A single-element extension of antimatroids

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Received 5 October 1999; received in revised form 13 September 2000; accepted 19 June 2001

Abstract

An antimatroid is a family of sets which is accessible, closed under union, and includes an empty set. A number of examples of antimatroids arise from various kinds of shellings and searches on combinatorial objects, such as, edge/node shelling of trees, poset shelling, node-search on graphs, etc. (Discrete Math. 78 (1989) 223; Geom. Dedicata 19 (1985) 247; Greedoids, Springer, Berlin, 1980) [1–3]. We introduce a one-element extension of antimatroids, called a lifting, and the converse operation, called a reduction. It is shown that a family of sets is an antimatroid if and only if it is constructed by applying lifting repeatedly to a trivial lattice. Furthermore, we introduce two specific types of liftings, 1-lifting and 2-lifting, and show that a family of sets is an antimatroid of poset shelling if and only if it is constructed from a trivial lattice by repeating 1-lifting. Similarly, an antimatroid of edge-shelling of a tree is shown to be constructed by repeating 2-lifting, and vice versa. © 2002 Elsevier Science B.V. All rights reserved.

1. Posets, lattices and antimatroids

We first present the definition of terminology. For a partially ordered set \(\mathcal{P} = (S, \leq)\), an ideal of \(\mathcal{P}\) is a subset \(K\) of \(S\) such that if \(x \in K\) and \(y \leq x\) for \(y \in S\), then \(y \in K\). A filter is the complement set of an ideal. \([x, y] = \{z \in S: x \leq z, z \leq y\}\) is the interval between \(x\) and \(y\). The lattice consisting only of an empty set is called a trivial lattice, and \(2^n\) denotes the Boolean algebra of all the subsets of an \(n\)-element set. For distinct elements \(x, y \in S\) with \(x \leq y\), if \(x \leq z \leq y\) necessarily implies \(x = z\) or \(z = y\), then \(x\) is covered by \(y\). A poset is called a forest if every element is covered by at most one element. In a forest, we call a maximal element a treetop. For the treetops \(t_1, \ldots, t_k\) of a forest \(Q = (S, \leq)\), clearly their principal ideals \(T_i = \{x \in Q: x \leq t_i\}\) for \(i = 1, \ldots, k\) form a partition of \(S\).

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Let $E$ denote a non-empty finite set, and $\mathcal{L}$ a family of subsets of $E$. For a set $X$ and an element $p$, we write $X \setminus p$, $X \cup p$ instead of $X \setminus \{p\}$, $X \cup \{p\}$ for the sake of simplicity. Also, We let $\mathcal{L} - p = \{X \setminus p: X \in \mathcal{L}\}$ and for a new element $q$ not in $E$, $\mathcal{L} + q = \{X \cup q: X \in \mathcal{L}\}$.

$\mathcal{L}$ is called an antimatroid on $E$ if it satisfies the following:

(L1) $\emptyset \in \mathcal{L}$,
(L2) if $X \neq \emptyset$ and $X \in \mathcal{L}$, then $X \setminus x \in \mathcal{L}$ for some $x \in X$,
(L3) if $X, Y \in \mathcal{L}$ and $X \not\subseteq Y$, then $Y \cup x \in \mathcal{L}$ for some $x \in X \setminus Y$.

The element of $\mathcal{L}$ is called a feasible set. Under the assumption (L2), (L3) is equivalent to (L3').

(L3') if $X, Y \in \mathcal{L}$, then $X \cup Y \in \mathcal{L}$.

The family of all the ideals of a poset is an antimatroid, which we call a poset-shelling antimatroid.

For a tree $T = (V, E)$, 
\[ \mathcal{L} = \{X \subseteq E: T - X \text{ is connected}\} \] (1)

is an antimatroid called an edge-shelling antimatroid of $T$.

2. Lifting and reduction of antimatroids

We shall define a one-element extension of antimatroids. Let $\mathcal{L}_1$, $\mathcal{L}_2$ be the subfamilies of an antimatroid $\mathcal{L}$. Suppose that they satisfy the following:

(E0) $\mathcal{L}_1 \cup \mathcal{L}_2 = \mathcal{L}$,
(E1) $\mathcal{L}_1$ is an antimatroid,
(E2) $\mathcal{L}_2$ is a filter in $\mathcal{L}$,
(E3) $\mathcal{L}_2 = \{Y \in \mathcal{L}: X \subseteq Y \text{ for some } X \in \mathcal{L}_1 \cap \mathcal{L}_2\}$.

Let $p$ be a new element not in $E$. Then we can define a one-rank higher lattice by
\[ (\mathcal{L} \uparrow p)_{(\mathcal{L}_1, \mathcal{L}_2)} = \mathcal{L}_1 \cup (\mathcal{L}_2 + p) = \mathcal{L}_1 \cup \{Y \cup p: Y \in \mathcal{L}_2\} \] (2)

which we call a lifting of $\mathcal{L}$ at $(\mathcal{L}_1, \mathcal{L}_2)$ by $p$. We write $\mathcal{L} \uparrow p$ to denote $(\mathcal{L} \uparrow p)_{(\mathcal{L}_1, \mathcal{L}_2)}$ when no confusion may occur.

Then we have the following theorem.

Theorem 2.1. A lifting $(\mathcal{L} \uparrow p)_{(\mathcal{L}_1, \mathcal{L}_2)}$ is an antimatroid on a set $E \cup p$.

Proof. $\emptyset \in \mathcal{L} \uparrow p$ is obvious. To see (L2), take any $X \in \mathcal{L} \uparrow p$. If $X \in \mathcal{L}_1$, (L2) is clear. Otherwise suppose $X = X' \cup p$ and $X' \in \mathcal{L}$. If $X'$ is not minimal in $\mathcal{L}_2$, there exists an element $x \in X'$ such that $X' \setminus x \in \mathcal{L}_2$, and we have $X \setminus x = (X' \setminus x) \cup p \in \mathcal{L} \uparrow p$.

If $X'$ is minimal in $\mathcal{L}_2$, $X' \in \mathcal{L}_1$ follows from (E3). Hence $X \setminus p = X' \in \mathcal{L}_1 \subseteq \mathcal{L} \uparrow p$.

So (L2) holds. Finally we shall show (L3'). The only interesting case is that $X \in \mathcal{L}_1$ and $Y = Y' + p \in \mathcal{L}_2 + p$. By (E2) $\mathcal{L}_2$ is a filter, and we have $X \cup Y' \in \mathcal{L}_2$. Hence $X \cup Y = (X \cup Y') + p \in \mathcal{L} \uparrow p$. ■
Next we introduce the converse operation of lifting. Take an element \( p \in E \). Then we have a one-rank lower lattice

\[
\mathcal{L} \downarrow p = \mathcal{L} - p = \{X \setminus p : X \in \mathcal{L}\}.
\]  

As is easy to observe, \( \mathcal{L} \downarrow p \) is an antimatroid on \( E \setminus p \). We call it a reduction of \( \mathcal{L} \) at \( p \).

The reduction and the lifting are the converse of each other.

**Theorem 2.2.** (a) For any \( p \in E \), we have

\[
((\mathcal{L} \downarrow p) \uparrow p)_{(\mathcal{L}_1, \mathcal{L}_2)} = \mathcal{L},
\]

where \( \mathcal{L}_1 = \{X : X \in \mathcal{L}, p \notin X\} \), \( \mathcal{L}_2 = \{X - p : X \in \mathcal{L}, p \in X\} \).

(b) Conversely, take a new element \( q \) not in \( E \) and suppose \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) satisfy (E0)–(E3). Then

\[
((\mathcal{L} \uparrow q)_{(\mathcal{L}_1, \mathcal{L}_2)}) \downarrow q = \mathcal{L}.
\]

**Proof.** We shall first show (a). Obviously, (E0) and (E1) hold for \( \mathcal{L}_1, \mathcal{L}_2 \) in \( \mathcal{L} \downarrow p \). To see that (E2) holds, take any \( X' \in \mathcal{L} \downarrow p = \mathcal{L}_1 \cup \mathcal{L}_2 \) such that \( X \subseteq X' \) for some \( X \in \mathcal{L}_2 \), and we shall show that \( X' \in \mathcal{L}_2 \). Suppose contrarily \( X' \notin \mathcal{L}_2 \). Then \( X' \in \mathcal{L}_1 \), so we have \( X' \in \mathcal{L} \), while \( X \cup p \in \mathcal{L} \) holds from the assumption. It follows from (L3) that \( X' \cup p \in \mathcal{L} \). Hence \( X' \in \mathcal{L}_2 \), a contradiction. Accordingly, \( \mathcal{L}_2 \) is a filter in \( \mathcal{L} \downarrow p \).

To show (E3), take any \( X \in \mathcal{L}_2 \). Let \( Z \) be a minimal element of \( \mathcal{L}_2 \) such that \( Z \subseteq X \). We shall show that \( Z \) belongs to \( \mathcal{L}_1 \). By assumption, \( Z' = Z \cup p \in \mathcal{L} \). By (L2), there exists \( a \in Z' = Z \cup p \) such that \( Z' \setminus a \in \mathcal{L} \). If \( a = p \), we have \( Z = Z' \setminus a \in \mathcal{L} \) and \( Z \in \mathcal{L}_1 \) follows. If \( a \neq p \), then \( Z' \setminus a = (Z \setminus a) \cup p \in \mathcal{L} \). Hence, we have \( Z \setminus a \in \mathcal{L}_2 \), which contradicts the minimality of \( Z \). Hence we have \( Z \setminus a \in \mathcal{L}_2 \), and (E3) follows. Since it is easy to check that the lifting of \( \mathcal{L} \downarrow p \) at \( (\mathcal{L}_1, \mathcal{L}_2) \) is equal to \( \mathcal{L} \), (a) readily follows.

Similarly (b) can be shown.

From Theorems 2.1 and 2.2, we have the following.

**Corollary 2.1.** Let \( \mathcal{L} \) be a family of subsets of \( E \). Then \( \mathcal{L} \) is an antimatroid if and only if it can be constructed from a trivial lattice by applying lifting repeatedly.

**Proof.** Order arbitrarily the elements of \( E \) as \( p_1, p_2, \ldots, p_n \). Then \( (\mathcal{L} \downarrow p_1 \downarrow p_2 \cdots) \downarrow p_n \) is a trivial lattice, and repeating the reverse lifting \( n \) times gives \( \mathcal{L} \).

3. Characterizations of poset-shelling antimatroids and edge-shelling antimatroids of trees

In this section, we shall present the characterizations of poset-shelling and tree edge-shelling antimatroids in terms of certain special liftings.
Let \( \mathcal{L} \) be an antimatroid on \( E \), and \( A \) a feasible set of \( \mathcal{L} \). When we define \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) by
\[
\mathcal{L}_1 = \mathcal{L}, \quad \mathcal{L}_2 = [A, E],
\]
then (E0)–(E3) are trivially satisfied, and the resultant lifting is a 1-lifting. If \( E \setminus A \in \mathcal{L} \) is further satisfied, we call it a self-dual 1-lifting.

The poset-shelling antimatroids are characterized by 1-lifting.

**Theorem 3.1.** Let \( \mathcal{L} \) be a family of subsets of \( E \). Then \( \mathcal{L} \) is a poset-shelling antimatroid if and only if it can be constructed from a trivial lattice by repeating 1-lifting.

**Proof.** First, suppose \( \mathcal{L} \) is a poset-shelling antimatroid on \( E \), we shall prove that \( \mathcal{L} \) can be constructed by 1-lifting. We use induction on \( n = |E| \). If \( n = 0 \), the assertion is trivial. Suppose the assertion holds until \( n = k \), and let \( \mathcal{L}' \) be a poset-shelling antimatroid on the underlying set \( E' \) with \( |E'| = k + 1 \). Take a maximal element \( p \) of \( E' \) and set \( A' = \{ x \in E' : x \leq p \} \). Then the reduction \( \mathcal{L} = \mathcal{L}' \setminus p \) is easily seen to be equal to the shelling antimatroid of the poset on \( E = E' \setminus p \). Obviously \( A = A' \setminus p \) is an ideal in \( E \).

Hence, we can define a 1-lifting \( \mathcal{L}'' = (\mathcal{L} \uparrow p)(\mathcal{L}, [A, E]) \) of \( \mathcal{L} \) and it is easy to check that \( \mathcal{L}'' \) is equal to \( \mathcal{L} \). This completes the induction step.

Conversely, suppose \( \mathcal{L} \) is constructed from a trivial lattice by applying 1-lifting \( n \) times. We shall show \( \mathcal{L} \) is a poset-shelling antimatroid. We use induction on \( n \). If \( n = 0 \) then the assertion is trivial. Let \( p \) be a new element not in \( E \). Take a feasible set \( A \in \mathcal{L} \), and consider 1-lifting \( \mathcal{L}' = (\mathcal{L} \uparrow p)(\mathcal{L}, [A, E]) \). We extend the partial order to that on \( E' = E \cup p \) by
\[
\begin{align*}
\{ x \leq p \text{ for } x \in A, \\
x \text{ and } p \text{ are incomparable in } E' \text{ for } x \in E \setminus A,
\end{align*}
\]
(In \( E' \), the other relations of elements are the same as those in \( E \).) Now it is an easy routine to check that \( \mathcal{L}' \) is the poset-shelling antimatroid of \( (E', \leq) \). This completes the proof.

An antimatroid of shelling of a forest, which is a special case of posets, can be characterized by self-dual 1-lifting.

**Corollary 3.1.** \( \mathcal{L} \) is a poset-shelling antimatroid of a forest if and only if it is constructed from a trivial lattice by repeating self-dual 1-lifting.

**Proof.** We shall show the sufficiency part first. Let \( \mathcal{L} \) be an antimatroid on \( E \) obtained by repeating self-dual 1-lifting. We use induction on \( n = |E| \). The case of \( n = 0 \) is trivial. Take a feasible set \( A \in \mathcal{L} \) such that \( E \setminus A \) is also feasible. Let \( \mathcal{L}' = (\mathcal{L} \uparrow q)(\mathcal{L}, [A, E]) \) be the associated self-dual 1-lifting. By induction hypothesis, \( \mathcal{L} \) is a shelling antimatroid of a certain forest \( F = (E, \leq) \). Let \( S \) be the set of the treetops of \( F \). Since \( A \) and \( E \setminus A \) are both feasible sets, they are ideals of \( F \). So if \( x \leq y \in F \), then either \( x, y \in A \) or...
Suppose we have:

\[ A = \bigcup_{t \in S_1} T(t), \quad E \setminus A = \bigcup_{t \in S_2} T(t), \]

where \( T(t) = \{ x \in E : x \leq t \} \). We define a relation between a new element \( q \) and the elements of \( E \) by

\[
\begin{align*}
    x \preceq q & \quad \text{if } x \in T(t) \text{ for some } t \in S_1, \\
    x \text{ and } q \text{ are incompatible} & \quad \text{otherwise.}
\end{align*}
\]

This gives a well-defined partial order on \( E \cup q \), which is again a forest. And it is easy to check that \( \mathcal{L}' \) is a shelling antimatroid of this forest on \( E \cup q \).

Next we shall show the necessity part. Let \( \mathcal{L} \) be a shelling antimatroid of a forest \( F = (E, \preceq) \). And we shall show that \( \mathcal{L} \) can be constructed by self-dual 1-lifting. We use induction on \( n = |E| \). The case of \( n = 0 \) is trivial. Let \( S \) be the set of the treetops of \( F \). Take a treetop \( p \in S \), and let \( T \) be the set of elements covered by \( p \) in \( F \). Deleting \( p \) from \( F \), we have a forest \( F' = (E', \preceq') \) where \( E' = E \setminus p \). Then, clearly, the post-shelling antimatroid of \( F' \) is equal to the reduction \( \mathcal{L}' = \mathcal{L} \setminus p \). By induction hypothesis, \( \mathcal{L}' \) is constructed from a trivial lattice applying self-dual 1-lifting \( n-1 \) times. The set of treetops of \( F' \) is a disjoint union of \( S \setminus p \) and \( T \). Then \( A = \{ x \in E' : x \preceq t \text{ for some } t \in T \} \) and \( E \setminus A = \{ x \in E' : x \preceq s \text{ for some } s \in S \setminus p \} \) are ideals of \( F' \) and hence feasible sets of \( \mathcal{L}' \). Then the lifting \( (\mathcal{L}' \uparrow p)_{(\mathcal{L}', [A, E'])} \) is a self-dual 1-lifting, and is equal to the original \( \mathcal{L} \). Hence, the induction step is completed. \( \square \)

Now we introduce another type of lifting. Suppose \( A \) and \( E \setminus A \) are both non-empty feasible sets of \( \mathcal{L} \). Then the families

\[
\mathcal{L}_1 = \mathcal{L}, \quad \mathcal{L}_2 = [A, E] \cup [E \setminus A, E]
\]

satisfy conditions (E0)–(E3), and define a lifting of \( \mathcal{L} \), which we call a 2-lifting.

The 2-lifting characterizes the edge-shelling antimatroids of trees. More precisely, we have:

**Theorem 3.2.** Suppose \( k \geq 0, m \geq 1 \). \( \mathcal{L} \) is an antimatroid of edge-shelling of a tree of \( m \) end edges and \( k \) interior edges if and only if \( \mathcal{L} \) can be constructed from a Boolean algebra \( 2^{[m]} \) by applying 2-lifting \( k \) times.

**Proof.** Sufficiency part: Suppose \( \mathcal{L} \) is an antimatroid obtained by applying 2-lifting \( k \) times starting from a Boolean algebra \( 2^{[m]} \). We shall show that \( \mathcal{L} \) is an edge-shelling antimatroid of a tree. We use induction on \( k \), and the case for \( k = 0 \) is obvious since the edge-shelling antimatroid of a star graph of \( m \) edges is just a Boolean algebra \( 2^{[m]} \). (Here a star graph is a tree consisting of \( m + 1 \) vertices \( \{w, u_1, \ldots, u_m\} \) and \( m \) edges \( \{wu_1, \ldots, wu_m\} \).) Suppose \( k \geq 1 \). Let \( \mathcal{L}' \) be an antimatroid constructed from \( \mathcal{L} \) by 2-lifting. That is, let \( A \) be a non-empty feasible set of \( \mathcal{L} \) such that \( E \setminus A \) is also a non-empty feasible set, and suppose \( \mathcal{L}' = (\mathcal{L} \uparrow p)_{(\mathcal{L}, [A, E] \cup [E \setminus A, E])} \). By induction
hypothesis, $\mathcal{L}$ is an edge-shelling antimatroid of a tree $T = (V,E)$. Since $A$ and $E \setminus A$ are non-empty feasible sets, the subgraphs $T_A$ and $T_{E\setminus A}$ spanned by $A$ and $E \setminus A$ in $T$ are both connected subgraphs. And if $V(T_A)$ and $V(T_{E\setminus A})$ have two vertices in common, then there would be a path between them in $A$ as well as another path in $E \setminus A$, and we have a circuit in $T$, which is a contradiction. Hence $V(T_A)$ and $V(T_{E\setminus A})$ have a unique common vertex, say $v$. Clearly, $v$ is not an end-node. Let $X = \{x \in V(T_A): xv \in E\}$ and $Y = \{y \in V(T_{E\setminus A}): yv \in E\}$. We extend tree $T$ to $T'$ by replacing $v$ with two new nodes $v_1, v_2$ and inserting a new edge $p = v_1v_2$, and we put edges $uv_1$ for $u \in X$ and $uv_2$ for $u \in Y$. We denote the resultant tree by $T' = ((V\setminus \{v\}) \cup \{v_1, v_2\}, E \cup \{p\})$, and the edge-shelling antimatroid of tree $T'$ is denoted by $\mathcal{L}_{T'}$. It is an easy routine to check that $\mathcal{L}_{T'}$ is equal to $\mathcal{L}'$. This completes the proof of sufficiency.

**Necessity part:** Suppose $\mathcal{L}$ is an edge-shelling antimatroid of a tree $T = (V,E)$ with $m$ end-edges and $k$ interior edges. We use induction on $k$. If $k = 0$, the assertion is obvious. Suppose $k \geq 1$. Take any interior edge $p = xy$ of $T$. Deleting edge $p$ from $T$ gives two separate subtrees $T_1$ and $T_2$. Let $A_1, A_2$ be the set of edges of $T_1, T_2$, respectively. Let $T/p$ denote the tree obtained from $T$ by contracting edge $p$. By induction hypothesis, the edge-shelling antimatroid of $T/p$, which we denote by $\mathcal{L}_p$, is constructed from a Boolean algebra $2^{[m]}$ applying 2-lifting $k - 1$ times. Obviously we have $\mathcal{L}_p = \mathcal{L} \downarrow p$. Also it is easy to show that

$$\mathcal{L} = ((\mathcal{L} \downarrow p) \uparrow p) \cup [A_1,E \setminus p] \cup [A_2,E \setminus p]).$$

That is, $\mathcal{L}$ is constructed from $\mathcal{L} \downarrow p$ by applying 2-lifting once. This completes the step of induction and the proof is completed.

**References**