (1 - q^{|v|-|z|}) (1 - q^{|v|-|x|}) \Gamma(v, z, x).
\end{aligned}$$

□

**Remark 12.** The confluent of two vertices  $z, x \in X$  is the common vertex of  $[o, x]$  and  $[o, z]$  farthest from  $o$ , denoted by  $z \wedge x$ . It is possible to see that the value of the kernel  $K$  at  $(z, x) \in X \times X$  depends only on the values of  $|x|$ ,  $|z|$  and  $|z \wedge x|$ . Indeed, from (15) and the fact that  $\Gamma(v, z, x)$  does not vanish if and only if  $v \in [o, z] \cap [o, x] = [o, z \wedge x]$ , we have

$$K_z(x) = \frac{1}{B_\sigma} + \frac{q^2}{(q-1)^2} \sum_{t=0}^{|z \wedge x|-1} \frac{1}{b_t} a_{|z|-t-1} a_{|x|-t-1} (1 - q^{t-|z|}) (1 - q^{t-|x|}). \quad (17)$$

Furthermore, it is clear that  $K$  is symmetric, that is  $K(z, x) = K(x, z)$ .

In the following sections we restrict our attention to the family of the exponentially decreasing radial measures  $\mu_\alpha$ ,  $\alpha > 1$ , defined in Example 5.

We shall use the notation  $L_\alpha^p$  and  $\mathcal{A}_\alpha^p$  for the Lebesgue and Bergman spaces with respect to  $\mu_\alpha$ , respectively. Furthermore, we denote by  $K_\alpha: X \times X \rightarrow \mathbb{R}$  the reproducing kernel of  $\mathcal{A}_\alpha^2$ . It will be useful to keep track of the weight in the constants introduced in (6), so we denote them by  $b_{\alpha, n}$ . In particular observe that in this case there is a relation between the constants: for every  $n \in \mathbb{N}$

$$\begin{aligned}
b_{\alpha, n} &= \sum_{m=n+1}^{+\infty} \left[ q^{-\alpha m} \left( \sum_{k=0}^{m-n-1} q^k \right) \left( \sum_{j=0}^{m-n-1} q^{-j} \right) \right] \\
&= \sum_{\ell=1}^{+\infty} \left[ q^{-\alpha(\ell+n)} \left( \sum_{k=0}^{\ell-1} q^k \right) \left( \sum_{j=0}^{\ell-1} q^{-j} \right) \right] = q^{-\alpha n} b_{\alpha, 0}.
\end{aligned} \quad (18)$$

Furthermore we set  $B_\alpha = \mu_\alpha(X)$ .

Now we show that the kernel  $K_\alpha$  satisfies an integral condition which will be formalized in Section 4 (see Definition 30).

**Proposition 13.** *The following holds*

$$\sup_{v \in X \setminus \{o\}} \sup_{x, y \in T_v} \sum_{z \in X \setminus T_v} |K_\alpha(z, x) - K_\alpha(z, y)| q^{-\alpha|z|} < +\infty. \quad (19)$$

*Proof.* Let  $v \in X \setminus \{o\}$ . We start by proving that (19) holds for  $y = v$ . Consider  $x \in T_v$  and observe that if  $z \in X \setminus T_v$ , then  $z \wedge x = z \wedge v$  and  $\Gamma(u, z, x) = \Gamma(u, z, v)$  for every  $u \in [o, z \wedge v]$ . Hence, if we put  $[o, z \wedge v] = \{u_t\}_{t=0}^{|z \wedge v|}$ , then from (17) we have

$$\begin{aligned} K_\alpha(z, x) - K_\alpha(z, v) &= \frac{q^2 b_{\alpha,0}}{(q-1)^2} \sum_{t=0}^{|z \wedge v|} q^{\alpha t} \Gamma(u_t, z, v) (1 - q^{t-|z|}) (q^{t-|v|} - q^{t-|x|}) \\ &= \frac{q^2 b_{\alpha,0}}{(q-1)^2} \sum_{t=0}^{|z \wedge v|} q^{(1+\alpha)t} \Gamma(u_t, z, v) (1 - q^{t-|z|}) (q^{-|v|} - q^{-|x|}). \end{aligned}$$

Then, since  $|\Gamma(u_t, z, v)| < 1$  and  $0 < q^{-|v|} - q^{-|x|} \leq q^{-|v|}$ , we have

$$\sum_{z \in X \setminus T_v} |K_\alpha(z, x) - K_\alpha(z, v)| q^{-\alpha|z|} \lesssim \sum_{z \in X \setminus T_v} \sum_{t=0}^{|z \wedge v|} q^{(1+\alpha)t} q^{-|v|} q^{-\alpha|z|}.$$

Observe that, since  $z \in X \setminus T_v$ , then  $|z \wedge v| \in \{0, \dots, |v| - 1\}$ . Then, for every  $\ell \in \{0, \dots, |v| - 1\}$ , we put

$$Y_\ell = \{z \in X : |z \wedge v| = \ell\} = T_{u_\ell} \setminus T_{u_{\ell+1}}.$$

Since  $\mu_\alpha$  is radial,

$$\mu_\alpha(Y_\ell) = \mu_\alpha(u_\ell) + (\#s(u_\ell) - 1) \mu_\alpha(T_{u_{\ell+1}}) = \begin{cases} \frac{1}{1 - q^{1-\alpha}}, & \text{if } \ell = 0; \\ q^{-\alpha\ell} \frac{1 - q^{-\alpha}}{1 - q^{1-\alpha}}, & \text{if } 1 \leq \ell < |v|. \end{cases}$$

Hence we have that

$$\begin{aligned} \sum_{z \in X \setminus T_v} |K_\alpha(z, x) - K_\alpha(z, v)| q^{-\alpha|z|} &\lesssim \sum_{\ell=0}^{|v|-1} q^{(1+\alpha)\ell} q^{-|v|} \sum_{z \in Y_\ell} q^{-\alpha|z|} \\ &\lesssim \sum_{\ell=0}^{|v|-1} q^{(1+\alpha)\ell} q^{-|v|} \\ &\simeq \sum_{\ell=0}^{|v|-1} q^{\ell-|v|} \leq \frac{q}{q-1}. \end{aligned}$$

Finally, by the triangular inequality we have (19) for every  $y \in T_v$ .  $\square$

### 3 Boundedness of the Bergman projector on $L^p_\alpha$

In this section we study the boundedness properties of the extension of the Bergman projector to  $L^p_\alpha$  spaces. For the class of exponentially decreasing radial measures we are able to prove that the extension of the Bergman projector to the relative weighted  $L^p$ -space is bounded if and only if  $p > 1$  (see Theorem ??).

In analogy with the operators studied by Zhu in Section 3.4 of [18], we introduce two families of operators. For any real parameters  $a, b$  and for  $c > 1$ , we define the integral operators

$$S_{a,b,c}f(z) = q^{-a|z|} \sum_{x \in X} |K_c(z, x)| f(x) q^{-b|x|},$$

$$T_{a,b,c}f(z) = q^{-a|z|} \sum_{x \in X} K_c(z, x) f(x) q^{-b|x|}.$$

We prove two results that imply the boundedness properties of the Bergman projectors. Theorem 14 is devoted to the study of the boundedness of  $S_{a,b,c}$  and  $T_{a,b,c}$  on weighted  $L^p$ -spaces for  $p > 1$ ; the case  $p = 1$  needs different arguments and for this reason is treated apart in Theorem 15. The two theorems are the analogues of Theorem 3.11 and Theorem 3.12 in [18], respectively. The proofs of both theorems are postponed to Subsection 3.1.

**Theorem 14.** *Let  $\alpha \in \mathbb{R}$ ,  $c > 1$  and  $1 < p < \infty$ . The following conditions are equivalent:*

- (i) *the operator  $S_{a,b,c}$  is bounded on  $L^p_\alpha$ ;*
- (ii) *the operator  $T_{a,b,c}$  is bounded on  $L^p_\alpha$ ;*
- (iii) *the parameters satisfy*

$$c \leq a + b, \quad -pa < \alpha - 1 < p(b - 1).$$

**Theorem 15.** *Let  $\alpha \in \mathbb{R}$  and  $c > 1$ . The following conditions are equivalent:*

- (i) *the operator  $S_{a,b,c}$  is bounded on  $L^1_\alpha$ ;*
- (ii) *the operator  $T_{a,b,c}$  is bounded on  $L^1_\alpha$ ;*
- (iii) *the parameters either satisfy*

$$c = a + b, \quad -a < \alpha - 1 < b - 1,$$

*or satisfy*

$$c < a + b, \quad -a < \alpha - 1 \leq b - 1.$$

We state a corollary which is simply a reformulation of the previous theorems when  $c = a + b$ .

**Corollary 16.** *Let  $1 \leq p < \infty$  and  $\alpha \in \mathbb{R}$ . If  $a, b \in \mathbb{R}$  are such that  $a + b > 1$ , then the following conditions are equivalent:*

- (i) *the operator  $S_{a,b,a+b}$  is bounded on  $L^p_\alpha$ ;*
- (ii) *the operator  $T_{a,b,a+b}$  is bounded on  $L^p_\alpha$ ;*



(iii) the parameters satisfy

$$-pa < \alpha - 1 < p(b - 1).$$

Let  $\beta > 1$ . Since  $\mathcal{A}_\beta^2$  is a closed subspace of  $L_\beta^2$ , there exists an orthogonal projection  $P_\beta: L_\beta^2 \rightarrow \mathcal{A}_\beta^2$ . Observe that by the reproducing property of  $K_{\beta,z} = K_\beta(z, \cdot)$ ,  $z \in X$ , we can write the projection  $P_\beta f$  of  $f \in L_\beta^2$  as follows

$$P_\beta f(z) = \langle P_\beta f, K_{\beta,z} \rangle_{\mathcal{A}_\beta^2} = \langle f, P_\beta K_{\beta,z} \rangle_{L_\beta^2} = \langle f, K_{\beta,z} \rangle_{L_\beta^2},$$

where we used the orthogonality of  $P_\beta$ . Hence we can rewrite  $P_\beta$  as the integral operator on  $L_\beta^2$  associated to the reproducing kernel  $K_\beta$ , that is

$$P_\beta f(z) = \sum_{x \in X} K_\beta(z, x) f(x) q^{-\beta|x|}, \quad f \in L_\beta^2, z \in X.$$

Since  $\mu_\beta$  is finite,  $L_\beta^p \subseteq L_\beta^2$  whenever  $p \geq 2$ . It is then natural to investigate whether the restriction of  $P_\beta$  to  $L_\beta^p$  is bounded. Furthermore, when  $1 \leq p < 2$  one has  $L_\beta^2 \subsetneq L_\beta^p$  and we shall study whether  $P_\beta$  admits a bounded extension to  $L_\beta^p$ . A more general question that we want to answer is whether the integral operator  $\mathcal{K}_\beta^\alpha$ ,  $\alpha > 1$ , with kernel  $K_\beta(z, x) q^{(\alpha-\beta)|x|}$  with respect to the measure  $\mu_\alpha$ , that is

$$\mathcal{K}_\beta^\alpha f(z) = \sum_{x \in X} K_\beta(z, x) f(x) q^{(\alpha-\beta)|x|} q^{-\alpha|x|}, \quad f \in L_\alpha^p \cap L_\beta^2, z \in X, \quad (20)$$

extends to a bounded operator from  $L_\alpha^p$  to  $\mathcal{A}_\alpha^p$ . The following result answers the above questions.

**Theorem 17.** *Let  $1 \leq p < \infty$ ,  $\alpha, \beta > 1$ . The operator  $\mathcal{K}_\beta^\alpha$  extends to a bounded operator from  $L_\alpha^p$  to  $\mathcal{A}_\alpha^p$  if and only if*

$$p(\beta - 1) > \alpha - 1.$$

*In particular,  $P_\alpha$  is bounded on  $L_\alpha^p$  if and only if  $p > 1$ .*

*Proof.* It is sufficient to observe that from (20),  $\mathcal{K}_\beta^\alpha = T_{0,\beta,\beta}$  on  $L_\alpha^p \cap L_\beta^2$ . Hence, the result follows from Corollary 16.  $\square$

**Remark 18.** It is worthwhile observing that the unboundedness of  $P_\alpha$  on  $L_\alpha^1$  may be seen directly with the following example. We make use of Lemma 23 that will be proved in the next subsection.

For every  $n \in \mathbb{N}$ , we fix a vertex  $v_n$  in  $S(o, n)$ , and define

$$f_n(x) = \mathbb{1}_{\{v_n\}}(x) q^{\alpha|x|}, \quad x \in X.$$

Clearly,  $\|f_n\|_{L_\alpha^1} = 1$  and  $f_n \in L_\alpha^2$ . Hence,  $P_\alpha f_n(z) = K_\alpha(z, v_n)$  and by Lemma 23

$$\|P_\alpha f_n\|_{L_\alpha^1} = \sum_{z \in X} |K_\alpha(z, v_n)| q^{-\alpha|z|} \gtrsim |v_n| = n.$$

This shows that  $P_\alpha$  does not admit a bounded extension to  $L_\alpha^1$ .

As a direct application of Theorem 17, we deduce the following result on the dual of Bergman spaces.

**Corollary 19.** *Let  $1 < p < \infty$  and  $\alpha > 1$ . Then*

$$(\mathcal{A}_\alpha^p)^* = \mathcal{A}_\alpha^{p'},$$

*with equivalent norms under the pairing*

$$\langle f, g \rangle_{\mathcal{A}_\alpha^p \times \mathcal{A}_\alpha^{p'}} = \sum_{z \in X} f(z)g(z)q^{-\alpha|z|} \quad f \in \mathcal{A}_\alpha^p, g \in \mathcal{A}_\alpha^{p'}. \quad (21)$$

*Proof.* Let  $g \in \mathcal{A}_\alpha^{p'}$ . By Hölder inequality we have that

$$|\langle f, g \rangle_{\mathcal{A}_\alpha^p \times \mathcal{A}_\alpha^{p'}}| \leq \|g\|_{\mathcal{A}_\alpha^{p'}} \|f\|_{\mathcal{A}_\alpha^p},$$

for every  $f \in \mathcal{A}_\alpha^p$  so that  $g$  defines a functional in  $(\mathcal{A}_\alpha^p)^*$ . Conversely, for  $\Phi \in (\mathcal{A}_\alpha^p)^*$ , by the Hahn-Banach theorem, there exists  $\tilde{\Phi} \in (L_\alpha^p)^*$  such that  $\tilde{\Phi}|_{\mathcal{A}_\alpha^p} = \Phi$  and  $\|\Phi\|_{(\mathcal{A}_\alpha^p)^*} \geq \|\tilde{\Phi}\|_{(L_\alpha^p)^*}$ . Then, there exists  $h \in L_\alpha^{p'}$  such that

$$\Phi(f) = \tilde{\Phi}(f) = \langle f, h \rangle_{L_\alpha^p \times L_\alpha^{p'}},$$

for every  $f \in \mathcal{A}_\alpha^p$ . By the orthogonality of  $P_\alpha$  and Theorem 17,

$$\Phi(f) = \langle P_\alpha f, P_\alpha h \rangle_{\mathcal{A}_\alpha^p \times \mathcal{A}_\alpha^{p'}} = \langle f, P_\alpha h \rangle_{\mathcal{A}_\alpha^p \times \mathcal{A}_\alpha^{p'}}.$$

Hence  $\Phi$  corresponds to  $P_\alpha h \in \mathcal{A}_\alpha^2$  under the pairing (21).  $\square$

### 3.1 Proofs of Theorems 14 and 15

This subsection is devoted to the proofs of Theorems 14 and 15, splitting them up in various steps. In both statements it is obvious that (i) implies (ii). For the rest of the section  $\alpha, a, b, c$  denote real parameters with  $c > 1$ .

#### 3.1.1 Proof that (ii) implies (iii)

In this subsection we suppose that the operator  $T_{a,b,c}$  is bounded on  $L_\alpha^p$  and we deduce necessary conditions on the parameters  $a, b, c, \alpha$ .

**Proposition 20.** *Let  $1 \leq p < \infty$ . If  $T_{a,b,c}f \in L_\alpha^p$  for every  $f \in L_\alpha^p$ , then  $-pa < \alpha - 1$ .*

*Proof.* Consider, for  $x \in X$ ,  $f(x) = q^{-R|x|}$  with  $R \in \mathbb{R}$  such that

$$R > \max \left\{ \frac{1 - \alpha}{p}, 1 - b \right\}.$$

Since  $Rp > 1 - \alpha$  we have that  $f \in L_\alpha^p$  and for every  $z \in X$

$$\begin{aligned} T_{a,b,c}f(z) &= q^{-a|z|} \sum_{x \in X} K_c(z, x) q^{-(b+R)|x|} \\ &= q^{-a|z|} \sum_{n=0}^{+\infty} q^{-(b+R)n} \sum_{|x|=n} K_c(z, x) \\ &= q^{-a|z|} \sum_{n=0}^{+\infty} q^{-(b+R)n} \#S(o, n) K_c(z, o) \end{aligned}$$

by (3) applied to the harmonic function  $K_c(z, \cdot)$ . Hence, since  $R > 1 - b$ ,

$$T_{a,b,c}f(z) = q^{-a|z|} \frac{1}{B_c} \left[ 1 + \frac{q+1}{q} \sum_{n=1}^{+\infty} q^{(-b-R+1)n} \right] = \frac{B_{b+R}}{B_c} q^{-a|z|}, \quad z \in X.$$

Now observe that  $T_{a,b,c}f \in L_\alpha^p$  implies

$$\sum_{z \in X} q^{-(ap+\alpha)|z|} = 1 + \frac{q+1}{q} \sum_{n=1}^{+\infty} q^{(1-ap-\alpha)n} < +\infty,$$

which holds if and only if  $-pa < \alpha - 1$ , as required.  $\square$

From now on we write, for  $1 \leq p < \infty$

$$\|e_{v,j}\|_p = \left( \sum_{y \in s(v)} |e_{v,j}(y)|^p \right)^{1/p}, \quad v \in X, j \in I_v.$$

**Proposition 21.** *Let  $1 \leq p < \infty$ . If  $T_{a,b,c}$  is bounded on  $L_\alpha^p$ , then  $a + b \geq c$ .*

*Proof.* Fix  $R \in \mathbb{R}$  such that

$$R > \max \left\{ \frac{1-\alpha}{p}, c-b \right\}.$$

For every  $v \in X \setminus \{o\}$  and  $j \in I_v$ , we define  $g_{v,j}(x) = f_{v,j}(x)q^{-R|x|}$ ,  $x \in X$ , where  $f_{v,j} \in \mathcal{B}$  are defined in (7). Since  $R > \frac{1-\alpha}{p}$ , we have that  $g_{v,j} \in L_\alpha^p$ . Thus,

$$\begin{aligned} T_{a,b,c}g_{v,j}(z) &= q^{-a|z|} \sum_{x \in X} K_c(z, x) f_{v,j}(x) q^{-(b+R)|x|} \\ &= q^{-a|z|} \langle f_{v,j}, K_{c,z} \rangle_{L_{b+R}^2} \end{aligned}$$

which implies  $K_{c,z} \in L_c^2 \subseteq L_{b+R}^2$  because  $R > c - b$ . Now we use the decomposition (12) of  $K_{c,z}$  on the orthonormal basis of  $\mathcal{A}_c^2$  and obtain

$$\begin{aligned} \langle K_{c,z}, f_{v,j} \rangle_{L_{b+R}^2} &= \left\langle \frac{1}{B_c} + \sum_{u \in X} \sum_{k \in I_u} \frac{\overline{f_{u,k}(z)} f_{u,k}}{b_{c,|u|}}, f_{v,j} \right\rangle_{L_{b+R}^2} \\ &= \frac{\overline{f_{v,j}(z)}}{b_{c,|v|}} \langle f_{v,j}, f_{v,j} \rangle_{L_{b+R}^2} \\ &= \frac{b_{b+R,|v|}}{b_{c,|v|}} \overline{f_{v,j}(z)}, \end{aligned}$$

where we use the orthogonality of  $\mathcal{B}$  and (10). The norm of  $T_{a,b,c}g_{v,j}$  in  $L_\alpha^p$  is

$$\begin{aligned} \|T_{a,b,c}g_{v,j}\|_{L_\alpha^p}^p &= \left( \frac{b_{b+R,|v|}}{b_{c,|v|}} \right)^p \sum_{z \in X} |f_{v,j}(z)|^p q^{-(ap+\alpha)|z|} \\ &= \left( \frac{b_{b+R,|v|}}{b_{c,|v|}} \right)^p \sum_{n=0}^{+\infty} q^{-(ap+\alpha)n} \sum_{|z|=n} |f_{v,j}(z)|^p. \end{aligned}$$

Since  $\text{supp}(f_{v,j}) \subseteq T_v \setminus \{v\}$ , the sum of  $|f_{v,j}|^p$  on the sphere  $S(o, n)$  vanishes for every  $n \leq |v|$ . If  $n > |v|$ , then the sum is on  $S(o, n) \cap T_v$  and if  $z \in T_v$  is such that  $|z| = n$

then  $p^{|z|-|v|-1}(z)$  is the unique vertex in  $s(v)$  such that  $z$  lies in its sector. Hence by (7) we have

$$\begin{aligned}
\sum_{|z|=n} |f_{v,j}(z)|^p &= \sum_{\substack{|z|=n \\ z \in T_v}} |e_{v,j}(p^{|z|-|v|-1}(z))|^p a_{n-|v|-1}^p \\
&= a_{n-|v|-1}^p \sum_{\substack{|z|=n \\ z \in T_v}} |e_{v,j}(p^{|z|-|v|-1}(z))|^p \\
&= a_{n-|v|-1}^p q^{n-|v|-1} \sum_{y \in s(v)} |e_{v,j}(y)|^p \\
&= a_{n-|v|-1}^p q^{n-|v|-1} \|e_{v,j}\|_p^p.
\end{aligned}$$

For simplicity, for every  $s \in \mathbb{R}$  and  $1 \leq p < \infty$ , we put

$$C(s, p) := \sum_{m=1}^{+\infty} q^{(1-s)m-1} a_{m-1}^p,$$

which is finite whenever  $s > 1$ . The above computation yields

$$\begin{aligned}
\|T_{a,b,c}g_{v,j}\|_{L_\alpha^p}^p &= \left(\frac{b_{b+R,|v|}}{b_{c,|v|}}\right)^p \sum_{n=|v|+1}^{+\infty} q^{-(ap+\alpha)n} a_{n-|v|-1}^p q^{n-|v|-1} \|e_{v,j}\|_p^p \\
&= \|e_{v,j}\|_p^p \left(\frac{b_{b+R,|v|}}{b_{c,|v|}}\right)^p \sum_{m=1}^{+\infty} q^{-(ap+\alpha)(m+|v|)} a_{m-1}^p q^{m-1} \\
&= \|e_{v,j}\|_p^p \left(\frac{b_{b+R,|v|}}{b_{c,|v|}}\right)^p q^{-(ap+\alpha)|v|} \sum_{m=1}^{+\infty} q^{(1-(ap+\alpha))m-1} a_{m-1}^p \\
&= \|e_{v,j}\|_p^p C(ap + \alpha, p) \left(\frac{b_{b+R,|v|}}{b_{c,|v|}}\right)^p q^{-(ap+\alpha)|v|},
\end{aligned}$$

where  $C(ap + \alpha, p)$  converges because  $ap + \alpha > 1$ , by Proposition 20. Furthermore,

$$\begin{aligned}
\|g_{v,j}\|_{L_\alpha^p}^p &= \sum_{x \in X} |f_{v,j}(x)|^p q^{-(Rp+\alpha)|x|} \\
&= \sum_{n=0}^{+\infty} q^{-(Rp+\alpha)n} \sum_{|x|=n} |f_{v,j}(x)|^p \\
&= \sum_{n=|v|+1}^{+\infty} q^{-(Rp+\alpha)n} a_{n-|v|-1}^p q^{n-|v|-1} \|e_{v,j}\|_p^p \\
&= \|e_{v,j}\|_p^p q^{-(Rp+\alpha)|v|} \sum_{m=1}^{+\infty} q^{(1-(Rp+\alpha))m-1} a_{m-1}^p \\
&= \|e_{v,j}\|_p^p C(Rp + \alpha, p) q^{-(Rp+\alpha)|v|},
\end{aligned}$$

where  $C(Rp + \alpha, p) \rightarrow 1$  when  $R \rightarrow +\infty$ . From the boundedness of  $T_{a,b,c}$  and by (18), it follows that for every  $v \in X \setminus \{o\}$ :

$$\begin{aligned}
\frac{\|T_{a,b,c}g_{v,j}\|_{L_\alpha^p}^p}{\|g_{v,j}\|_{L_\alpha^p}^p} &\simeq \left(\frac{b_{b+R,|v|}}{b_{c,|v|}}\right)^p q^{-(ap+\alpha-Rp-\alpha)|v|} \\
&\simeq q^{-p(R+b-c)|v|} q^{-(ap-Rp)|v|} \\
&= q^{(c-a-b)p|v|},
\end{aligned}$$

which is bounded if and only if  $c \leq a + b$ .  $\square$

**Proposition 22.** *Let  $1 < p < \infty$ . If  $T_{a,b,c}$  is bounded on  $L_\alpha^p$ , then  $\alpha - 1 < p(b - 1)$ .*

*Proof.* The boundedness of  $T_{a,b,c}$  on  $L_\alpha^p$  is equivalent to the boundedness of the adjoint operator  $T_{a,b,c}^*$  on  $L_\alpha^{p'}$ . It is easy to see that

$$T_{a,b,c}^*g(x) = q^{-(b-\alpha)|x|} \sum_{z \in X} K_c(x, z)g(z)q^{-(a+\alpha)|z|} = T_{b-\alpha, a+\alpha, c}g(x), \quad g \in L_\alpha^{p'}.$$

Hence, the fact that  $T_{a,b,c}^*$  is bounded on  $L_\alpha^{p'}$  implies, by Proposition 20, that  $-p'(b - \alpha) < \alpha - 1$ , that is  $\alpha - 1 < p(b - 1)$ .  $\square$

Propositions 20, 21, 22 show that (ii) implies (iii) in Theorem 14. Now we focus on the case  $p = 1$ , and we prove that (ii) implies (iii) in Theorem 15.

**Lemma 23.** *Let  $\alpha > 1$ . Then:*

$$\sum_{z \in X} |K_\alpha(x, z)|q^{-\alpha|z|} \gtrsim |x|, \quad x \in X.$$

*Proof.* The case  $x = o$  is trivial. For every  $x \in X \setminus \{o\}$ , we put  $\{v_t\}_{t=0}^{|x|} = [o, x]$ . Then, by (17)

$$\begin{aligned} \sum_{z \in X} |K_\alpha(x, z)|q^{-\alpha|z|} &\geq \sum_{t=1}^{|x|} |K_\alpha(x, v_t)|q^{-\alpha t} \\ &= \sum_{t=1}^{|x|} \left( \frac{1}{B_\alpha} + \frac{q^2}{(q+1)^2} \sum_{v \in X} \frac{1}{b_{\alpha, |v|}} \Gamma(v, v_t, x) (1 - q^{|v|-t}) (1 - q^{|v|-|x|}) \right) q^{-\alpha t} \\ &\gtrsim b_{\alpha, o}^{-1} \sum_{t=1}^{|x|} \sum_{v \in X} q^{\alpha(|v|-t)} \Gamma(v, v_t, x) (1 - q^{|v|-t}) (1 - q^{|v|-|x|}) \\ &= \sum_{t=1}^{|x|} \sum_{\ell=0}^{t-1} q^{\alpha(\ell-t)} \Gamma(v_\ell, v_t, x) (1 - q^{\ell-t}) (1 - q^{\ell-|x|}) \\ &\gtrsim \sum_{t=1}^{|x|} \sum_{\ell=0}^{t-1} q^{\alpha(\ell-t)} \simeq \sum_{t=1}^{|x|} q^{-\alpha t} q^{\alpha t} = |x|, \end{aligned}$$

where we used the fact that  $\text{supp}(\Gamma(\cdot, v_t, x)) = [o, v_{t-1}] = [v_0, v_{t-1}]$  and the function is greater than or equal to  $\frac{q-1}{q}$  there.  $\square$

**Proposition 24.** *If  $T_{a,b,c}$  is bounded on  $L_\alpha^1$ , then*

$$\begin{aligned} \alpha &< b, \quad \text{when } c = a + b; \\ \alpha &\leq b, \quad \text{when } c < a + b. \end{aligned}$$

*Proof.* From Proposition 21, if  $T_{a,b,c}$  is bounded on  $L_\alpha^1$ , then  $c \leq a + b$ . The boundedness of  $T_{a,b,c}$  on  $L_\alpha^1$  implies the boundedness of the adjoint operator  $T_{a,b,c}^*$  on  $L_\alpha^\infty$  which is given by

$$T_{a,b,c}^*g(x) = q^{-(b-\alpha)|x|} \sum_{z \in X} K_c(x, z)g(z)q^{-(a+\alpha)|z|}, \quad g \in L_\alpha^\infty.$$

In particular, by (3)

$$\begin{aligned}
T_{a,b,c}^* \mathbb{1}_X(x) &= q^{-(b-\alpha)|x|} \sum_{z \in X} K_c(x, z) q^{-(a+\alpha)|z|} \\
&= q^{-(b-\alpha)|x|} \sum_{n=0}^{+\infty} q^{-(a+\alpha)n} \sum_{|z|=n} K_c(x, z) \\
&= q^{-(b-\alpha)|x|} \frac{1}{B_c} \sum_{n=0}^{+\infty} \#S(o, n) q^{-(a+\alpha)n} = \frac{B_{a+\alpha}}{B_c} q^{-(b-\alpha)|x|},
\end{aligned}$$

which belongs to  $L_\alpha^\infty$  if and only if  $\alpha \leq b$ .

Suppose now that  $a + b = c$ . We know that  $\alpha \leq b$  and we want to prove that  $\alpha < b$ . Suppose by contradiction that  $\alpha = b$ . For every  $x \in X$  define

$$g_x(z) = \begin{cases} |K_c(z, x)| K_c(z, x)^{-1}, & \text{if } K_c(z, x) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then  $\|g_x\|_{L_\alpha^\infty} = 1$  and

$$T_{a,b,c}^* g_x(x) = \sum_{z \in X} |K_c(x, z)| q^{-c|z|} \gtrsim |x|,$$

by Lemma 23. Thus  $T_{a,b,c}^*$  is unbounded on  $L_\alpha^\infty$  and consequently  $T_{a,b,c}$  is unbounded on  $L_\alpha^1$  for  $\alpha = b$  and  $c = a + b$ . □

Propositions 20, 21, 24 show that (ii) implies (iii) in Theorem 15.

### 3.1.2 Proof that (iii) implies (i)

We start by stating a technical lemma, which will be useful both in Proposition 26 and Proposition 27, that are devoted to prove that (iii) implies (i) in the case  $p > 1$  and  $p = 1$ , respectively.

**Lemma 25.** *Let  $\beta, \gamma > 1$ . There exist  $C_1, C_2 > 0$  depending only on  $\beta$  and  $\gamma$  such that*

$$\sum_{x \in X} |K_\gamma(z, x)| q^{-\beta|x|} \leq \begin{cases} C_1(1 + q^{-(\beta-\gamma)|z|}), & \text{if } \gamma \neq \beta, \\ C_2(1 + |z|), & \text{if } \gamma = \beta. \end{cases}$$

*Proof.* We start by observing that the orthogonal bases  $\{e_{v,j}\}_{j \in I_v}$  of  $W_v$ ,  $v \in X$ , involved in the construction of the functions in  $\mathcal{B}$  are such that their 1-norms in  $W_v$  are uniformly bounded from above, namely

$$\|e_{v,j}\|_1 \leq \sqrt{|s(v)|} \|e_{v,j}\|_2 = \sqrt{|s(v)|} \leq \sqrt{q+1}, \quad v \in X, j \in I_v.$$

Hence, by (12),

$$\begin{aligned}
\sum_{x \in X} |K_\gamma(z, x)| q^{-\beta|x|} &= \sum_{x \in X} \left| \frac{1}{B_\gamma} + \sum_{v \in X} \sum_{j \in I_v} \frac{\overline{f_{v,j}(z)} f_{v,j}(x)}{b_{\gamma,|v|}} \right| q^{-\beta|x|} \\
&\leq \frac{B_\beta}{B_\gamma} + \sum_{x \in X} \sum_{v \in X} \sum_{j \in I_v} \left| \frac{\overline{f_{v,j}(z)} f_{v,j}(x)}{b_{\gamma,|v|}} \right| q^{-\beta|x|} \\
&= \frac{B_\beta}{B_\gamma} + \sum_{v \in X} \frac{1}{b_{\gamma,|v|}} \sum_{j \in I_v} |f_{v,j}(z)| \sum_{x \in X} |f_{v,j}(x)| q^{-\beta|x|} \\
&\leq \frac{B_\beta}{B_\gamma} + \sum_{v \in X} \frac{1}{b_{\gamma,|v|}} \sum_{j \in I_v} |f_{v,j}(z)| B_\beta^{-\frac{1}{2}} \|f_{v,j}\|_{L_\beta^2} \\
&= \frac{B_\beta}{B_\gamma} + B_\beta^{-\frac{1}{2}} \sum_{v \in X} \frac{b_{\beta,|v|}}{b_{\gamma,|v|}} \sum_{j \in I_v} |f_{v,j}(z)|,
\end{aligned}$$

where we use the fact that the measure  $\mu_\beta$  is finite on  $X$  and thus  $\|\cdot\|_{L_\beta^1} \leq B_\beta^{-\frac{1}{2}} \|\cdot\|_{L_\beta^2}$ , by Cauchy-Schwarz inequality. Now observe that from (7), we have that  $f_{v,j}(z) = 0$  if  $z \notin T_v \setminus \{v\}$ . Hence, if we denote by  $\{v_\ell\}_{\ell=0}^{|z|}$  the path  $[o, z]$ , then

$$\sum_{j \in I_v} |f_{v,j}(z)| = \begin{cases} a_{|z|-\ell-1} \sum_{j \in I_v} |e_{v_\ell,j}(v_{\ell+1})|, & \text{if } v = v_\ell, 0 \leq \ell < |z|; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, since  $a_n \leq \frac{q}{q-1}$  and by (18), we have

$$\begin{aligned}
\sum_{x \in X} |K_\gamma(z, x)| q^{-\beta|x|} &\leq \frac{B_\beta}{B_\gamma} + B_\beta^{-\frac{1}{2}} \frac{q}{q-1} \sum_{\ell=0}^{|z|-1} \frac{b_{\beta,\ell}}{b_{\gamma,\ell}} \sum_{j \in I_{v_\ell}} |e_{v_\ell,j}(v_{\ell+1})| \\
&\leq \frac{B_\beta}{B_\gamma} + B_\beta^{-\frac{1}{2}} \frac{q}{q-1} \frac{b_{\beta,0}}{b_{\gamma,0}} \sum_{\ell=0}^{|z|-1} q^{-(\beta-\gamma)\ell} \sup_{\substack{v \in X \\ j \in I_v}} \|e_{v,j}\|_1 \\
&\leq \begin{cases} C_1(1 + q^{-(\beta-\gamma)|z|}), & \text{if } \gamma \neq \beta, \\ C_2(1 + |z|), & \text{if } \gamma = \beta, \end{cases}
\end{aligned}$$

as required.  $\square$

**Proposition 26.** *Let  $1 < p < \infty$ . If  $a + b \geq c > 1$  and  $-pa < \alpha - 1 < p(b - 1)$ , then  $S_{a,b,c}$  is bounded on  $L_\alpha^p$ .*

*Proof.* We set

$$H(z, x) = |K_c(z, x)| q^{-a|z|} q^{-(b-\alpha)|x|},$$

so that the operator  $S_{a,b,c}$  becomes

$$S_{a,b,c}f(z) = \sum_{x \in X} H(z, x) f(x) q^{-\alpha|x|}.$$

Our purpose is to apply Schur's test (see Theorem 3.6 in [18]) to the integral operator with positive kernel  $H: X \times X \rightarrow [0, +\infty)$ . To do so, we have to show that there exists a positive function  $h$  on  $X$  such that

$$\sum_{z \in X} H(z, x) h(z)^{p'} q^{-\alpha|z|} \lesssim h(x)^{p'}, \quad \sum_{x \in X} H(z, x) h(x)^p q^{-\alpha|x|} \lesssim h(z)^p. \quad (22)$$

Observe that the two inequalities assumed for  $\alpha$  are equivalent to

$$-\frac{a + \alpha - 1}{p} < \frac{a}{p'}, \quad -\frac{b - 1}{p'} < \frac{b - \alpha}{p}.$$

Hence, since  $a + b > 1$ , it is possible to choose an element

$$\gamma \in \left(-\frac{b - 1}{p'}, \frac{a}{p'}\right) \cap \left(-\frac{a + \alpha - 1}{p}, \frac{b - \alpha}{p}\right) \neq \emptyset. \quad (23)$$

We want to show that  $h(x) = q^{-\gamma|x|}$  satisfies (22). Let  $z \in X$ . We suppose  $\gamma \neq \frac{c-b}{p'}$ . We can apply Lemma 25 since  $b + \gamma p' > 1$  by (23), obtaining

$$\begin{aligned} \sum_{x \in X} H(z, x) h(x) p' q^{-\alpha|x|} &= q^{-a|z|} \sum_{x \in X} |K_c(z, x)| q^{-(b+\gamma p')|x|} \\ &\lesssim q^{-a|z|} (1 + q^{-(b+\gamma p'-c)|z|}) \\ &\lesssim q^{-\gamma p'|z|} = h(z) p', \end{aligned}$$

where we used  $a + b - c \geq 0$  and  $a > \gamma p'$ . Similarly, when  $\gamma = \frac{c-b}{p'}$  we can apply again Lemma 25 and conclude by using  $a > \gamma p'$ . On the other hand, we have that if  $\gamma \neq \frac{c-a-\alpha}{p}$ , by  $a + \gamma p + \alpha > 0$  and by Lemma 25,

$$\begin{aligned} \sum_{z \in X} H(z, x) h(z) p q^{-\alpha|z|} &= q^{-(b-\alpha)|x|} \sum_{z \in X} |K_c(z, x)| q^{-(a+\gamma p+\alpha)|z|} \\ &\lesssim q^{-(b-\alpha)|x|} (1 + q^{-(a+\gamma p+\alpha-c)|z|}) \\ &\lesssim q^{-\gamma p|z|} = h(z) p, \end{aligned}$$

since  $a + b \geq c$  and, by (23),  $b - \alpha > \gamma p$ . Similarly when  $\gamma = \frac{c-a-\alpha}{p}$ .

In conclusion, (22) holds and by Schur's test the operator  $S_{a,b,c}$  is bounded on  $L_\alpha^p(X)$ .  $\square$

Notice that Proposition 26 shows that (iii) implies (i) in Theorem 14.

**Proposition 27.** *If  $a + b \geq c$  and*

$$\begin{aligned} -a < \alpha - 1 < b - 1, & \quad \text{when } c = a + b; \\ -a < \alpha - 1 \leq b - 1, & \quad \text{when } c < a + b, \end{aligned}$$

*then  $S_{a,b,c}$  is bounded on  $L_\alpha^1$ .*

*Proof.* Let  $f \in L_\alpha^1$ . We suppose  $c \neq a + \alpha$  and we observe that, since  $a + \alpha > 1$ , by Lemma 25

$$\begin{aligned} \|S_{a,b,c} f\|_{L_\alpha^1} &= \sum_{z \in X} \left| \sum_{x \in X} |K_c(z, x)| |f(x)| q^{-b|x|} \right| q^{-(a+\alpha)|z|} \\ &\leq \sum_{x \in X} |f(x)| q^{-b|x|} \sum_{z \in X} |K_c(z, x)| q^{-(a+\alpha)|z|} \\ &\lesssim \sum_{x \in X} |f(x)| q^{-b|x|} (1 + q^{-(a+\alpha-c)|x|}) \\ &\lesssim \sum_{x \in X} |f(x)| q^{-\alpha|x|} = \|f\|_{L_\alpha^1}, \end{aligned}$$

where we used the fact that  $a + b - c \geq 0$  and  $b \geq \alpha$ . The case  $c = a + \alpha$  follows similarly using again Lemma 25 and  $b > \alpha$ . Hence,  $S_{a,b,c}$  is bounded on  $L_\alpha^1$ .  $\square$

Proposition 27 shows that (iii) implies (i) in Theorem 15.



## 4 Calderón-Zygmund decomposition and Hörmander type kernels

In this section, we provide a Calderón-Zygmund decomposition of functions in  $L_\alpha^1$  and we formulate an integral condition for kernels  $K$  on the tree which guarantees the weak type (1,1) boundedness of the integral operators associated to  $K$  which are bounded on  $L_\alpha^2$ . As byproduct, we have that  $P_\alpha$  is of weak type (1,1) for every  $\alpha > 1$ .

We start with a preliminary geometrical result that allows us to obtain an infinite family of partitions of a sector. In particular, the partition at a given scale is a refinement of the partition at the previous scale, and the measure of a partitioning set is comparable with the measure of the set which contains it in the previous partition.

**Lemma 28.** *Let  $v \in X \setminus \{o\}$ . For every  $m \in \mathbb{N}$ , there exists  $I_m \in \mathbb{N}$  and sets  $Q_{k,m} \subseteq T_v$  for every  $k \in \mathcal{I}_m := \{0, \dots, I_m\}$  such that*

- (i)  $Q_{k,m} \cap Q_{k',m} = \emptyset$  for every  $k \neq k'$ ;
- (ii) the sector  $T_v$  is the disjoint union of the sets  $Q_{k,m}$ ,  $k \in \mathcal{I}_m$ ;
- (iii) the partition at scale  $m > 0$  is a refinement of the partition at scale  $m - 1$ , that is, for every  $k' \in \mathcal{I}_{m-1}$  there exists  $\mathcal{I}_{m,k'} \subseteq \mathcal{I}_m$  such that

$$Q_{k',m-1} = \bigsqcup_{k \in \mathcal{I}_{m,k'}} Q_{k,m};$$

- (iv) for every  $k \in \mathcal{I}_m$  and  $k' \in \mathcal{I}_{m-1}$  for which  $Q_{k,m} \subseteq Q_{k',m-1}$ , we have

$$\mu_\alpha(Q_{k,m}) \leq \mu_\alpha(Q_{k',m-1}) \leq C_\alpha \mu_\alpha(Q_{k,m}),$$

where  $C_\alpha = \max\{q^\alpha, (1 - q^{1-\alpha})^{-1}\}$ .

*Proof.* We start by computing for every  $u \in X \setminus \{o\}$

$$\mu_\alpha(T_u) = \sum_{\ell=0}^{+\infty} q^\ell q^{-\alpha(\ell+|u|)} = q^{-\alpha|u|} \frac{1}{1 - q^{1-\alpha}}. \quad (24)$$

For every  $m \in \mathbb{N}$  we set

$$I_m = \frac{q^{m+1} - q}{q - 1}.$$

We label the vertices of  $T_v$  in such a way that  $v_0 = v$  and  $s(v_k) = \{v_{qk+\ell} : \ell \in \{1, \dots, q\}\}$  for every  $k \in \mathbb{N}$ . Since  $\mathcal{I}_0 = \{0\}$  it is sufficient to set  $Q_{0,0} = T_v$ . Then for every  $m \in \mathbb{N} \setminus \{0\}$  we set

$$\begin{aligned} Q_{k,m} &:= \{v_k\}, & \text{if } k \in \mathcal{I}_{m-1}, \\ Q_{k,m} &:= T_{v_k}, & \text{if } k \in \mathcal{I}_m \setminus \mathcal{I}_{m-1}. \end{aligned}$$

In this way, (i), (ii), and (iii) easily follow by construction. If  $k \in \mathcal{I}_{m-2}$ , then (iv) follows from the fact that  $Q_{k,m} = Q_{k',m-1}$ . If  $k \in \mathcal{I}_{m-1} \setminus \mathcal{I}_{m-2}$ , then by (24)

$$\frac{\mu_\alpha(Q_{k',m-1})}{\mu_\alpha(Q_{k,m})} = \frac{\mu_\alpha(T_{v_k})}{\mu_\alpha(\{v_k\})} = \frac{q^{-\alpha|v_k|}(1 - q^{1-\alpha})^{-1}}{q^{-\alpha|v_k|}} = \frac{1}{1 - q^{1-\alpha}}.$$

Finally, if  $k \in \mathcal{I}_m \setminus \mathcal{I}_{m-1}$ , then

$$\frac{\mu_\alpha(Q_{k',m-1})}{\mu_\alpha(Q_{k,m})} = \frac{\mu_\alpha(T_{p(v_k)})}{\mu_\alpha(T_{v_k})} = \frac{q^{-\alpha(|v_k|-1)}(1-q^{1-\alpha})^{-1}}{q^{-\alpha|v_k|}(1-q^{1-\alpha})^{-1}} = q^\alpha.$$

□

The previous result leads to a Calderón-Zygmund decomposition for integrable functions on the tree at a level  $t \in \mathbb{R}^+$  sufficiently large w.r.t the  $L_\alpha^1$ -norm of the function.

**Proposition 29.** *Let  $f \in L_\alpha^1$  and  $t > \|f\|_{L_\alpha^1}/\mu_\alpha(X)$ . There exist two families  $\mathcal{Q}$  and  $\mathcal{F}$  of disjoint sets of the form  $Q_{k,m}$  such that, if we denote by  $\Omega$  and  $F$  the disjoint union of all the sets in  $\mathcal{Q}$  and  $\mathcal{F}$ , respectively, the following properties hold:*

- (i)  $X = \Omega \sqcup F$ ;
- (ii)  $|f(z)| \leq t$  for every  $z \in F$ ;
- (iii) there exist  $g, b: X \rightarrow \mathbb{C}$  and  $C > 0$  such that  $f = g + b$ ,  $\text{supp } b \subseteq \Omega$ , and  $\|g\|_{L_\alpha^2}^2 \lesssim t\|f\|_{L_\alpha^1}$ . Moreover, if we set  $b_Q = b\mathbb{1}_Q$  for every  $Q \in \mathcal{Q}$ , then

$$\sum_{z \in Q} b_Q(z)q^{-\alpha|z|} = 0, \quad \sum_{Q \in \mathcal{Q}} \|b_Q\|_{L_\alpha^1} \leq C\|f\|_{L_\alpha^1}, \quad Q \in \mathcal{Q}.$$

*Proof.* For every  $v \in S(o, 1)$  we consider the decomposition of the sector  $T_v$  given by Lemma 28. We define two families of subsets  $\mathcal{Q}_v$  and  $\mathcal{F}_v$  following the steps below. Starting from  $Q_{k,m} = Q_{0,0} = T_v$ :

1) if

$$\frac{1}{\mu_\alpha(Q_{k,m})} \sum_{z \in Q_{k,m}} |f(z)|q^{-\alpha|z|} > t,$$

then we put  $Q_{k,m} \in \mathcal{Q}_v$  and we stop. Otherwise,

2a) if  $\#Q_{k,m} = 1$  then  $Q_{k,m} \in \mathcal{F}_v$  and we stop;

2b) if  $\#Q_{k,m} > 1$  then for each set in the family

$$Q_{k,m+1} \cup \{Q_{kq+j,m+1} : j \in 1, \dots, q\}$$

we repeat the procedure, starting from 1).

We define

$$\mathcal{Q} := \begin{cases} \bigsqcup_{v \in S(o,1)} \mathcal{Q}_v, & \text{if } |f(o)| \leq t; \\ \{o\} \cup \bigsqcup_{v \in S(o,1)} \mathcal{Q}_v, & \text{otherwise,} \end{cases} \quad \mathcal{F} := \begin{cases} \{o\} \cup \bigsqcup_{v \in S(o,1)} \mathcal{F}_v, & \text{if } |f(o)| \leq t; \\ \bigsqcup_{v \in S(o,1)} \mathcal{F}_v, & \text{otherwise.} \end{cases}$$

We denote by  $\Omega$  and  $F$  the (disjoint) union of all the subsets in  $\mathcal{Q}$  and  $\mathcal{F}$ , respectively. The sets  $\Omega$  and  $F$  clearly satisfy (i) and (ii). Let  $Q \in \mathcal{Q}$ . If  $Q = \{o\}$ , then

$$t < |f(o)| \leq \mu_\alpha(X) \frac{\|f\|_{L_\alpha^1}}{\mu_\alpha(X)} < \mu_\alpha(X)t.$$

If  $Q \neq \{o\}$ , then there exist  $v \in S(o, 1)$ ,  $m \in \mathbb{N}$  and  $k \in \mathcal{I}_m$  such that  $Q = Q_{k,m} \subseteq T_v$ . If  $m = 0$  then

$$t < \frac{1}{\mu_\alpha(T_v)} \sum_{z \in T_v} |f(z)|q^{-\alpha|z|} \leq \frac{1}{\mu_\alpha(T_v)} \|f\|_{L_\alpha^1} < \frac{\mu_\alpha(X)}{\mu_\alpha(T_v)} t.$$

If  $m > 0$ , then by (iv) in Lemma 28 there exist  $k' \in \mathcal{I}_{m-1}$  and  $C_\alpha > 0$  such that  $Q_{k,m} \subseteq Q_{k',m-1}$  and  $\mu_\alpha(Q_{k',m-1}) \leq C_\alpha \mu_\alpha(Q_{k,m})$ . By the previous construction, we have that

$$t < \frac{1}{\mu_\alpha(Q_{k,m})} \sum_{z \in Q_{k,m}} |f(z)|q^{-\alpha|z|} \leq \frac{C_\alpha}{\mu_\alpha(Q_{k',m-1})} \sum_{z \in Q_{k',m-1}} |f(z)|q^{-\alpha|z|} \leq C_\alpha t.$$

Hence by choosing  $C = \max\{C_\alpha, \mu_\alpha(X)/\mu_\alpha(T_v), \mu_\alpha(X)\}$ , we have

$$t < \frac{1}{\mu_\alpha(Q)} \sum_{z \in Q} |f(z)|q^{-\alpha|z|} \leq Ct, \quad Q \in \mathcal{Q}. \quad (25)$$

It is easy to see that

$$\mu_\alpha(\Omega) \leq \frac{1}{t} \sum_{Q \in \mathcal{Q}} \sum_{x \in Q} \frac{1}{\mu_\alpha(Q)} \left( \sum_{z \in Q} |f(z)|q^{-\alpha|z|} \right) q^{-\alpha|x|} \leq \frac{\|f\|_{L_\alpha^1}}{t}. \quad (26)$$

We now define  $b = f - g$ , where

$$g(z) = \begin{cases} f(z), & z \in F; \\ \frac{1}{\mu_\alpha(Q)} \sum_{x \in Q} f(x)q^{-\alpha|x|}, & z \in Q. \end{cases}$$

It is obvious that  $\text{supp } b \subseteq \Omega$ . We show next that  $\|g\|_{L_\alpha^2}^2 \leq (1 + C^2)t\|f\|_{L_\alpha^1}$ . Indeed, by (25),

$$\begin{aligned} \|g\|_{L_\alpha^2}^2 &= \sum_{z \in F} |g(z)|^2 q^{-\alpha|z|} + \sum_{z \in \Omega} |g(z)|^2 q^{-\alpha|z|} \\ &= \sum_{z \in F} |f(z)|^2 q^{-\alpha|z|} + \sum_{Q \in \mathcal{Q}} \sum_{z \in Q} \left| \frac{1}{\mu_\alpha(Q)} \sum_{x \in Q} f(x)q^{-\alpha|x|} \right|^2 q^{-\alpha|z|} \\ &\leq \sum_{z \in F} t|f(z)|q^{-\alpha|z|} + \mu_\alpha(\Omega)C^2t^2 \leq (1 + C^2)t\|f\|_{L_\alpha^1} < +\infty, \end{aligned}$$

where we used (26). The fact that  $b_Q = b\mathbb{1}_Q$ ,  $Q \in \mathcal{Q}$ , has vanishing mean on  $Q$  follows by construction. Furthermore, since  $|b(z)| \leq |f(z)| + |g(z)|$  we have

$$\begin{aligned} \sum_{Q \in \mathcal{Q}} \sum_{z \in Q} |b_Q(z)|q^{-\alpha|z|} &\leq \sum_{z \in \Omega} |f(z)|q^{-\alpha|z|} + \sum_{Q \in \mathcal{Q}} \sum_{z \in Q} |g(z)|q^{-\alpha|z|} \\ &\leq \|f\|_{L_\alpha^1} + \mu_\alpha(\Omega)Ct \lesssim \|f\|_{L_\alpha^1}, \end{aligned}$$

by (26). □

Let  $F: X \rightarrow \mathbb{C}$ . We define the distribution function of  $F$  with respect to  $\mu_\alpha$  as

$$d_F(t) = \mu_\alpha(\{z \in X: |F(z)| > t\}), \quad t > 0.$$

The Lorentz space  $L_\alpha^{1,\infty}$  is defined as the space of functions  $F$  such that

$$\|F\|_{L_\alpha^{1,\infty}} := \sup_{t>0} \{td_F(t)\} < \infty.$$

**Definition 30.** We say that a kernel  $K: X \times X \rightarrow \mathbb{C}$  satisfies the Hörmander type condition relative to the measure  $\mu_\alpha$ , if for every  $v \in X \setminus \{o\}$

$$\sup_{v \in X \setminus \{o\}} \sup_{x, y \in T_v} \sum_{z \in X \setminus T_v} |K(z, x) - K(z, y)| q^{-\alpha|z|} < +\infty. \quad (27)$$

**Remark 31.** Condition (27) should be compared with the classical Hörmander integral condition (see [15] and formula (10) Ch.I in [17]). In our case balls or cubes are replaced by sectors and the sum is taken over the complement of the sector itself, instead of the complement of an enlarged sector.

**Theorem 32.** Fix  $\alpha > 1$  and let  $K: X \times X \rightarrow \mathbb{C}$  be a kernel satisfying the Hörmander type condition (27) with respect to  $\mu_\alpha$ . If the integral operator defined on functions  $f \in L_\alpha^2$  by

$$\mathcal{K}f(z) = \sum_{x \in X} K(z, x)f(x)q^{-\alpha|x|}$$

is bounded on  $L_\alpha^2$ , then  $\mathcal{K}$  is of weak type (1,1). Furthermore,  $\mathcal{K}$  admits a bounded extension  $\mathcal{K}$  on  $L_\alpha^p$ , for every  $1 < p < 2$ .

*Proof.* It is enough to show that for every  $f \in L_\alpha^1$

$$d_{\mathcal{K}f}(t) \lesssim \frac{\|f\|_{L_\alpha^1}}{t}, \quad t > 0. \quad (28)$$

Since  $\mu_\alpha$  is a finite measure, we have that (28) is satisfied for  $t \leq \|f\|_{L_\alpha^1}/\mu_\alpha(X)$ , indeed

$$d_{\mathcal{K}f}(t) = \mu_\alpha(\{z \in X : |\mathcal{K}f(z)| > t\}) \leq \mu_\alpha(X) \leq \frac{\|f\|_{L_\alpha^1}}{t}.$$

From now on we suppose  $t > \|f\|_{L_\alpha^1}/\mu_\alpha(X)$ . Fix a positive integer  $n \in \mathbb{N} \setminus \{0\}$ . We consider the Calderón-Zygmund decomposition of  $f$ , that is the data  $\mathcal{Q}, \mathcal{F}, \Omega, F, g$ , and  $b$  defined in Proposition 29. Observe that  $f = g + b$  yields  $\mathcal{K}f = \mathcal{K}g + \mathcal{K}b$  and

$$d_{\mathcal{K}f}(t) \leq d_{\mathcal{K}g}\left(\frac{t}{2}\right) + d_{\mathcal{K}b}\left(\frac{t}{2}\right). \quad (29)$$

By Chebychev's inequality, by the boundedness of  $\mathcal{K}$  on  $L_\alpha^2$  and by the estimate  $\|g\|_{L_\alpha^2} \lesssim t\|f\|_{L_\alpha^1}$ , we have

$$d_{P_\alpha g}(t) \leq \frac{1}{t^2} \sum_{z \in X} |P_\alpha g(z)|^2 q^{-\alpha|z|} \leq \frac{1}{t^2} \sum_{z \in X} |g(z)|^2 q^{-\alpha|z|} \lesssim \frac{\|f\|_{L_\alpha^1}}{t}.$$

A set  $Q \in \mathcal{Q}$  can either be a singleton or a (non-trivial) sector. From (27), if we choose  $v_Q := \operatorname{argmin}\{|v| : v \in Q\}$ , we then have that

$$\sup_{Q \in \mathcal{Q}} \sup_{x \in Q} \sum_{z \in X \setminus Q} |K(z, x) - K(z, v_Q)| q^{-\alpha|z|} < +\infty$$

because the summation vanishes whenever  $Q$  is a singleton. Hence, by (27),

$$\begin{aligned}
\sum_{z \in X \setminus \Omega} |\mathcal{K}b(z)|q^{-\alpha|z|} &\leq \sum_{Q \in \mathcal{Q}} \sum_{z \in X \setminus Q} |\mathcal{K}b_Q(z)|q^{-\alpha|z|} \\
&\leq \sum_{Q \in \mathcal{Q}} \sum_{z \in X \setminus Q} \sum_{x \in Q} |K(z, x) - K(z, v_Q)| |b_Q(x)|q^{-\alpha|x|}q^{-\alpha|z|} \\
&= \sum_{Q \in \mathcal{Q}} \sum_{x \in Q} |b_Q(x)|q^{-\alpha|x|} \sum_{z \in X \setminus Q} |K(z, x) - K(z, v_Q)|q^{-\alpha|z|} \\
&\lesssim \sum_{Q \in \mathcal{Q}} \sum_{x \in Q} |b_Q(x)|q^{-\alpha|x|} \lesssim \|f\|_{L_\alpha^1}, \tag{30}
\end{aligned}$$

where we used Proposition 29. Finally,

$$d_{\mathcal{K}b}(t) \leq \mu_\alpha(\{z \in X \setminus \Omega : |\mathcal{K}b(z)| > t\}) + \mu_\alpha(\Omega) \lesssim \frac{\|f\|_{L_\alpha^1}}{t},$$

from (30) and (26). The statement follows from (29). The boundedness on  $L_\alpha^p$ ,  $1 < p < 2$ , follows by interpolation.  $\square$

The following result is obtained as byproduct of Proposition 13 and Theorem 32.

**Corollary 33.** *The Bergman projector  $P_\alpha$  is of weak type  $(1, 1)$ , for every  $\alpha > 1$ .*

### Acknowledgment

The authors are grateful to Marco M. Peloso for suggesting references on the weak type  $(1,1)$  boundedness of the Bergman projector in the case of the hyperbolic disk, and to Matteo Levi for useful comments.

### References

- [1] D. Békollé. Inégalité à poids pour le projecteur de Bergman dans la boule unité de  $\mathbb{C}^n$ . *Studia Math.*, 71(3):305–323, 1981/82.
- [2] D. Békollé and A. Bonami. Inégalités à poids pour le noyau de Bergman. *C. R. Acad. Sci. Paris Sér. A-B*, 286(18):A775–A778, 1978.
- [3] D. Békollé, A. Bonami, G. Garrigós, C. Nana, M. M. Peloso, and F. Ricci. Lecture notes on Bergman projectors in tube domains over cones: an analytic and geometric viewpoint. *IMHOTEV J. Afr. Math. Pures Appl.*, 5, 2004.
- [4] T. Boiko and W. Woess. Moments of Riesz measures on Poincaré disk and homogeneous tree—a comparative study. *Expo. Math.*, 33(3):353–374, 2015.
- [5] P. Cartier. Harmonic analysis on trees. In *Proceedings of Symposia in Pure Mathematics*, volume 26, pages 419–424, 1973.
- [6] J. M. Cohen and F. Colonna. Embeddings of trees in the hyperbolic disk. *Complex Variables Theory Appl.*, 24(3-4):311–335, 1994.
- [7] J. M. Cohen, F. Colonna, M. A. Picardello, and D. Singman. Bergman spaces and Carleson measures on homogeneous isotropic trees. *Potential Anal.*, 44, 05 2016.

- [8] J. M. Cohen, F. Colonna, M. A. Picardello, and D. Singman. Fractal functions with no radial limits in Bergman spaces on trees. *Hokkaido Math. J.*, 47:269–289, 06 2018.
- [9] J. M. Cohen, F. Colonna, M. A. Picardello, and D. Singman. Carleson measures for non-negative subharmonic functions on homogeneous trees. *Potential Anal.*, 52(1):41–67, 2020.
- [10] M. Cowling, S. Meda, and A. G. Setti. An overview of harmonic analysis on the group of isometries of a homogeneous tree. *Expo. Math.*, 16(5):385–423, 1998.
- [11] M. Cowling, S. Meda, and A. G. Setti. Estimates for functions of the Laplace operator on homogeneous trees. *Trans. Amer. Math. Soc.*, 352, 01 2000.
- [12] Y. Deng, L. Huang, T. Zhao, and D. Zheng. Bergman projection and Bergman spaces. *J. Operator Theory*, 46(1):3–24, 2001.
- [13] A. Figà-Talamanca and C. Nebbia. *Harmonic analysis and representation theory for groups acting on homogenous trees*, volume 162. Cambridge University Press, 1991.
- [14] F. Forelli and W. Rudin. Projections on spaces of holomorphic functions in balls. *Indiana Univ. Math. J.*, 24(6):593–602, 1974.
- [15] L. Hörmander. Estimates for translation invariant operators in  $L^p$  spaces. *Acta Math.*, 104:93–140, 1960.
- [16] E. M. Stein. Singular integrals and estimates for the Cauchy-Riemann equations. *Bull. Amer. Math. Soc.*, 79(2):440 – 445, 1973.
- [17] E. M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993.
- [18] K. Zhu. *Operator Theory in Function Spaces*. Mathematical surveys and monographs. American Mathematical Society, 2007.