Letters

Fuzzy Smoothing Algorithms for Variable Structure Systems
Yean-Ren Hwang and Masayoshi Tomizuka

Abstract—A variable structure system (VSS) is a control system implementing different control laws in different regions of the state space divided by a set of boundary manifolds. The control input switches from one control law to another when the state crosses the boundary manifolds. In general, the control input may not be smooth when switching at these boundary manifolds and may excite high frequency dynamics. This paper proposes two fuzzy rule based algorithms for smoothing the control input. The merits of these fuzzy smoothing control algorithms are illustrated by two examples: a semiactive suspension system based on optimal control and a direct drive robot arm under discrete time sliding mode control. The controller design for these two examples is a blend of traditional control theoretic approaches and fuzzy rule based approaches.

Index Terms—Fuzzy, smoothing, control, variable structure system, VSS, suspension, robot

I. INTRODUCTION

It is normal practice that control algorithms, whether they are based on mathematical control theory or engineering intuition, require various modifications at implementation stage to enhance performance. It is rather unusual that a particular theory, when coded into computer software, provide the ultimate solution in any application. Fuzzy rule based control has been demonstrated to provide a powerful tool for fine tuning of control algorithms based on conventional control theoretic approaches. It is expected that such blending of conventional theory and fuzzy control will receive increased popularity and acceptance among control practitioners. In this paper, we will show how variable structure systems (VSS’s), the structure of which is based on mathematical analysis, can be made more appropriate for actual implementation by introduction of fuzzy rules.

A VSS is a control system implementing different control laws in different regions of the state space. The regions of the VSS are divided by their boundary manifolds which represent the checking conditions of the VSS. The control law of the VSS is altered or switched when its state crosses these boundary surfaces. In general, the control input is not smooth, sometimes even discontinuous, at switching points. A drawback of having un-smoothed switching control input is that unmodelled dynamics of the VSS may be excited. During the past two decades, many methods have been proposed (see [1] and [2] for examples) to smooth the control input. Recently, fuzzy logic control is considered as a simple alternative way to design a smoothed controller for the VSS. Kawai and Matsunaga [3] showed a simple fuzzy logic controller that is identical to a sliding mode controller with adequate boundary layers, where the Cartesian distance between the switching surface and the state is used as the input for the fuzzy controller. However, if the VSS is linear within each region, the time required for the state to travel from any point to the switching surface of the VSS is independent of its modulus. Therefore, for this type of VSS, the Cartesian distance is not a suitable input for the fuzzy logic controller. In this paper, we will develop a methodology for designing a fuzzy rule based controller to smooth the control input for a general class of VSS’s.

The remainder of this letter is organized as follows. The basic concepts of fuzzy sets and fuzzy rule based controllers are discussed in Section II. A fuzzy tuning low-pass filter is developed in Section III-A. A fuzzy region controller is developed in Section III-B and the conclusion is in Section IV.

II. FUZZY SETS AND FUZZY RULE BASED CONTROL

Fuzzy set theory, first introduced by Zadeh [4] in 1965, has been applied to numerous systems such as system control and identification. A fuzzy set can be considered as a generalization of a crisp set, to which an element can either belong or not belong. For example, if a crisp set is defined as \( D = \{1\} \) then an element \( x_1 = 1 \) belongs to \( D \) but \( x_2 = 1 \) does not belong to \( D \). In contrast, a fuzzy set considers elements having certain degree of membership as to its belonging to a particular fuzzy set. For instance, consider a fuzzy set \( A = \{x | x \in R \text{ and } x \text{ is close to } 1\} \), and an element \( x_2 = 1.1 \) can be considered belonging to \( A \) to some degree. This degree of belonging is ranged from 0 to 1, and the fuzzy set \( A \) can be characterized by a membership function which is a mapping from the universe of discourse \( U \) to the interval \([0,1]\), namely, \( \mu_A(x) : U \rightarrow [0,1] \). The problem of choosing membership functions has been studied by several researchers such as Procyk and Mamdani [5], Mizumoto [6], and Pedrycz [7]. Although the membership functions can take any suitable form as long as the range interval is \([0,1]\), the triangular/trapezoidal function, as shown in Fig. 1, is most commonly used due to its simplicity.
Set operations, such as intersection and union, have been defined for fuzzy sets (see [7] and [8] for examples). The algebraic product and algebraic sum are widely used as the intersection operation and union operation, respectively. Since fuzzy sets are characterized by their membership functions, the set operations on fuzzy sets are also defined in terms of membership functions. Therefore, if \( \mu_\text{A}(x) \) and \( \mu_\text{B}(x) \) are the degree of membership that \( x \) belongs to \( \text{A} \) and \( \text{B} \), respectively, then the degree of membership that \( x \) belongs to “\( \text{A AND B} \)” and “\( \text{A OR B} \)” are defined as follows:

\[
\begin{align*}
\mu_\text{A AND B}(x) &= \mu_\text{A}(x) \cdot \mu_\text{B}(x) \\
\mu_\text{A OR B}(x) &= \mu_\text{A}(x) + \mu_\text{B}(x) - \mu_\text{A}(x) \cdot \mu_\text{B}(x).
\end{align*}
\]

Details about fuzzy set operations can be found in [7]-[9]. The fuzzy set operations are the mathematical tool required to process the inference rules in a fuzzy rule-based controller. These inference rules are presented as linguistic statements such as:

**IF \( \hat{p} \) is \( \Lambda_i \), AND ** \( \text{IF } \hat{q} \) is \( \overline{\text{B}_j} \), THEN \( \hat{n} \) is \( \overline{\text{U}_l} \),

where \( i = 1, 2, \ldots \) is the rule index, \( \hat{p} \) and \( \hat{q} \) are linguistic variables representing the process state variables or outputs, \( \hat{n} \) is a linguistic variable representing the control variable, and \( \Lambda_i, \overline{\text{B}_j}, \text{ and } \overline{\text{U}_l} \) are fuzzy sets over \( \hat{p}, \hat{q} \) and \( \hat{n} \), respectively. The truth value of a rule is the degree to which the antecedent is true. This value can be calculated by the fuzzy set operations.

Once the truth value of each rule has been determined, the final step is defuzzification, which is the procedure to determine a crisp control for \( \hat{n} \). Basically, defuzzification is a mapping from a space of fuzzy control actions into a space of crisp control actions. There are many defuzzification strategies such as the maximum criterion, the mean of maximum, the center of area and the weighted average method. The weighted average method is commonly used by researchers and engineers because it is simple and easy to implement. By computing the weighted average of the control actions from all rules, the crisp control input can be derived as follows:

\[
n = \frac{\sum_{i=1}^{m} \mu_i \cdot u_i}{\sum_{i=1}^{m} \mu_i}
\]

where \( \mu_i \) is the truth value of \( i \)-th rule and \( u_i \) can be chosen in many ways such as the mean of all the outputs at which \( \mu_i \) are equal to its maximum value.

III. FUZZY SMOOTHING CONTROLLERS

A continuous-time single input VSS can be expressed by the following equations.

\[
\dot{x} = f(x, n, t)
\]

and

\[
\begin{align*}
\hat{n} &= \gamma_1(x) \quad \text{if } x \in \Sigma_1 \\
\hat{n} &= \gamma_2(x) \quad \text{if } x \in \Sigma_2 \\
&\vdots \\
\hat{n} &= \gamma_m(x) \quad \text{if } x \in \Sigma_m
\end{align*}
\]

where \( x \in \mathbb{R}^n, a \in R, \) and \( \Sigma_i \)'s are mutually exclusive regions of the state space, and \( \gamma_i(x) \) is the control law in \( \Sigma_i \). The union of all regions is equal to the entire state space \( \mathbb{R}^n \), i.e.,

\[
\Sigma_1 \cup \Sigma_2 \cup \ldots \cup \Sigma_m = \mathbb{R}^n
\]

Note that these regions are divided by their boundary manifolds which represent the checking conditions of the VSS. Let \( S_{ij} \) denote the boundary manifold between the regions \( \Sigma_i \) and \( \Sigma_j \). In general, the degree of freedom of each region is equal to \( n \) and the degree of freedom of each manifold is equal to \( n - 1 \). Note that these regions are crisp sets because the state can only either be in or not in one particular region. Also note that each state belongs to one and only one region.

Each region of the VSS is defined by their boundary manifolds. Therefore, the belonging of the state to a particular region can be checked by the “sign distance” (denoted by \( \zeta \)) between the state and the boundary manifolds of that region. There are many ways to define \( \zeta \); two of them are given as below:

- **the \( l_p \)-norm signed distance** \( d_p(x, S_{ij}) \), defined as

\[
\zeta = d_p(x, S_{ij}) = \left| sign_{ij}(x) \right| \min_{\gamma \in S_{ij}} \{\|x - \gamma\|_p\}
\]

\[
= sign_{ij}(x) \cdot \min_{\gamma \in S_{ij}} \{\|x_1 - y_1\|^p + \|x_2 - y_2\|^p \ldots \|x_n - y_n\|^p\}
\]

where \( sign_{ij}(x) \) is positive if \( x \) is on one particular side of \( S_{ij} \), and negative if \( x \) is on the other side of \( S_{ij} \). The most common use is the \( d_2(x, S_{ij}) \) function because it represents the distance in Cartesian sense. For example, if \( S_{ij} \) is a linear manifold then \( d_2(x, S_{ij}) \) is equal to \( x^T s \), where \( s \) is the unit normal vector of \( S_{ij} \). Because of its physical meaning and simple calculation, \( d_2(x, S_{ij}) \) is frequently used, e.g., Kawaji and Matsumasa [3].

- **the angle between \( x \) and the normal vector of \( S_{ij} \)**. For all linear systems, the time required for a state to travel to any linear manifold containing the origin is independent of the modulus of the state. This fact is important for discrete time control system because the sampling time is usually constant. Jabbari and Tomizuka [10] proposed a discrete time sliding mode controller which guarantees the state to reach a cone around the sliding surface. The previous definition of \( d_2(x, S_{ij}) \) is not suitable because, for the same \( d_2 \) values, \( x \) with larger modulus requires
less time to hit the manifold than \( x \) with smaller modulus.

In this case, the angle \( \theta \) between \( x \) and \( \delta \) and its cosine value are suitable measurements of \( \zeta \). When the state \( x \) is close to the boundary manifold, \( \theta \) is close to 90 degree and \( \cos \theta \) is close to zero.

Note that the sign of \( \zeta \) shows the side of the manifold that the state belongs to. Due to the fact that a VSS has different dynamics at different sides of the manifold, the sign of \( \zeta \) is important.

For the VSS defined by (2), the state can only be in one particular region. Therefore, when the state passes through a boundary manifold, the state enters a different region instantly and the control input also switches from one law to another instantly. These switchings generally are not smooth and may cause undesired dynamics of the VSS. A typical example is the chattering phenomenon of a sliding mode control system which has discontinuous control inputs on different sides of the sliding surface. To smooth the control input, we propose two methods to design a fuzzy smoothing controller. First, a low-pass filter with fuzzy tuning bandwidth is used. Second, a fuzzy logic controller with fuzzy regions is proposed to replace the original crisp regions of the VSS.

### A. Fuzzy Tuning Bandwidth Low-Pass Filter

A common approach to smooth the control inputs is to add a low-pass filter:

\[
\dot{u} = -\lambda(u - u_i) \quad i f \quad x \in \Sigma_i
\]

where \( \lambda \) is the bandwidth of the filter, \( u_i \) is the control command computed from \( g_i(x) \), and \( u \) is the filtered input of the VSS. When \( \lambda \) is small, abrupt changes of \( u \) can be prevented. However, if \( \lambda \) is too small, the difference between \( u \) and \( u_i \) will become large and the performance achieved by the original VSS may be lost. Intuitively, when the state is extremely close to one of the boundary manifolds, \( \lambda \) should be small since the change of \( u \) is expected to be abrupt. Otherwise, \( \lambda \) can be made large so that the performance achieved by the original VSS will not be lost. Therefore, the following fuzzy rules are used to select \( \lambda \).

**IF**  
\( \zeta_i \) is \( M_i \)  
**THEN** \( \lambda \) is \( N_i \)

where \( \zeta_i \) is the “signed distance” presented in the previous section, \( M_i \)'s and \( N_i \)'s are fuzzy sets over \( \zeta \) and \( \lambda \), respectively. Figs. 1 and 2 show a set of membership functions of \( M_i \) and \( N_i \), and Table I gives a set of selecting rules for \( \lambda \). According to (1), the crisp value for \( \lambda \) can be calculated by the following equation.

\[
\lambda = \sum_{i=1}^{m} \mu_{M_i} \lambda_i / \sum_{i=1}^{m} \mu_{M_i}
\]

where \( \lambda_i \) is the value such that \( \mu_{N_i}(\lambda_i) = \max \{ \mu_{N_i} \} \).

The discrete time version of this low-pass filter is as follows.

\[
\Delta u(k) = -\lambda \Delta t(u(k - 1) - u_i(k - 1))
\]

where \( \Delta t \) represents the sampling time and \( \Delta u(k) = u(k) - u(k - 1) \). Therefore, the fuzzy inference laws for selecting \( \lambda \) can be generalized to the fuzzy inference laws for controlling \( \Delta u(k) \). Because \( \Delta u(k) \) is equal to the multiplication of \( \lambda \Delta t \) and \( (u(k - 1) - u_i(k - 1)) \), the inference laws for controlling \( \Delta u(k) \) may be further generalized to depend both on \( \zeta_i(k - 1) \) and \( (u(k - 1) - u_i(k - 1)) \) as follows.

**IF** \( \zeta_i(k - 1) \) is \( \hat{M}_i \)  
**AND IF** \( (u(k - 1) - u_i(k - 1)) \) is \( \hat{P}_i \)  
**THEN** \( \Delta u(k) \) is \( \hat{Q}_i \)

where \( \hat{M}_i, \hat{P}_i \) and \( \hat{Q}_i \) are suitable fuzzy sets. Note that this type of inference rules is often used when designing a fuzzy logic controller.

Due to the fact that the overall system (2) and (3) is one order higher than the original VSS, the low-pass filter introduces additional delay into the system. When applied to a sliding mode control system, whose boundary manifold is an attractor and control input switches at very high frequency, this type of smoothing controller may make the system less robust and even cause instability. However, when the boundary manifold is not an attractor and the state always passes through the boundary manifold, this type of low-pass filter can be effective.

**Example 1: Semiactive Suspension System:** Vehicle suspensions systems consist of a sprung mass, an unsprung mass, a damping element and a spring element (see Fig. 3). If an actuator (e.g., a hydraulic actuator) is placed between the sprung mass and unsprung mass, the suspension characteristics may be arbitrarily adjusted, and such a system is called an active suspension system. While the active suspension system is known to improve performance and is available in several luxurious models, it is expensive and may be unstable. A cost effective alternative is to place a damper with variable damping coefficient as shown in Fig 3, which is referred to as a semiactive suspension system. Note that the variable damper is effective for generating a pushing force for \( z_a - z_u < 0 \) and a pulling force for \( z_s - z_u > 0 \).
Fig. 3 shows the dynamic model of a quarter car semiactive suspension system, which has the variable damping coefficient $u$ as its control input. The state equation of a quarter car semiactive suspension system is described as:

$$\dot{x} = Ax + (N x) u + L w$$

(7)

where $A \in R^{2 \times 2}$, $N \in R^{2 \times 4}$ are constant matrices, $L \in R^1$ is a constant vector, $w \in R$ is the road disturbance, $u \in R$ is the control input and corresponds to the adjustable damping coefficient taking a value in $[u_{\text{min}}, u_{\text{max}}]$. The state $x = (x_1, x_2, x_3, x_4)^T$, and $x_1$, $x_2$, $x_3$, and $x_4$ represent the suspension deflection, sprung mass velocity, tire deflection, and unsprung mass velocity, respectively. The performance index of the semiactive suspension system is defined as follows.

$$J = \lim_{t \to \infty} \frac{1}{2} \int_0^t (\dot{x}_2^2 + 400x_1^2 + 16x_2^2 + 400x_3^2 + 16x_4^2)dt$$

where $\dot{x}_2$ represents the sprung mass acceleration. The suboptimal control law [12], [13] is described as:

$$u = u_{\text{min}} = \begin{cases} \alpha^T x > 0 & \text{and} \quad \gamma_1^T x > 0 \\ \alpha^T x < 0 & \text{and} \quad \gamma_1^T x < 0 \end{cases}$$

$$u = \alpha = \begin{cases} \alpha^T x > 0 & \text{and} \quad \gamma_1^T x \leq 0 \leq \gamma_2^T x \\ \alpha^T x < 0 & \text{and} \quad \gamma_1^T x \geq 0 \geq \gamma_2^T x \end{cases}$$

$$u = u_{\text{max}} = \begin{cases} \alpha^T x > 0 & \text{and} \quad \gamma_2^T x < 0 \\ \alpha^T x < 0 & \text{and} \quad \gamma_2^T x > 0 \end{cases}$$

where

$$a^\dagger = \frac{\alpha^T x}{\alpha^T x} - b$$

is the optimal control for nonconstrained input (i.e., $u^\dagger \in [-\infty, \infty]$), $b$ is the damping coefficient of the passive damper (see Fig. 3) and $x$, $a^\dagger$ and $a$ are the normal vectors of the following three linear boundary manifolds.

$$\alpha^T x = 0: \text{with normal vector } = \alpha$$

$$\gamma_1^T x = 0: \text{with normal vector } = \gamma_1$$

$$\gamma_2^T x = 0: \text{with normal vector } = \gamma_2$$

Note that $a^\dagger = (u_{\text{min}} + b)\alpha + \beta$, $a = (u_{\text{max}} + b)\alpha + \beta$ and $a$ are all in the same second order subspace defined by $a$ and $\beta$. Therefore, these three boundary manifolds intersect at the same $R^2$ subspace and divide the state space into six regions. The values of $u_{\text{min}}$, $u_{\text{max}}$, $A$, $N$, $L$, $\alpha$, $\beta$, $\gamma_1$ and $\gamma_2$ can be found in the appendix.

Fig. 4 shows the conceptual picture of the three boundary manifolds, their normal vectors and the six regions divided by them. In Region (I) (where $\alpha^T x > 0$ and $\gamma_1^T x > 0$) and Region (IV) (where $\alpha^T x < 0$ and $\gamma_1^T x < 0$), $u$ is assigned to be $u_{\text{min}}$. In Region (II) (where $\alpha^T x > 0$ and $\gamma_1^T x \leq 0 \leq \gamma_2^T x$) and Region (V) (where $\alpha^T x < 0$ and $\gamma_1^T x \geq 0 \geq \gamma_2^T x$), $u$ is assigned to be $u_{\text{max}}$. In Region (III) (where $\alpha^T x > 0$ and $\gamma_2^T x < 0$) and Region (VI) (where $\alpha^T x < 0$ and $\gamma_2^T x > 0$), $u$ is assigned to be $u_{\text{max}}$. It is shown in [13] that the system is linear within each region and all boundary manifolds are not attractive sets.

As the state $x$ crosses these boundary manifolds, the controller switches from one law to another and the dynamic system switches from one linear system to another. It has been shown by simulation that high sprung mass jerk is created when $x$ crosses the manifold $\alpha^T x = 0$. To smooth the control input and reduce the sprung mass jerk, a fuzzy tuning bandwidth low-pass filter is added.

In constructing the fuzzy tuning bandwidth low-pass filter, the signed distance, $\zeta$, is defined as the cosine value of the angle between $x$ and $a$. The membership functions for $M_i$'s and $N_i$'s are defined as shown in Figs. 1 and 2, respectively, the inference rules are given in Table I, and $\lambda$ is calculated by (5).

During the first simulation, the initial condition is chosen $x(0) = (0.0, 0.0, 0.0, 0.0)^T$ and the disturbance $w$ is set to be zero. The control inputs of two different controllers for semiactive suspension are shown in Fig. 5. The dashed line shows the control input of the unsmoothed suboptimal controller, and the solid line shows the control input with a fuzzy smoother. Apparently, the control input is smoothed by the fuzzy smoother. Fig. 6 shows that the suspension deflection $x_1$ changes only slightly for a suboptimal controller with a fuzzy smoother. Similarly, Fig. 7 shows that the sprung mass acceleration does not change much between these two controllers. However, Fig. 8 shows the reduction of the maximum sprung mass jerk from $130 m/s^2$ to $150 m/s^2$ when the fuzzy smoother is used. Fig. 9 shows the control input of two different controllers.
when the vehicle travels over a bump described by

$$w(t) = \begin{cases} 
0.1 \sin 10\pi(t - 0.1) & \text{when } t \in [0.1, 0.3] \\
0 & \text{otherwise} 
\end{cases}$$

(8)

and the initial condition is set to zero. Figs. 10, 11, and 12 show the time response of the suspension deflection, sprung mass acceleration and sprung mass jerk respectively. Similar to the previous simulation results, the control input is smoothed by the fuzzy smoother and the maximum absolute sprung mass jerk is reduced from 70m/s$^3$ to 40m/s$^3$ at the price of increasing of the maximum absolute sprung mass acceleration from 0.8 m/s$^2$ to 0.92 m/s$^2$.

**B. Fuzzy Regions and Fuzzy Boundaries**

Based on the control laws of the VSS defined in (2), the following fuzzy rules can be generated:

- IF $x$ is in $\Sigma_i$ THEN $u_i = g_i(x)$ for $i = 1, 2, \ldots, m$

where $\Sigma_i$ are the fuzzy regions generalized from $\Sigma_i$ while $g_i(x)$ remain to be crisp functions. These fuzzy regions can be defined on the “sign distance” as follows:

$$\mu_{\Sigma_i}(\zeta) = \begin{cases} 
L((\zeta_i - \zeta)/p) & \text{if } \zeta < \zeta_i; \\
1 & \text{if } \zeta_i \leq \zeta \leq \zeta_{2i}; \\
R((\zeta - \zeta_{2i})/q) & \text{if } \zeta > \zeta_{2i} 
\end{cases}$$

where $\zeta_i$ and $\zeta_{2i}$ are parameters defining the boundary of $\Sigma_i$; $p$ and $q$ parameterize the left and right characteristic function $L(.)$ and $R(.)$. Fig. 13 shows two simple membership functions for $\Sigma_i$. Note that the membership functions shown in Fig. 13 have the following property:

$$\sum_{i=1}^{m} \mu_{\Sigma_i}(\zeta) = 1$$

which imply that the fuzzy classification is compatible with the feature based classification in terms of classical sets (Langari and Tomizuka [14]).
Example 2: Direct Drive Robot Arm A variable structure controller with fuzzy region smoother is tested by simulation for a single axis direct drive robot arm, which is modeled by the following equation:

$$\dot{x}(t) = Ax(t) + Bu(t) + d(t)$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{b} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{K}{b} \end{bmatrix} u(t) + d(t)$$

(10)

where $x_1(t)$ and $x_2(t)$ represent the angular position and velocity of the arm, and

- $u(t) = \text{input}$,
- $K = \text{torque constant}$,
- $b = \text{damping}$,
- $J = \text{inertia} = J_0 + \Delta J$
- $J_0 = \text{nominal inertia}$, $\Delta J = \text{uncertainty in inertia}$.
- $d(t) = \text{disturbance}$ and $|d(t)| \leq d_{\text{max}}$

The values of $b$ and $K$ are assumed to be $1.4 \text{ Nm/sec}$ and $39.0 \text{ Nm/volt}$ respectively. The nominal inertia $J_0$ is assumed to be $1.89 kg.m^2$ and the variation of the inertia $\Delta J$ is assumed bounded by $1.06 kg.m^2$. The sliding surface is defined as $s = 0$ where

$$s = C^T x \quad \text{and} \quad C = [20, 1]^T.$$

When the input $u$ is preceded by a zero order hold, the following discrete time model can be obtained:

$$x(k + 1) = A_dx(k) + B_du(k) + \int_0^{\Delta t} e^{A\tau} d(\tau) B_d\tau$$

where

$$A_d = e^{A\Delta t}, \quad B_d = \int_0^{\Delta t} e^{A\tau} B_d\tau$$

The sampling time $\Delta t$ is chosen as $4 \text{ ms}$.

For the position control, Jabbari and Tomizuka [10] proposed a digital sliding mode controller which guaranteed that the state will reach a cone around the sliding surface. Following [10], we define a cone $\Psi_C$ and a ball $\Psi_B$ in Fig. 16 as follows:

$$\Psi_C = \{ x| \nu M(x) \geq |C^T x| \} = \{ x| \nu \|C\| \|x\| \cos \theta_c \geq |C^T x| \},$$

$$\Psi_B = \{ x| \phi \geq \|x\| \}.$$

where

$$M(x) = \|C\| \|x\| e^{\gamma \Delta t - 1}, \quad \nu = \frac{(C^T B_d)_{\text{left}} + (C^T B_d)_{\text{right}}}{2(C^T B_d)_{\text{max}}},$$

$$\cos \theta_c = \cos^{-1}(e^{\gamma \Delta t - 1}), \quad \phi = 2(C^T B_d)_{\text{max}} d_{\text{max}}.$$

and $\gamma$ is the maximum singular value of $A$. We partition the state space into four regions as follows (see Fig. 16).

- $\Sigma_1 = \{ x| \|x\| > \phi \}$ and $\cos \theta > \cos \theta_c$
- $\Sigma_2 = \{ x| \|x\| > \phi \}$ and $\cos \theta < -\cos \theta_c$
- $\Sigma_3 = \{ x| \|x\| > \phi \}$ and $-\cos \theta_c \leq \cos \theta \leq \cos \theta_c$
- $\Sigma_4 = \{ x| \|x\| \leq \phi \}$

where $\theta$ is the angle between the state vector and the normal vector of the sliding surface. The control laws in Region $\Sigma_1$ and $\Sigma_2$ are defined according to [10] as follows.

- $u(k) = -M(x(k)) - d_{\text{max}}$ if $s > 0 \quad$ (i.e. $x \in \Sigma_1$)
- $u(k) = M(x(k)) + d_{\text{max}}$ if $s < 0 \quad$ (i.e. $x \in \Sigma_2$)
Note that this control law satisfies the conditions, developed by Jabbari and Tomizuka in [10], which guarantee $|s_1(t)|$ to decrease if $x$ is in $\Sigma_1$ and $\Sigma_2$. However, these two control laws do not guarantee that $x$ approaches the sliding surface if the state is in Region $\Sigma_3$ and $\Sigma_4$. Therefore, we define the following control law for Region $\Sigma_3$:

$$u(k) = -\frac{1}{C^T B_d} \left( C^T A_d x(k) - s(k) + K_1 \Delta t \frac{\cos \theta(k)}{\nu \cos \theta_r} \right)$$

where $K_1$ is a positive number. If there is no disturbance and parameter uncertainty, this control law guarantees that the state will stay inside the cone and approach to the origin. However, when there exist disturbance and parameter uncertainties, this controller does not work around the origin because the disturbance will push the state out of the cone. Therefore, we define another control law in Region $\Sigma_4$ as follows:

$$u(k) = -\frac{1}{C^T B_d} \left( C^T A_d x(k) - s(k) + K_2 \Delta t \frac{s(k)}{\phi} \right)$$

where $K_2$ is a positive number. Note that $K_1$ and $K_2$ should be large enough to cover the disturbance and parameter uncertainties.

To design the fuzzy smoothing controller, we define the following fuzzy regions:

- $\tilde{\Sigma}_1 = \{ x|\zeta_2(x) \text{ is } \text{PB} \text{ AND } \zeta_3(x) \text{ is } \text{B} \}$
- $\tilde{\Sigma}_2 = \{ x|\zeta_2(x) \text{ is } \text{NB} \text{ AND } \zeta_3(x) \text{ is } \text{B} \}$
- $\tilde{\Sigma}_3 = \{ x|\zeta_2(x) \text{ is } \text{S} \text{ AND } \zeta_3(x) \text{ is } \text{B} \}$
- $\tilde{\Sigma}_4 = \{ x|\zeta_3(x) \text{ is } \text{S} \}$

where

$$\zeta_2(x) = \frac{C^T x}{\|C\|\|x\|} \text{ and } \zeta_3(x) = \|x\|$$

and B and S represent linguistic terms as big and small, respectively, and P and N represent positive and negative.

Fig. 17. Membership functions for characterizing $\zeta_2$ and $\zeta_3$.

Note that $\zeta_2$ is the cosine value of the angle between $x$ and the normal vector of the sliding surface. The membership functions for characterizing $\zeta_2$ and $\zeta_3$ are chosen as shown in Fig. 17. The fuzzy sets $\tilde{\Sigma}_1$, $\tilde{\Sigma}_2$, and $\tilde{\Sigma}_3$ can be represented as the intersection of two fuzzy sets, i.e.,

$$\tilde{\Sigma}_1 = \tilde{E}_{PB} \cap \tilde{E}_B,$$
$$\tilde{\Sigma}_2 = \tilde{E}_{NB} \cap \tilde{E}_B,$$
$$\tilde{\Sigma}_3 = \tilde{E}_S \cap \tilde{E}_B$$

where

$$\tilde{E}_{PB} = \{ x|\zeta_2(x) \text{ is } \text{PB} \}, \quad \tilde{E}_{NB} = \{ x|\zeta_2(x) \text{ is } \text{NB} \},$$
$$\tilde{E}_S = \{ x|\zeta_3(x) \text{ is } \text{S} \}, \quad \tilde{E}_B = \{ x|\zeta_3(x) \text{ is } \text{B} \}$$

Therefore, by applying the fuzzy set operation,

$$\mu_{\tilde{\Sigma}_1} = \mu_{\tilde{E}_{PB}} \mu_{\tilde{E}_B},$$
$$\mu_{\tilde{\Sigma}_2} = \mu_{\tilde{E}_{NB}} \mu_{\tilde{E}_B},$$
$$\mu_{\tilde{\Sigma}_3} = \mu_{\tilde{E}_S} \mu_{\tilde{E}_B}.$$}

The final control input can be obtained by (9).

During simulation, the initial condition $x(0)$ is equal to $[-1,0]^T$, the disturbance $d(t)$ is specified as $0.1 \cos 6\pi t$ and $\Delta F$ is assumed as $-1.06kg.m^2$. The constants $K_1$ and $K_2$ are chosen to be 10 and 4 respectively. Fig. 18 shows that the reaching time of the VSS with fuzzy regions is almost the same as that of the VSS with crisp regions. However, in Fig. 19, the control input of the original VSS chatter when the state crosses two boundary manifolds: the manifold between Region $\Sigma_1$ and $\Sigma_3$ and the manifold between $\Sigma_3$ and $\Sigma_4$. Fig. 20 shows that the state also chatters at these boundary manifolds. The solid lines in Figs. 19 and 20 represent the control input and state trajectory of the fuzzy region VSS. Apparently, both the control input and the state trajectory are smoothed.

**IV. CONCLUSION**

In developing the fuzzy smoothing controller, a new distance measure $\zeta$ is proposed, which achieves better control result. It is found that the angle between the state and the normal vector of the boundary manifold ($\theta$) and its cosine value can be used as $\zeta$. For discrete time control of linear dynamic systems with constant sampling time, $\theta$ and $\cos \theta$ are better choices.

Two fuzzy smoothing controllers have been developed: 1) a fuzzy tuning low-pass filter and 2) a fuzzy region controller.
The second controller is applied to smooth the control input of a direct drive robot arm under sliding mode control. Simulation results show that the chattering of the control input is eliminated by the second controller. In either of these two examples, the controller structure was derived based on conventional control theoretic consideration. The fuzzy tuning low-pass filter and fuzzy region controller were used to enhance performance of the controller. In this sense, the paper has presented two examples of blending conventional and fuzzy rule based approaches to the design of control systems.

**Appendix**

\[
\begin{align*}
A &= \begin{pmatrix} 
0 & 1 & 0 & -1 \\
-k_p/m_s & -b/m_s & 0 & b/m_s \\
0 & 0 & 0 & 1 \\
k_s/m_u & b/m_u & -k_t/m_u & -b/m_u \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1/m_u & 0 & -1/m_u \\
0 & 0 & 0 & 0 \\
\end{pmatrix}; \\
N &= \begin{pmatrix} 
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}; \\
L &= (0 \ 0 \ -1 \ 0)^T; \\
\alpha &= (0 \ 1.7 \times 10^{-5} \ 0 \ -1.7 \times 10^{-5})^T; \\
\beta &= (0.1944 \ -0.0313 \ 0.0418 \ 0.0167)^T; \\
\gamma_1 &= (0.1944 \ -0.0140 \ 0.0418 \ -0.0007)^T; \\
\gamma_2 &= (0.1944 \ 0.0381 \ 0.0418 \ -0.0528)^T; \\
\end{align*}
\]

\[u_{\text{max}} = 3000. \quad \text{and} \quad u_{\text{min}} = 0\]

**References**


