Application of Improved Grey Prediction Model to Short Term Load Forecasting

Guo-Dong Li*, Daisuke Yamaguchi**, Masatake Nagai***

Abstract – The grey model GM(1,1) based on the grey system theory has recently emerged as a powerful tool for short term load forecasting (STLF) problem. Since GM(1,1) is only first order dynamic grey model, the accuracy is not satisfactory when original data show great randomness. In this paper, we proposed improved dynamic mode GM(2,1) to enhance forecasted accuracy. Then it is applied to improve STLF performance. The improved procedure is shown as follows briefly: We presented a grey interval analysis, then this analysis based whitening coefficients were presented. Furthermore, these coefficients were combined with cubic spline function to establish GM(2,1) model. Finally, Taylor approximation method is presented to optimize these whitening coefficients and make forecasted error reduce to minimum. The improved GM(2,1) model is defined as T-3spGM(2,1) and it can overcome above mentioned shortcomings. The power system load data of ordinary and special days were used to test the proposed model. The experimental results showed that the proposed T-3spGM(2,1) model has better performance for STLF problem.

Keywords: short term load forecasting (STLF), grey forecasting model GM(2,1), cubic spline function, Taylor approximation method, grey interval analysis.

1. Introduction

The short term load forecasting (STLF) problem has been widely studied in the field of electrical power and energy systems. The reason is that accurate forecasting helps in the real-time power generation, efficient energy management, and economic cost saving. Up to present, proposed methods for STLF problem can be roughly divided into four types: time series method, regression method, expert-based method and neural network based method. These methods need not only a large amount of history data but also typical distributions, and use the statistic method to analyze the characteristics of the system. Thus, these methods are limited by the used history data. Therefore, they are often difficult to carry on and even not approachable due to cost consideration.

In order to reduce the amount of sample data and obtain high accuracy, Deng presented the grey system theory [1]. The grey model (GM) based on the grey system theory is a forecasting dynamic model and has been applied to many forecasting fields. The GM has three properties: First, it does not need a large amount of sample data. Second, its calculation is simple. Third, it can use random sample data. Since 1990s, the approach based on GM is getting more and more attention for its promising results in STLF. In the beginning, researchers are trying to demonstrate the feasibility of applying GM to STLF problem in power engineering. Recently, efforts are put to improve the forecasting performance of GM [2]. A wide variety of methods to improve STLF performance have been reported which include combining with ARIMA model or neural network, error compensation and data preprocessing etc.. But these methods were proposed only based on GM(1,1) which stands for the first order grey dynamic model with one variable. It is analyzed and discussed that GM(1,1) is only first dynamic differential equation model, the accuracy is not satisfactory when original data shown great randomness. Even if the improved GM(1,1) is used to resolve STLF problem, but the basic weak point cannot be settled. Up to present, GM(2,1) which stands for the second order grey dynamic model with one variable is not applied to STLF problem in power engineering still. It has pointed out that the differential equation model of GM(2,1) has very serious morbidity problem and suggested that it must be careful to use GM(2,1). However, there is not an effective method for using GM(2,1). Since the change of power load include second order factors, if GM(2,1) is not used, the superior properties of grey dynamic model cannot be expressed completely.

In this paper, we proposed an improved GM(2,1) model, then it is applied to perform load forecasting. The improved procedure is shown as follows briefly: First, we presented a grey interval analysis. Second, this analysis based whitening coefficients were presented, furthermore, these coefficients were combined with cubic spline function to establish GM(2,1) model. Finally, Taylor approximation method is presented to optimize these whitening coefficients and make forecasted error reduce to minimum. The improved GM(2,1) model is defined as T-3spGM(2,1).

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The power system load data of ordinary and special days were used to verify the effectiveness of proposed model. The experimental results showed that the proposed T-3spGM(2,1) model has better performance for STLF problems.

2. Preliminaries

In recent years, grey system theory has become a very effective method of solving uncertainty problems under discrete data and incomplete information. The theory includes five major parts, which include grey forecasting, grey relation, grey decision, grey programming and grey control. The grey forecasting dynamic model has the advantages of establishing a model with few data and uncertain data and has become the core of grey system theory.

2.1 Basic Definitions

Definition 1. A grey system [1] is defined as a system containing uncertain information presented by grey intervals and grey variables.

Definition 2. In grey system, when a forecasting model uses an observed data set, there will be a numerical interval accompanying it. This numerical interval will contain the accuracy and the other sources of uncertainty that are associated with the observed values in the data set. The numerical interval is defined as grey interval.

Definition 3. The number of grey interval is defined as grey number. Grey number means that the certain value is unknown, but the rough range is known. The grey interval can be taken as a special grey number $\otimes X_g$, with bound values $X_d$ and $X_u$:

$$\otimes X_g = [X_d, X_u] \quad (1)$$

where $X_d$ is the lower limit and $X_u$ is the upper limit.

Definition 4. The whitening method of grey number is shown as following:

$$X_g = (1 - \mu)X_d + \mu X_u \quad (2)$$

where $\mu = [0, 1], \mu$ is called whitening coefficient.

2.2 Grey GM(1,1) model

The GM based on grey system theory do not requires lots of history data to make grey forecasting. The most critical feature of GM is the use of grey generating approaches to reduce the variation of the original data series by transforming the data series linearly. The most commonly seen and applied grey generating approaches are the accumulative generating operation (AGO) and the inverse accumulative generating operation (IAGO). The AGO converts a series lacking any obvious regularity into a strictly monotonically increasing series to reduce the randomness of the series, increase the smoothness of the series, and minimize interference from the random information.

Definition 5. Assume that $x^{(0)}=\{x^{(0)}(1), x^{(0)}(2), \cdots, x^{(0)}(n)\}$ is original data series of real numbers with irregular distribution.

Then $x^{(i)}$ is viewed as 1-AGO generation series for $x^{(0)}$, if $\forall x^{(i)}(j) \in x^{(i)}$ can satisfy

$$x^{(i)}(j) = \sum_{k=1}^{j} x^{(0)}(i) \quad (3)$$

Then $x^{(i)} = \{\sum_{k=1}^{n} x^{(0)}(i), \sum_{k=1}^{n} x^{(0)}(i), \cdots, \sum_{k=n-i}^{n} x^{(0)}(i)\}$, which is the first order AGO series obtained from $x^{(0)}$.

Definition 6. From (3), it is obvious that the original data $x^{(0)}(i)$ can be easily recovered from $x^{(1)}(i)$ as

$$x^{(0)}(i) = \sum_{j=1}^{i} x^{(1)}(i-1) \quad (4)$$

where $x^{(0)}(1) = x^{(1)}(i)$, $x^{(0)}(i) \in x^{(1)}$. This operation is called first order IAGO.

Definition 7. The grey forecasting GM(1,1) model can be expressed by one variable, and first order differential equation.

$$\frac{dx^{(1)}(t)}{dt} + ax^{(1)}(t) = b \quad (5)$$

The grey derivative for the first order grey differential equation with 1-AGO is conventionally represented as

$$\frac{dx^{(1)}(t)}{dt} = \lim_{\Delta t \rightarrow 0} \frac{x^{(1)}(t + \Delta t) - x^{(1)}(t)}{\Delta t} \quad (6)$$

Let $\Delta t \rightarrow 1$ and obtain

$$\frac{dx^{(1)}(t)}{dt} = x^{(1)}(t + 1) - x^{(1)}(t) = x^{(0)}(t + 1) \quad (7)$$

Its discrete form are shown by

$$x^{(0)}(i) + ax^{(0)}(i) = b \quad (8)$$

where $z^{(i)}(i)$ is called background value of $\frac{dx^{(1)}(t)}{dt}$. It is obtained by

$$z^{(0)}(i) = \frac{1}{2} \left( x^{(0)}(i-1) + x^{(0)}(i) \right) \quad (9)$$

From Definition 7, we know that first derivative $dx^{(1)}(t)/dt$ is calculated by difference of whitening data, and background value $z^{(i)}(i)$ is calculated by mean value of whitening data in each interval $[i-1,i]$ for $i=1,2,\cdots,n$. Similarly, these problems are applied to the 2nd, 3rd, ..., nth orders grey model respectively. Actually, these intervals consist of uncertain grey set which including non-precise information. Only using each whitening discrete data of intervals to calculate derivative $dx^{(1)}(t)/dt$ and background value $z^{(i)}$, thus it will disregard the existences of grey information in the intervals [3]. Since the irrational problems, when GM is applied to predict the future characteristics of system, the forecasted error decreases more and more according to the increase of derivative. Therefore, the accuracy of GM(2,1) model is low and is not
applied.

3. T-3spGM(2,1) model

We can view the interval $[i-1, i]$ for $i=1,2,\ldots,n$ as grey interval, derivatives $d^{(1)}x(i)/dt$, $d^{(2)}x(i)/dt^2$ and background value $z^{(0)}$ are a serial unclear numbers in all grey intervals [3]. In this paper, we presented whitening coefficients based on grey interval to establish new GM(2,1) model which is defined as 3spGM(2,1). But 3spGM(2,1) is only as a middle model. When whitening coefficients of 3spGM(2,1) are set initial values, we presented Taylor approximation method to optimize the whitening coefficients and made forecasted error reduce to minimum. This generated improved model is defined as T-3spGM(2,1).

3.1 Grey interval analysis

As shown in Fig.1, in interval $[i-1, i]$ for $i=1,2,\ldots,n$, we assume that derivative value $d^{(1)}x(i)/dt$ is grey derivative value $\odot d^{(1)}x(i)/dt$ and background value $z^{(0)}$ is grey background value $\odot z^{(0)}$. Through the grey interval operation, we can obtain their value. By connecting between 1-AGO discretion point $x^{(0)}(i-1)$ and $x^{(0)}(i)$. According to cubic spline function, the smooth curve $H_1(i)$ is created. Therefore, the condition in which the derivative of $H_1(i)$ is established, and we can obtain the derivative value of $x^{(0)}(i-1)$ and $x^{(0)}(i)$. As a result, the derivative value $H_1'(i-1)$ and $H_1'(i)$ of the discretion point is obtained. When $i=1,2,\ldots,n$, we can obtain all derivative values of primitive sequence $x^{(0)}(i)$. Because cubic spline function is used, the second derivative value $H_1''(i-1)$ and $H_1''(i)$ can be acquired too.

Then, according to (1), grey derivative value $\odot d^{(1)}x(i)/dt$ in interval $[i-1, i]$ is shown as follows:

\[
\odot d^{(1)}x(i)/dt = \left[ d^{(1)}x(i)/dt - d^{(1)}x(i-1)/dt \right]
\]  

(10)

where \[ \frac{dx^{(i)}}{dt} = \frac{H_1'(i)}{H_1'(i-1)} + \frac{H_1''(i)}{H_1''(i-1)} \]  \( i \leq 1 \) \( i \leq 1 \)

\[ \frac{dx^{(i)}}{dt} = \frac{H_1'(i)}{H_1'(i-1)} + \frac{H_1''(i)}{H_1''(i-1)} \]  \( i \leq 1 \)

According to (2), we introduce coefficients $\alpha$ and $\gamma$ to calculate the whitening value of grey derivative value $\odot d^{(1)}x(i)/dt$:

\[ \frac{dx^{(i)}}{dt} = (1-\alpha)\frac{dx^{(i)}}{dt} + \alpha \frac{dx^{(i)}}{dt} \]

(12)

where $\alpha = [0,1]$. We introduce a coefficient $\beta$ to calculate the whitening value of grey derivative value $\odot z^{(0)}$:

\[ z^{(0)} = (1-\beta)x^{(0)}(i-1) + \beta x^{(0)}(i) \]

(15)

where $\beta = [0,1]$.

Spline interpolation function is used so that the data of whole interval can be expressed by different polynomials for every small interval and the smooth curve can be generated in each small interval. Therefore, in GM, the derivative of each discrete whitening data can be obtained according to spline interpolation function. In this paper, cubic spline function [4] is presented based on cubic Hermite polynomial to calculate the first and second orders derivatives for accumulated data series $x^{(0)}$. Then, we presented the whitening coefficients to calculate grey derivatives $\odot d^{(1)}x(i)/dt$, $\odot d^{(2)}x(i)/dt^2$ and grey background $\odot z^{(0)}$ in grey interval.

3.2 3spGM(2,1) model

For original accumulated data series $x^{(i)}$, we can form the whitening second order differential equation is

\[ \frac{d^2x^{(i)}}{dt^2} + a_1 \frac{dx^{(i)}}{dt} + a_2 x^{(i)} = b \]

(16)

It can be shown that the solution for $x^{(i)}(i)$ is

\[ x^{(1)}(i) = x^{(0)}(i) + \frac{b}{a_2} \]

(17)

where $x^{(0)}(i)$ is called general solution, it has three kinds of types. The detail is omitted here. Coefficients $a_1, a_2$ and b can be obtained

\[ \begin{bmatrix} a_1 \\ a_2 \\ b \end{bmatrix} = (A^T A)^{-1} A^T X_n \]

(18)

where
\[ A = \begin{bmatrix} -\frac{dx^{(1)}}{dt}(1) & -\frac{dz^{(1)}}{dt}(1) & 1 \\ -\frac{dx^{(1)}}{dt}(2) & -\frac{dz^{(1)}}{dt}(2) & 1 \\ \vdots & \vdots & \vdots \\ -\frac{dx^{(1)}}{dt}(n) & -\frac{dz^{(1)}}{dt}(n) & 1 \end{bmatrix} \] (19)

\[ X_n = \begin{bmatrix} \frac{d^2 x^{(1)}}{dt^2}(1) \\ \frac{d^2 x^{(1)}}{dt^2}(2) \\ \vdots \\ \frac{d^2 x^{(1)}}{dt^2}(n) \end{bmatrix} \] (20)

\[ \frac{dx^{(1)}}{dt}(i), \frac{d^2 x^{(1)}}{dt^2}(i) \text{ and } \frac{dz^{(1)}}{dt}(i) \text{ for } i = 1, 2, \ldots, n \] are expressed by (10), (11) and (14) respectively. Their whitening values are obtained by (12), (13) and (15) respectively. The first and second order derivatives shown in (10) and (11) are calculated by cubic spline function, they are mentioned in Appendixes A and B.

### 3.3 T-3spGM(2,1) model

After whitening coefficients \( \alpha, \beta \) and \( \gamma \) of proposed middle model 3spGM(2,1) are set initial values and these values as the initial parameters of Taylor approximation method [5] can be adjusted repeatedly until reaches the optimal values and make the forecasted error reduce to the minimum. The generated model is defined as T-3spGM(2,1) [6].

**Algorithm of T-3spGM(2,1):**

**Step 1:** Initialization

Approximation times \( K = 0 \).

Setting target function vector \( G \):

\[ G = \{x^{(0)}(i), x^{(0)}(i+1), \ldots, x^{(0)}(n)\} \] (21)

where \( x^{(0)}(i) \in G \{x^{(0)}(i), i=0,1,\ldots,n\} \) is original data series.

Setting approximation function vector \( F^{(K)} \):

\[ F^{(K)} = [x^{(0K)}(0), x^{(0K)}(1), \ldots, x^{(0K)}(n)] \] (22)

where \( x^{(0K)}(i) \in F^{(K)} \{x^{(0K)}(i), i=0,1,\ldots,n\} \) is the \( K \) times generated forecasted data series of 3spGM(1,1).

Setting initial parameters:

\[ \hat{\alpha}^{(K)} = [\alpha^{(K)}, \beta^{(K)}, \gamma^{(K)}] = [0.5,0.5,0.5] \] (23)

where \( \hat{\alpha}^{(K)} \) are \( K \) times generated parameters, \( \hat{\alpha}^{(0)} \) are the initial coefficients \( \alpha, \beta \) and \( \gamma \) of 3spGM(2,1).

**Step 2:** Calculation of approximation function vector \( F^{(K+1)} \) according to first order Taylor development.

\[ F^{(K+1)} = F^{(K)} + F^{(K)}(\alpha^{(K+1)} - \alpha^{(K)}) + F^{(K)}(\beta^{(K+1)} - \beta^{(K)}) + F^{(K)}(\gamma^{(K+1)} - \gamma^{(K)}) \] (24)

where \( F^{(K)} = \frac{\partial F^{(K)}}{\partial \alpha^{(K)}}, F^{(K)} = \frac{\partial F^{(K)}}{\partial \beta^{(K)}}, F^{(K)} = \frac{\partial F^{(K)}}{\partial \gamma^{(K)}} \).

**Step 3:** Setting evaluation function \( Q^{(K)} \):

\[ Q^{(K)} = \left( \frac{E^{(K)} - F^{(K)}}{E^{(K)}} \right)^2 \] (25)

where \( E^{(K)} = G - F^{(K)}, \eta_1^{(K)} = \alpha^{(K+1)} - \alpha^{(K)}, \eta_2^{(K)} = \beta^{(K+1)} - \beta^{(K)}, \eta_3^{(K)} = \gamma^{(K+1)} - \gamma^{(K)} \).

**Step 4:** Detect stop criterion

If \( Q^{(K)} \leq \varepsilon \), stop; otherwise, go to Step 5.

where \( \varepsilon \) is allowable error.

**Step 5:** Update approximation parameter \( \hat{\alpha}^{(K)} \):

\[ \hat{\alpha}^{(K)} = \hat{\alpha}^{(K)} + \hat{A}^{(K)} + F^{(K)} \] (26)

where \( \hat{A}^{(K)} = [A^{(K)}], F^{(K)} \).

**Step 6:** Increase approximation times \( K = K+1 \); go to Step 2.

By the optimization process, the parameter \( \hat{\alpha}^{(K)} \) are updated for \( K \) times, the evaluation function \( Q^{(K)} \) as the convergent error is reduced. When \( Q^{(K)} \leq \varepsilon \), we can find the optimal parameters and convergent error is reduced to the minimum. At this time, vector \( F^{(K)} \) become the \( K \) forecasted generated data series \( \{x^{(0K)}(i), i = 0,1,\ldots,n\} \) as the result of approximated calculation.

### 4. Simulation results

In order to verify the effectiveness of proposed T-3spGM(2,1) model, the STLF problem for ordinary and special days are used. For two forecasting problems, the power system load data of just past four weeks are used to forecast the power load of next week. Power load for five weeks have been collected. Two criteria are used for evaluating proposed model. They are the mean square error (MSE) and absolute error (AE) which are calculated as

\[ \text{MSE} = \frac{1}{n} \sum_{i=1}^{n} e^2(i) \] (27)

\[ \text{AE} = |e(i)| \] (28)

where \( e(i) = x^{(0)}(i) - x^{(0)}(i) \), \( x^{(0)}(i) \) is the forecasted value for time \( i \).

The improved rate \( \sigma[\%] \) of proposed model is given as

\[ \sigma[\%] = \frac{\text{MSE}(\text{GM}(1,1)) - \text{MSE}(\text{T-3spGM(2,1)})}{\text{MSE}(\text{GM}(1,1))} \times 100\% \] (29)

For ordinary daily load forecasting, the STLF data set were obtained from Taiwan electric power system in the summer of 1992. The four daily loads of 1992.7.20, 1992.7.27, 1992.08.03 and 1992.08.10 are used to establish GM(1,1) model and the power load of 1992.08.17 is forecasted. The forecasted results from GM(1,1) model based on four data modeling are shown in Fig. 2. The forecasted curve cannot keep track of the actual one, especially at those extremes. The absolute error (AE) is also shown in Fig. 2. We can see that the AE is still very big for
GM(1,1) and it cannot exactly match the STLF system dynamics. The T-3spGM(2,1) model is then adopted to improve the STLF performance. Fig. 3 shows the results obtained from T-3spGM(2,1) model. It is obvious that the extreme-effect has been somewhat removed and the forecasted curve is on the right track. The AE is also shown in Fig. 3. This results verify the effectiveness of proposed T-3spGM(2,1) model. The accuracy comparison of two models are listed in Table I. Then we repeat the forecasting problem of special day. The four daily loads of 1992.7.18, 1992.7.25, 1992.08.01 and 1992.08.08 are used to forecast the power load of 1992.08.15. The forecasting results and absolute error (AE) are shown in Figs. 4-5. The accuracy comparison of two models are listed in Table. II.

Table I  Accuracy comparison of ordinary day

<table>
<thead>
<tr>
<th>Models</th>
<th>MSE[MW]^2</th>
<th>σ [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>GM(1,1)</td>
<td>2.52×10^6</td>
<td>-</td>
</tr>
<tr>
<td>T-3spGM(2,1)</td>
<td>2.52×10^6</td>
<td>94.94</td>
</tr>
</tbody>
</table>

Table II  Accuracy comparison of special day

<table>
<thead>
<tr>
<th>Models</th>
<th>MSE[MW]^2</th>
<th>σ [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>GM(1,1)</td>
<td>6.38×10^6</td>
<td>-</td>
</tr>
<tr>
<td>T-3spGM(2,1)</td>
<td>8.67×10^6</td>
<td>86.42</td>
</tr>
</tbody>
</table>

5. Conclusion

The major purpose of this paper is to enhance the STLF performance based on improved grey dynamic model. A power system can be regarded as a grey system because the relation between the power demand and the weather conditions or business fluctuations, etc. is not necessarily clear although the power demand is influenced by them. The grey dynamic model based on grey system theory can consider various factors of fluctuation, the algorithm is relatively easy and few calculations are needed. But when original data show great randomness, the accuracy of GM(1,1) is not satisfactory. The superior properties of grey dynamic model cannot be expressed completely. Therefore, We proposed improved T-3spGM(2,1) model and applied to improve STLF performance. The effects are achieved more than conventional GM(1,1) model. The proposed model can help in more accurate forecasting, such as weather conditions, economy changes and society information etc..

Appendix A

In (10), the first order derivative \( \frac{dx^{(1)}}{dt} \) for accumulated whitening series \( \{x^{(1)}(i), i = 0,1, \cdots, n\} \) can be obtained by differential calculus directly. Assume that accumulated whitening \( \{x^{(1)}(i), i = 0,1, \cdots, n\} \) as node
function value \( y_j, i=0,1,\ldots,n \) of cubic spline function, that is \( y_j = x^{(3)}(i) \).

Letting

\[
\frac{dx^{(3)}}{dt} = \begin{bmatrix} y'_{0} \\ y'_{1} \\ \vdots \\ y'_{n} \end{bmatrix} = Y'
\]

(A-1)

Algorithm of \( Y' \):

\[
Y' = A_3^T B
\]

(A-2)

where

\[
A_3 = \begin{bmatrix} 2 & 1 & 0 & \cdots & \cdot & 0 \\ 1 & 2 & \frac{1}{2} & \cdot & \cdot & \cdot \\ 0 & 1 & 2 & \frac{1}{2} & \cdot & \cdot \\ \vdots & \vdots & \vdots & \ddots & \cdot & \cdot \\ 0 & 0 & 1 & 2 & \frac{1}{2} & \cdot \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix}
\]

\[
B = [B_0, B_1, \ldots, B_n]
\]

(A-3)

\[
B_0 = 3(y_1 - y_0)
\]

\[
B_i = \frac{3}{2}(y_{i+1} - y_{i-1}), i=1,2,\ldots,n-1
\]

\[
B_n = 3(y_n - y_{n-1})
\]

(A-4)

\[
C = \begin{bmatrix} -4 & 2 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdot & \cdot \\ 0 & 1 & 0 & -1 & \cdot \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdot & 1 & 0 & -1 \\ 0 & \cdots & \cdots & \cdots & 2 & 4 \end{bmatrix}
\]

(A-5)

\[
D = \begin{bmatrix} 6(y_1 - y_0) \\ 3(y_2 - 2y_1 + y_0) \\ 3(y_4 - 2y_3 + y_2) \\ \vdots \\ 3(y_{n} - 2y_{n-1} + y_{n-2}) \end{bmatrix}
\]

(A-6)

\[
Y^* = CY' + D
\]

(A-7)

\[
Y'' = \frac{d^2x^{(3)}}{dt^2} = \begin{bmatrix} y''_{0} \\ y''_{1} \\ \vdots \\ y''_{n} \end{bmatrix} = Y''
\]

(A-8)

Proof: Assume that \( \{x_i, i=0,1,\ldots,n \} \) is node and \( y_j \) is node function value, then the cubic Hermite polynomial is expressed by

\[
H_j(x) = y_{i-1}F_0(t) + y_1F_1(t) + h_i[y_jG_0(t) + y_i'G_1(t)]
\]

(A-9)

where

\[
F_0(t) = (1-t)^2(1+2t), \quad F_1(t) = t^2(3-2t)
\]

\[
G_0(t) = t(1-t)^2, \quad G_1(t) = -t^2(1-t)
\]

\[ t = \frac{x - x_{i-1}}{h_i}, \quad (x_{i-1} \leq x \leq x_i), \quad x \text{ is node, } h_i (i=0,1,\ldots,n) \text{ is interpolation distance.}
\]

The character of the Hermite polynomial is shown in following:

\[
\begin{align*}
H_j(x_i) &= y_j, \\
\lim_{s \to x_j^-} H_j^{(p)}(s) &= \lim_{s \to x_j^+} H_j^{(p)}(s), p = 0,1,2 \\
H_j'(x_i) &= H_j'(x_i) = 0
\end{align*}
\]

By (A-8),(A-9), we can obtain the first order derivative \( Y' \) for accumulated whitening series \( \{x^{(3)}(i), i=0,1,\ldots,n \} \).

Appendix B

In (11), the second order derivative \( \frac{d^2x^{(3)}}{dt^2} \) for accumulated whitening series \( \{x^{(3)}(i), i=0,1,\ldots,n \} \) can be also obtained by differential calculus directly.

\[
\frac{d^2x^{(3)}}{dt^2} = \begin{bmatrix} y''_{0} \\ y''_{1} \\ \vdots \\ y''_{n} \end{bmatrix} = Y''
\]

(A-10)

Algorithm of \( Y'' \):

\[
Y'' = CY' + D
\]

(A-11)

where \( Y' \) is same to (A-2). C and D are obtain by

\[
C = \begin{bmatrix} 6(y_1 - y_0) \\ 3(y_2 - 2y_1 + y_0) \\ 3(y_4 - 2y_3 + y_2) \\ \vdots \\ 3(y_{n} - 2y_{n-1} + y_{n-2}) \end{bmatrix}
\]

(A-12)

\[
D = \begin{bmatrix} 6(y_1 - y_0) \\ 3(y_2 - 2y_1 + y_0) \\ 3(y_4 - 2y_3 + y_2) \\ \vdots \\ 3(y_{n} - 2y_{n-1} + y_{n-2}) \end{bmatrix}
\]

(A-13)

\[
Y'' = \frac{d^2x^{(3)}}{dt^2} = \begin{bmatrix} y''_{0} \\ y''_{1} \\ \vdots \\ y''_{n} \end{bmatrix} = Y''
\]

(A-14)

\[
Y'' = \frac{d^2x^{(3)}}{dt^2} = \begin{bmatrix} y''_{0} \\ y''_{1} \\ \vdots \\ y''_{n} \end{bmatrix} = Y''
\]

(A-15)

By the above equation, we can obtain the second order derivative \( Y'' \) for accumulated whitening series \( \{x^{(3)}(i), i=0,1,\ldots,n \} \).

References