Equivocations, Exponents and Second-Order Coding Rates under Various Rényi Information Measures

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Abstract

In this paper, we evaluate the asymptotics of equivocations, their exponents as well as their second-order coding rates under various Rényi information measures. Specifically, we consider the effect of applying a hash function on a source and we quantify the level of non-uniformity and dependence of the compressed source from another correlated source when the number of copies of the sources is large. Unlike previous works that use Shannon information measures to quantify randomness, information or uniformity, in this paper, we define our security measures in terms of a more general class of information measures—the Rényi information measures and their Gallager-type counterparts. A special case of these Rényi information measure is the class of Shannon information measures. We prove tight asymptotic results for the security measures and their exponential rates of decay. We also prove bounds on the second-order asymptotics and show that these bounds match when the magnitudes of the second-order coding rates are large. We do so by establishing new classes non-asymptotic bounds on the equivocation and evaluating these bounds using various probabilistic limit theorems asymptotically.

Index Terms

Information-theoretic security, Conditional Rényi entropies, Equivocation, Error exponents, Secrecy Exponents, Second-order coding rates,

I. INTRODUCTION

In this paper, we consider the situation where we are given \( n \) independent and identically distributed (i.i.d.) copies of a joint source \((A^n, E^n)\). One of the central tasks in information-theoretic security is to understand the effect of applying a hash function [1] (binning operator) \( f \) on \( A^n \). This hash function is used to ensure that the compressed source \( f(A^n) \) is almost uniform on its alphabet and also almost independent of another discrete memoryless source \( E^n \). Mathematically, we want to understand the deviation of \( f(A^n) \in \{1, \ldots, [e^{nR}] \} \) from the uniform distribution \( P_{\text{mix}, f(A^n)} \) and the level of remaining dependence between \( f(A^n) \) and a correlated source \( E^n \). These two criteria can be described by equivocation measures. Traditionally in information-theoretic security [2], [3], equivocation is measured in terms of the Shannon-type quantities such as the Shannon entropy, relative entropy (Kullback-Leibler divergence), and mutual information. In particular, it is common to design \( f \) such that the following is small for any rate \( R \):

\[
D(P_{f(A^n) E^n} \| P_{\text{mix}, f(A^n)} \times P_{E^n}) = nR - H(f(A^n) | E^n).
\]  

(1)

Clearly if the above quantity is small in some sense, the message \( f(A^n) \) is close to uniform and almost independent of \( E^n \), two desirable traits of a hash function for security applications.

A. Motivations

In this paper, we depart from using Shannon information measures to quantify randomness and independence. It is known that the Shannon entropy \( H \) or the relative entropy \( D \) are special cases of a larger family of information measures known as Rényi information measures, denoted as \( H_{1+s} \) and \( D_{1+s} \) for \( s \in \mathbb{R} \). Thus, as expounded by Iwamoto and Shikata [4], we can quantify equivocation using these measures, gaining deeper insights into the

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fundamental limits of information leakage under the effect of hash functions. There may also be a possibility of the optimal key generation rate changing when we use alternative information measures. In addition, in the study of cryptography and quantum key distribution (QKD), the Rényi entropy of order 2 [5] (or collision entropy) \( H_2(A|P_A) := -\log \sum_{a \in A} P_A(a)^2 \) and the min-entropy \( H_{\min}(A|P_A) := -\log \max_{a \in A} P_A(a) \) play important roles in quantifying randomness. A case in point is the leftover hash lemma [6]–[8]. Another motivation stems from the recent study of overcoming weak expectations by Dodis and Yu [9] where cryptographic primitives are based on weak secrets, in which the only information about the secret is some fraction of min-entropy. The authors in [9] provided bounds on the weak expectation \( E \) in the asymptotic behavior of \( D \) therein), we show that if we measure security using \( D \) in Section II-B, the quantity above is closely related to the equivocation [12]. In Section III (particularly in Corollary 1), we quantify lengths of secret keys in terms of Rényi entropies of general orders. Motivated by these studies, the authors opine that it is of interest to study the performance of hashing under these generalized families of entropies (generalized uncertainty measures) and divergences (generalized distance measures).

B. Main Contributions

In this paper, we consider the scenario where a common source of randomness \( \{X_n \in \mathcal{X}_n\}_{n \in \mathbb{N}} \) is available to all parties as shown in the setup in Fig. 1. We aim to characterize the asymptotic behavior of

\[
D_{1+s}(P f(A^n) E^n X_n \| P_{\text{mix}, f(A^n)} \times P_{X_n})
\]

for a fixed rate \( R = \frac{1}{n} \log \| f \| \) where \( \| f \| := |f(A^n)| \) is the cardinality of the range of \( f \). As we shall see in Section II-B, the quantity above is closely related to the equivocation [12]. In Section III (particularly in Corollary 1 therein), we show that if we measure security using \( D_{1+s} \) with \( s > 0 \), the fundamental limits of key generation rates change relative to those for traditional Shannon-type measures \( D_1 \).

We are also interested in the speed of the exponential decay of (2) given a fixed rate \( R \). That is, we are interested in the asymptotic behavior of

\[
\frac{1}{n} \log D_{1+s}(P f(A^n) E^n X_n \| P_{\text{mix}, f(A^n)} \times P_{X_n}).
\]

This is likened to error exponent or reliability function analysis in classical information theory [13], [14]. We study this in Section IV.

Finally, in Section V, we also study the second-order asymptotics [15], [16] of the decay of \( D_{1+s} \) with the blocklength, i.e., the asymptotic behavior of

\[
\frac{1}{\sqrt{n}} D_{1+s}(P f(A^n) E^n X_n \| P_{\text{mix}, f(A^n)} \times P_{X_n}), \quad \text{and}
\]

\[
\frac{1}{\sqrt{n}} \log D_{1+s}(P f(A^n) E^n X_n \| P_{\text{mix}, f(A^n)} \times P_{X_n}),
\]

where the number of compressed symbols (size of the hash function) \( \| f \| \) equals \( e^{nR + \sqrt{n}L} \) for some first-order rate \( R \) (usually the conditional Rényi entropy) and second-order rate \( L \in \mathbb{R} \). For some cases (Rényi parameter less than one) where we cannot exactly determine the tight second-order asymptotics (i.e., the upper and lower bounds do not match), we study the asymptotic behavior of (2) when the second-order rate \( L \) tends to \( +\infty \) or \( -\infty \). In this case, the upper and lower bounds match up to and including a term quadratic in \( L \).

There are some subtle differences between the criteria in without the common randomness \( X_n \) in (1) and with \( X_n \) in (2). Both are employed in information theory but the former is used more in random coding arguments (such as the achievability proof for the wiretap channel) while the latter is used more for proofs the performance of constructive codes. Of course, if we can prove that \( D^{(n)}_{1+s} := D_{1+s}(P f(A^n) E^n X_n \| P_{\text{mix}, f(A^n)} \times P_{X_n}) \) is no larger than \( \varepsilon > 0 \), then by the usual random coding (or random selection) argument, since \( D^{(n)}_{1+s} \) is an average over \( X_n \), there exists an \( x^*_n \in \mathcal{X}_n \) indexing a deterministic protocol \( f_{X_n=x^*_n} \) that ensures that \( D_{1+s}(P f(A^n) E^n X_n=x^*_n \| P_{\text{mix}, f(A^n)} \times P_{E^n} \times 1\{X_n = x^*_n\}) \) is also no larger than \( \varepsilon \). When \( s = 0 \), the expectation of the former quantity in (1) under the common randomness \( X_n \) generating a universal hash function \( f_{X_n}(\cdot) \) is equivalent to the latter quantity in (2).
A hash function $f$ on the source $A^n$. Common randomness $X_n$, independent of a correlated source $E^n$, is available to all parties. We would like $f(A^n)$ to be uniform on its support $\{1, \ldots, \|f\|\}$ and almost independent of $E^n$ in the sense of ensuring that quantity in (2) is small. We examine (2) under different asymptotic regimes such as the equivocation, the exponential behavior (3), and the second-order asymptotics (4)–(5).

but for $s \neq 0$, they are, in general, different. In the sequel, we adopt the latter criterion in (2) to simplify the presentation of the results.

We believe the results contained herein may serve as logical starting points to derive tight exponential error bounds and second-order coding rates for the wiretap channel [12] (as was done in [17], [18]) and other information-theoretic security problems such as the secret key agreement [19] (as was done in [20], [21]) problem. The leakage rate for these problems may be measured using traditional Shannon measures or the Rényi measures (or their Gallager-type counterparts) studied extensively in this paper. Here, we are only concerned with the secrecy requirement rather than both the secrecy and reliability requirements of the wiretap problem. The reliability requirement can be handled using, by now, standard error exponent analyses [13], [14].

C. Related Works

In [17], [22], Hayashi generalized and strengthened the seminal privacy amplification analyses of Bennett et al. [7], Renner [23] and Renner and Wolf [24] to obtain exponential error bounds for the leakage rate of the discrete memoryless wiretap channel and the secrecy key agreement problems [2], [3], [12], [19]. The leakage rate was measured by the mutual information $I(A \wedge E|P_{AE})$ and the variational (or trace) distance $\|P_{AE} - P_A \times P_E\|_1$. However, the results contained in [17], and further generalized in [20], [21], are only achievability results. This paper, though it does not focus on the wiretap channel, derives tight bounds for a family of Rényi information measures (which contains the Shannon measures as special cases) for commonly employed hashing operators for security applications. In [18], [25], Hayashi focused on the variational distance criterion and derived tight exponential error bounds (both achievability and converse). Hayashi and Tsurumaru [26] proposed an efficient construction of hash functions for the purpose of privacy amplification with less random seeds, thus potentially realizing the system in Fig. 1 with less random resources. Other works along the lines of deriving exponential error bounds for information-theoretic security problems include those by Hou and Kramer [27], [28], Pierrot and Bloch [29], Bloch and Laneman [30], Han et al. [31] and Parizi and Telatar [32].

D. Paper Organization

The rest of the paper is organized as follows: In Section II, we state the relevant preliminaries and define relevant information measures and security criteria for understanding the rest of the paper. In Section III, we state our results for the asymptotics of the equivocation. In Section IV, we state our results for the exponential behavior of the Rényi-type security criteria. In Section V, we state our results for the second-order asymptotics of the equivocation. We also consider the case where the magnitudes of the second-order rates are large. These are proved using novel one-shot bounds which are stated in Section VI. The proofs of the asymptotic results are provided in Section VII. We conclude the paper in Section VIII by summarizing our key contributions and stating avenues for further investigations. The proofs of the one-shot bounds are rather technical and are thus relegated to the Appendices.

II. PRELIMINARIES AND INFORMATION MEASURES

A. Basic Shannon and Rényi Information Quantities

We now introduce some information measures that generalize Shannon’s information measures. Fix a normalized distribution $P_A \in \mathcal{P}(A)$ and a sub-distribution (a non-negative vector but not necessarily summing to one) $Q_A \in \mathcal{Q}_A$.
\( \mathcal{P}(A) \) supported on a finite set \( A \). Then the relative entropy and the Rényi divergence of order \( 1 + s \) are respectively defined as

\[
D(P_A\|Q_A) := \sum_{a \in A} P(a) \log \frac{P(a)}{Q(a)}
\]

\[
D_{1+s}(P_A\|Q_A) := \frac{1}{s} \log \sum_{a \in A} P_A(a)^{1+s} Q_A(a)^{-s},
\]

where throughout, \( \log \) is to the natural base \( e \). It is known that \( \lim_{s \to 0} D_{1+s}(P_A\|Q_A) = D(P_A\|Q_A) \) so a special case of the Rényi divergence is the usual relative entropy. It is known that the map \( s \mapsto D_{1+s}(P_A\|Q_A) \) is concave in \( s \in \mathbb{R} \) and hence \( D_{1+s}(P_A\|Q_A) \) is monotonically increasing for \( s \in \mathbb{R} \). Furthermore, the following data processing inequalities for Rényi divergences hold for \( s \in [-1, 1] \),

\[
D(P_A W\|Q_A W) \leq D(P_A\|Q_A) \]

\[
D_{1+s}(P_A W\|Q_A W) \leq D_{1+s}(P_A\|Q_A).
\]

Here \( W : A \to \mathcal{B} \) is any stochastic matrix (channel) and \( P_A W(b) := \sum_a W(b|a)P_A(a) \) is the output distribution induced by \( W \) and \( P_A \).

We use \( P_{\text{mix},A} \) to denote the uniform distribution on \( A \). We also introduce conditional entropies on the joint alphabet \( A \times E \). If \( P_{AE} \) is a distribution on \( A \times E \), the conditional entropy and the conditional Rényi entropy of order \( 1 + s \) relative to another normalized distribution \( Q_E \) on \( E \) as

\[
H(A|E|P_{AE}\|Q_E) := -D(P_{AE}|I_A \times Q_E),
\]

\[
H_{1+s}(A|E|P_{AE}\|Q_E) := -D_{1+s}(P_{AE}|I_A \times Q_E).
\]

Here \( I_A(a) = 1 \) for each \( a \in A \) and it is known that \( \lim_{s \to 0} H_{1+s}(A|E|P_{AE}\|Q_E) = H(A|E|P_{AE}\|Q_E) \). If \( Q_E = P_E \), we simplify the notation and denote the conditional entropy and the conditional Rényi entropy of order \( 1 + s \) as

\[
H(A|E|P_{AE}) := H(A|E|P_{AE}\|P_E) = -\sum_e P_E(e) \sum_a P_{A|E}(a|e) \log P_{A|E}(a|e)
\]

\[
H_{1+s}(A|E|P_{AE}) := H_{1+s}(A|E|P_{AE}\|P_E) = -\frac{1}{s} \log \sum_e P_E(e) \sum_a P_{A|E}(a|e)^{1+s}.
\]

The function \( s \mapsto sH_{1+s}(A|E|P_{AE}) \) is concave, and \( H_{1+s}(A|E|P_{AE}\|Q_E) \) is monotonically decreasing on \((0, \infty)\) and \((-\infty, 0)\).

We are also interested in the so-called Gallager form of the conditional Rényi entropy for a joint distribution \( P_{AE} \in \mathcal{P}(A \times E) \):

\[
H_{1+s}^\dagger(A|E|P_{AE}) := \frac{1}{s} \log \sum_e \left( \sum_a P_{AE}(a,e)^{1+s} \right)^{-\frac{1}{s}}.
\]

By defining the familiar Gallager function \([13], [33]\) (parametrized slightly differently)

\[
\phi(s|A|E|P_{AE}) := \log \sum_e \left( \sum_a P_{AE}(a,e)^{\frac{1}{1+s}} \right)^{-s}
\]

we can express (14) as

\[
H_{1+s}^\dagger(A|E|P_{AE}) = \frac{1}{s} \phi \left( \frac{s}{1+s} |A|E|P_{AE} \right),
\]

thus (loosely) justifying the nomenclature “Gallager form” of the conditional Rényi entropy in (14). The quantities \( H_{1+s} \) and \( H_{1+s}^\dagger \) can be shown to be related as follows:

\[
\max_{Q_E \in \mathcal{P}(E)} H_{1+s}(A|E|P_{AE}\|Q_E) = H_{1+s}^\dagger(A|E|P_{AE})
\]
for $s \in [-1, \infty) \setminus \{0\}$. The maximum on the left-hand-side is attained for the tilted distribution

$$Q_E(e) = \frac{\left(\sum_a P_{AE}(a, e)^{1+s}\right)^{\frac{1}{1+s}}}{\sum_e \left(\sum_a P_{AE}(a, e)^{1+s}\right)^{\frac{1}{1+s}}}.$$ (18)

The map $s \mapsto sH_{1+s}^t(A|E|P_{AE})$ is concave and the map $s \mapsto H_{1+s}^t(A|E|P_{AE})$ is monotonically decreasing for $s \in (-1, \infty)$. It can be shown by L'Hôpital’s rule that

$$\lim_{s \to 0} H_{1+s}^t(A|E|P_{AE}) = H(A|E|P_{AE}).$$ (19)

Thus, we regard $H_1(A|E|P_{AE})$ as $H(A|E|P_{AE})$, i.e., for Rényi parameter $\alpha = 1 + s = 1$, the conditional Rényi entropy and its Gallager form coincide. We also find it useful to consider a two-parameter family of the conditional Rényi entropy:

$$H_{1+s|1+t}(A|E|P_{AE}) := -\frac{1+t}{s} \log \sum_e P_E(e) \left(\sum_a P_{AE}(a, e)^{1+s}\right)^{\frac{1}{1+s}}.$$ (20)

This was first introduced in the work by Hayashi and Watanabe [34, Eq. (55)-(57)]. Clearly,

$$H_{1+s|1+s}(A|E|P_{AE}) = H_{1+s}^t(A|E|P_{AE})$$ (21)

so two-parameter conditional Rényi entropy is a generalization of the Gallager form of the conditional Rényi entropy in (14).

The Rényi entropies can be shown to satisfy a form of data processing inequality. In particular if $f : A \to \mathcal{M}$ is any function on the set $\mathcal{A}$, we have

$$H(f(A)|E|P_{AE}) \leq H(A|E|P_{AE}),$$ (22)

$$H_{1+s}(f(A)|E|P_{AE}) \leq H_{1+s}(A|E|P_{AE}),$$ (23)

$$H_{1+s}^t(f(A)|E|P_{AE}) \leq H_{1+s}^t(A|E|P_{AE}).$$ (24)

Inequalities (23) and (24) hold true for all $s > -1$. These inequalities say that processing the random variable $A$ cannot increase its randomness measured under any of the above conditional Rényi entropies.

**B. Rényi Security Criteria**

Now, we introduce various criteria that measure independence and uniformity jointly. The mutual information is

$$I(A \land E|P_{AE}) := D(P_{AE}\|P_A \times P_E).$$ (25)

This, together with its normalized version, has been traditionally used as measure of dependence in classical information-theoretic security [2], [3], going back to the seminal work of Wyner [12] for the wiretap channel. It is also used in Ahlswede and Csiszár for the key agreement problem [19]. However, it does not guarantee approximate uniformity of the source $P_A$ on $\mathcal{A}$. Thus, we introduce the modified mutual information

$$C(A|E|P_{AE}) := D(P_{AE}\|P_{\text{mix}, A} \times P_E)$$ (26)

$$= \log |\mathcal{A}| - H(A|E|P_{AE}).$$ (27)

This quantity was also considered by Csiszár and Narayan [35, Eq. (6)] in their work on secrecy capacities. An axiomatic justification of $C(A|E|P_{AE})$ was provided recently by Hayashi [36, Thm. 8] and it clearly satisfies

$$C(A|E|P_{AE}) = I(A \land E|P_{AE}) + D(P_A\|P_{\text{mix}, A}).$$ (28)

Hence, if $C(A|E|P_{AE})$ is small, $A$ is approximately independent of $E$ and $A$ is approximately uniform on its alphabet, desirable properties in information-theoretic security. We may further generalize the modified mutual information by considering Rényi information measures, introduced in Section II-A, as follows:

$$C_{1+s}(A|E|P_{AE}) := D_{1+s}(P_{AE}\|P_{\text{mix}, A} \times P_E)$$ (29)

$$= \log |\mathcal{A}| - H_{1+s}(A|E|P_{AE}).$$ (30)
This can be relaxed to give yet another security measure—the *Gallager-form of the modified mutual information*:

\[
C_{1+s}^\ddagger(A|E|P_{AE}) := \min_{Q_E \in \mathcal{P}(E)} D_{1+s}(P_{AE} \| P_{\text{mix},A} \times Q_E) \\
= \log |A| - H_{1+s}^\ddagger(A|E|P_{AE}).
\]  

(31)

We characterize these quantities asymptotically when \((A, E) \equiv (f(A^n), E^n)\) for some (classes of) hash functions \(f(\cdot)\). The quantities \(H_{1+s}\) and \(H_{1+s}^\ddagger\) can be regarded as equivocations [12] so \(C_{1+s}\) and \(C_{1+s}^\ddagger\) are the negative of the equivocations up to a shift. We work with \(C_{1+s}\) and \(C_{1+s}^\ddagger\) in the rest of the paper as it is more convenient.

### III. Asymptotics of the Equivocation

In this section we present our results concerning the asymptotic behavior of the equivocation. First we define precisely the notion of hash function. This is a generalization of the definition by Carter and Wegman [1].

**Definition 1.** A hash function \(f_X\) is a stochastic map from \(A\) to \(M := \{1, \ldots, M\}\), where \(X\) denotes a random variable describing its stochastic behavior. An ensemble of the functions \(f_X\) is called an \(\epsilon\)-almost universal hash function when it satisfies the following condition: For any distinct \(a_1, a_2 \in A\), the probability of a collision

\[
\Pr(f_X(a_1) = f_X(a_2)) \leq \frac{\epsilon}{M}.
\]

(33)

When \(\epsilon = 1\), we simply say that the ensemble of functions is a universal_2 hash function.

As an example, if we randomly and uniformly assign each element of \(a \in A\) into one of \(M\) bins indexed by \(m \in M\) (i.e., the familiar random binning process introduced by Cover [37]), then \(\Pr(f_X(a_1) = f_X(a_1)) = \frac{1}{M}\) so this is a universal_2 hash function, and furthermore, (33) is achieved with equality.

Let \(|t|^+ = \max(0, t)\). The following is our first main result.

**Theorem 1** (Asymptotics of the Equivocation). Let \(^1 M_n = e^{nR}.\) Let \(\epsilon\) be a fixed positive number. Assume that \(f_X^n : A^n \to M_n = \{1, \ldots, M_n\}\) is an \(\epsilon\)-almost universal_2 hash function. For any \(s \in [0, 1]\), we have

\[
\lim_{n \to \infty} \frac{1}{n} \inf_{f_X^n} C_{1+s}(f_X^n(A^n)|E^n X_n|P_{AE}^n \times P_{X_n}) = |R - H_{1+s}(A|E|P_{AE})|^+,
\]

(34)

\[
\lim_{n \to \infty} \frac{1}{n} \inf_{f_X^n} C_{1-s}^\ddagger(f_X^n(A^n)|E^n X_n|P_{AE}^n \times P_{X_n}) = |R - H_{1+s}^\ddagger(A|E|P_{AE})|^+.
\]

(35)

Furthermore, for any \(s \in (0, 1]\), we also have

\[
\lim_{n \to \infty} \frac{1}{n} \inf_{f_X^n} C_{1-s}(f_X^n(A^n)|E^n X_n|P_{AE}^n \times P_{X_n}) = \begin{cases} 
R - H_{1-s}(A|E|P_{AE}) & R \geq \hat{R}_s \\
\max_{t \in [0, s]} \frac{d}{dt} (R - H_{1-t}(A|E|P_{AE})) & R \leq \hat{R}_s 
\end{cases},
\]

(36)

\[
\lim_{n \to \infty} \frac{1}{n} \inf_{f_X^n} C_{1-s}^\ddagger(f_X^n(A^n)|E^n X_n|P_{AE}^n \times P_{X_n}) = \begin{cases} 
R - H_{1-s}^\ddagger(A|E|P_{AE}) & R \geq \hat{R}_s^\ddagger \\
\max_{t \in [0, s]} \frac{d}{dt} (R - H_{1-t}^\ddagger(A|E|P_{AE})) & R \leq \hat{R}_s^\ddagger
\end{cases},
\]

(37)

where

\[
\hat{R}_s := \frac{d}{dt} tH_{1-t}(A|E|P_{AE})\big|_{t=s},
\]

(38)

\[
\hat{R}_s^\ddagger := \frac{d}{dt} tH_{1-t}^\ddagger(A|E|P_{AE})\big|_{t=s}.
\]

(39)

Furthermore, the infima in (34)–(37) are achieved by some sequence of universal_2 hash functions \(f_X^n\).

This result is proved in Section VII-A.

We remark that the converse parts (lower bounds) to (34)–(35) hold for all \(s \geq 0\) (and not only being upper bounded by 1) owing to the data processing inequalities in (23)–(24).

The results in (34)–(37) imply that an optimum sequence of hash functions \(\{f_X^n\}_{n \in \mathbb{N}}\) is such that asymptotically the normalized security measure \(C_{1+s}\) and its Gallager-type counterpart \(C_{1+s}^\ddagger\) increase linearly with the rate \(R\) if

\(^1\)As is usual in information theory, we ignore the integer effects on the size of the hash function \(M_n = ||f||\) since this is inconsequential asymptotically. This imprecision is also employed in the sequel for notational convenience.
the rate is larger than the conditional Rényi entropy and its Gallager-type counterpart. However, note that this only holds for the case where $R$ is greater than the analogue of the critical rates, defined in (38)–(39) in the case where the Rényi parameter $\alpha = 1 - s$ is less than one. Observe that there is difference in behavior when we consider the other direction, i.e., the quantities $C_{1-s}$ and $C^\uparrow_{1-s}$ for $s \in [0, 1]$. Below the critical rate, the equivocation no longer increases linearly with $R$ but is nonetheless still convex in $R$. See Figs. 2 and 3 for illustrations of the asymptotics of the equivocations in Theorem 1. The behaviors of the normalized security measure $C_{1+s}, C_{1-s}$ and their Gallager-type counterparts $C^\uparrow_{1+s}, C^\uparrow_{1-s}$ are similar.

Finally, we examine the optimal (maximum) key generation rates, i.e., the largest rates $R$ for which there exists a sequence of functions from $\mathcal{A}^n$ to $\{1, \ldots, e^{nR}\}$ such that $\frac{1}{n}C_{1+s}$ or $\frac{1}{n}C^\uparrow_{1+s}$ tend to zero as the blocklength grows. We see from the following corollary that this cutoff rate depends strongly on the sign of $s$. In particular for $s \in (0, 1]$, the cutoff rates are $H_{1+s}(A|E|P_{AE})$ and $H^\uparrow_{1+s}(A|E|P_{AE})$ respectively, while for $s \in [-1, 0]$, the cutoff rates are both equal to the Shannon conditional entropy $H(A|E|P_{AE})$ independent of $s$. This difference between the behaviors of the optimal key generation rates depending on the sign of $s$ (also illustrated in Figs. 2 and 3) is somewhat surprising (at least to the authors).
Corollary 1 (Optimal key generation rates). We have

\[
\sup \left\{ R \in \mathbb{R}_+ : \lim_{n \to \infty} \inf_{f : A^n \to \{1, \ldots, e^{nR}\}} \frac{C_{1+s}(f(A^n))|E^n|P_{AE}^n}{n} = 0 \right\} = \begin{cases} H_{1+s}(A|E|P_{AE}) & \text{if } s \in (0, 1] \\ H(A|E|P_{AE}) & \text{if } s \in [-1, 0] \end{cases} \tag{40}
\]

\[
\sup \left\{ R \in \mathbb{R}_+ : \lim_{n \to \infty} \inf_{f : A^n \to \{1, \ldots, e^{nR}\}} \frac{C_{1+s}(f(A^n))|E^n|P_{AE}^n}{n} = 0 \right\} = \begin{cases} H_{1+s}(A|E|P_{AE}) & \text{if } s \in (0, 1] \\ H(A|E|P_{AE}) & \text{if } s \in [-1, 0] \end{cases} \tag{41}
\]

Proof: We only prove the statement for $C_{1+s}$ in (40) since that for $C_{1+s}^+$ in (41) is completely analogous. The case for $s \in (0, 1]$ is obvious from (34) in Theorem 1 since the limit is $|R - H_{1+s}(A|E|P_{AE})|$. Now, for the case $s \in [-1, 0]$, if $R \leq H(A|E|P_{AE})$, we know from the monotonically decreasing nature of $H_{1+s}(A|E|P_{AE})$ (in $s$) that $R - H_{1+s}(A|E|P_{AE})$ is non-positive for $t \in [0, s]$. Thus, referring to (36) in Theorem 1, the optimal $t$ in the optimization $\max_{t \in [0, s]} \frac{1}{s}(R - H_{1-s}(A|E|P_{AE}))$ is attained at $t = 0$ and consequently, the optimal objective value is 0. On the other hand, for $R \in (H(A|E|P_{AE}), R_0]$, the optimal $t \in (0, s]$ and so the optimal objective value is positive. Thus, for $s \in [-1, 0]$, the optimal key generation rate is the Shannon conditional entropy $H(A|E|P_{AE})$.

IV. EXponential Behavior OF OF THE SECURITY Measures

In this section, we evaluate the exponential rates of decay of the security measures $C_{1+s}$ and $C_{1+s}^+$ for fixed rates $R$ above an analogue of the critical rate.

Theorem 2 (Exponents of the Equivocation). Let $M_n = e^{nR}$ and let $\epsilon$ be a fixed positive number. Assume that $f_{X_n} : A^n \to M_n = \{1, \ldots, M_n\}$ is an $\epsilon$-almost universal2 hash function. For

\[
R \geq \hat{R}_+ := \frac{d}{ds} sH_{1+s}(A|E|P_{AE}) \big|_{s=1} \tag{42}
\]

and any $s \in [0, 1]$, we have

\[
\lim_{n \to \infty} -\frac{1}{n} \log \inf_{f_{X_n}} C_{1+s}(f_{X_n}(A^n)|E^nX_n|P_{AE}^n \times P_{X_n}) = \sup_{t \in [0, 1]} tH_{1+t}(A|E|P_{AE}) - tR, \tag{43}
\]

\[
\lim_{n \to \infty} -\frac{1}{n} \log \inf_{f_{X_n}} C_{1-s}(f_{X_n}(A^n)|E^nX_n|P_{AE}^n \times P_{X_n}) = \max_{t \in [0, 1]} tH_{1+t}(A|E|P_{AE}) - tR. \tag{44}
\]

For the Gallager-type counterparts of the Rényi quantities and

\[
R \geq \hat{R}_+^+ := \frac{d}{ds} sH_{1+s}^+(A|E|P_{AE}) \big|_{s=1} \tag{45}
\]

with any $s \in [0, 1]$, we also have

\[
\lim_{n \to \infty} -\frac{1}{n} \log \inf_{f_{X_n}} C_{1+s}^+(f_{X_n}(A^n)|E^nX_n|P_{AE}^n \times P_{X_n}) = \max_{t \in [0, 1]} tH_{1+t}(A|E|P_{AE}) - tR, \tag{46}
\]

\[
\lim_{n \to \infty} -\frac{1}{n} \log \inf_{f_{X_n}} C_{1-s}^+(f_{X_n}(A^n)|E^nX_n|P_{AE}^n \times P_{X_n}) = \max_{t \in [0, 1]} tH_{1+t}(A|E|P_{AE}) - tR. \tag{47}
\]

Similarly to Theorem 1, the infima in (43)–(47) are achieved by some sequence of universal2 hash functions $f_{X_n}$.

This result is proved in Section VII-B. The techniques for the direct parts are somewhat similar to those in [17], [21], [22] using improved versions of Bennett et al.’s [7] bound which was based on the Rényi entropy of order 2. However, the non-asymptotic bounds (e.g., Lemma 5) and asymptotic evaluations for the converse parts require new ideas. Different from the direct part, we need to convert the evaluation of $e^{-sC_{1-s}}$ and $e^{-\frac{1}{s}C_{1-s}}$ into information spectrum [38] quantities (involving the conditional entropy random variable) so that is is amenable to asymptotic evaluation. These information spectrum quantities are then evaluated using various large deviation [39] bounds, such as Cramer’s theorem and various forms of the Gärtnert-Ellis theorems.

Note that the derivative of the conditional Rényi entropy in (42) and (45) are the analogues of the critical rate in error exponent analysis [13], [14]. For the exponents, we have a complete characterization of the exponential rates of decay of both $C_{1+s}$ and $C_{1+s}^+$ for $s \in [0, 1]$ and they are given by optimization of quantities that are related
to the conditional Rényi entropy. We observe that the Gallager form results in larger exponents in general as the optimizations in (44) and (47) are larger than their non-Gallager counterparts in (43) and (46) respectively.

The exponents in (43) and (44) of Theorem 2 are illustrated in Fig. 4. We observe the same behavior for the exponents of the Gallager forms in (46) and (47) since the expressions are the same and so we omit these cases. We note (from the plot and from direct evaluations) that the zero-crossings for the exponents of \( C_{1/2} \), \( C_1 \), \( C_{3/2} \) and \( C_{7/4} \) occur at \( H_1, H_1, H_{3/2} \) and \( H_{7/4} \) respectively (\( H_1 \) being the Shannon entropy). This is in line with Corollary 1. Indeed, the exponent being positive implies that the normalized security measures \( \frac{1}{n} C_{1+s} \) and \( \frac{1}{n} C_{1+s}^7 \) vanish as blocklength grows.

V. SECOND-ORDER ASYMPTOTICS

In the previous sections, the security measures in terms of equivocations and their logarithms were normalized by the blocklength \( n \). In this section, we study different normalizations, e.g., by \( \sqrt{n} \). In addition, we examine the effect of changing the size of the hash function \( M_n \) from \( e^{nR} \) (considered in Sections III and IV) to \( e^{nR+\sqrt{n}L} \), where \( L \in \mathbb{R} \) is an arbitrary real number. To do, we first define the following important quantities.

**Definition 2.** Given a discrete joint source \( P_{AE} \in \mathcal{P}(A \times E) \), define the conditional varentropy \([40]\) or conditional source dispersion \([41], [42]\) to be

\[
V(A|E|P_{AE}) := \sum_{a,e} P_{AE}(a,e)(\log P_{A|E}(a|e) + H(A|E|P_{AE}))^2.
\]  

We also define the following variants of the conditional varentropy

\[
V_1(A|E|P_{AE}) := \sum_e P_E(e)(H(A|E|P_{AE}) - H(A|P_{A|E=e}))^2
\]

\[
V_2(A|E|P_{AE}) := V(A|E|P_{AE}) - V_1(A|E|P_{AE})
\]

\[
= \sum_{a,e} P_{AE}(a,e)(\log P_{A|E}(a|e) + H(A|P_{A|E=e}))^2.
\]

One can readily check that \( V = V_1 + V_2 \) from the definitions. This also follows immediately from the law of total variance. Let

\[
\Phi(t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} \, du
\]

be the cumulative distribution function of the standard Gaussian random variable. With these definitions, we are ready to state our results on the second-order asymptotics for the security measures \( C_{1+s} \) and \( C_{1+s}^7 \) which are
simple functions of the equivocation \( H_{1+s} \) and \( H_{1+s}^\dagger \) respectively. Note that apart from the second-order analysis of \( C_1 \) (\( s = 0 \) case), which builds on Hayashi’s work in [16, Theorem 8], the other results in Theorems 3 and 4 are novel.

**Theorem 3 (Second-Order Asymptotics).** Let \( \epsilon \) be a fixed positive number. Assume that \( f_{X_n} : \mathcal{A}^n \to \mathcal{M}_n = \{1, \ldots, M_n\} \) is an \( \epsilon \)-almost universally hash function. Consider the following three cases:

- **Case (A):** \( \alpha = 1 + s \) with \( s \in (0, 1] \): Suppose that the number of messages \( M_n = e^{nH_{1+s}(A)|E|P_{AE}^n} + \sqrt{nL} \) or \( M_n = e^{nH_{1+s}(A)|E|P_{AE}^n} + \sqrt{nL} \). When \( L \geq 0 \), we have

\[
\lim_{n \to \infty} \frac{1}{\sqrt{n}} \inf_{f_{X_n}} C_{1+s}(f_{X_n}(A^n)|E^nX_n|P_{AE}^n \times P_{X_n}) = L
\]

(53)

\[
\lim_{n \to \infty} \frac{1}{\sqrt{n}} \inf_{f_{X_n}} C_{1+s}^\dagger(f_{X_n}(A^n)|E^nX_n|P_{AE}^n \times P_{X_n}) = L.
\]

(54)

When \( L \leq 0 \), we have

\[
\lim_{n \to \infty} -\frac{1}{\sqrt{n}} \log \inf_{f_{X_n}} C_{1+s}(f_{X_n}(A^n)|E^nX_n|P_{AE}^n \times P_{X_n}) = -sL
\]

(55)

\[
\lim_{n \to \infty} -\frac{1}{\sqrt{n}} \log \inf_{f_{X_n}} C_{1+s}^\dagger(f_{X_n}(A^n)|E^nX_n|P_{AE}^n \times P_{X_n}) = -sL.
\]

(56)

- **Case (B):** \( \alpha = 1 \) (i.e., \( s = 0 \)): Suppose that \( M_n = e^{nH(A|E|P_{AE}^n)} + \sqrt{nL} \) for some \( L \in \mathbb{R} \), we have

\[
\lim_{n \to \infty} \frac{1}{\sqrt{n}} \inf_{f_{X_n}} C_1(f_{X_n}(A^n)|E^nX_n|P_{AE}^n \times P_{X_n}) = \int_{-\infty}^{L/\sqrt{V(A|E|P_{AE})}} L - \frac{\sqrt{V(A|E|P_{AE})}x}{\sqrt{2\pi}} e^{-x^2/2} \, dx.
\]

(57)

By (19), the same asymptotic behavior also holds true for the Gallager version of the security measure \( C_1^\dagger(f_{X_n}(A^n)|E^nX_n|P_{AE}^n \times P_{X_n}) \).

- **Case (C):** \( \alpha = 1 - s \) with \( s \in (0, 1] \): Suppose that \( M_n = e^{nH(A|E|P_{AE}^n)} + \sqrt{nL} \) for some \( L \in \mathbb{R} \), we have

\[
\max \left\{ -\frac{1}{s} \log \left( 2^{1-s} \frac{L}{s} \right) - \frac{1}{s} \log \Phi \left( -\frac{L}{\sqrt{V(A|E|P_{AE})}} \right), -\frac{1}{s} \log \Phi \left( -\frac{L}{\sqrt{V(A|E|P_{AE})}} \right) \right\}
\]

\[
\leq \liminf_{n \to \infty} C_{1-s}(f_{X_n}(A^n)|E^nX_n|P_{AE}^n \times P_{X_n})
\]

\[
\leq \limsup_{n \to \infty} C_{1-s}(f_{X_n}(A^n)|E^nX_n|P_{AE}^n \times P_{X_n})
\]

\[
\leq -\frac{1}{s} \log \Phi \left( -\frac{L}{\sqrt{V_2(A|E|P_{AE})}} \right).
\]

(58)

In addition, for the Gallager-type counterparts, with \( M_n = e^{nH(A|E|P_{AE}^n)} + \sqrt{nL} \) for some \( L \in \mathbb{R} \), we also have

\[
\max \left\{ -\frac{1}{s} \log \left( 2^{s+1} \frac{L}{s} \right) - \frac{1}{s} \log \int_{-\infty}^{\infty} \Phi \left( -\frac{L + x}{\sqrt{V_2(A|E|P_{AE})}} \right) \right\}
\]

\[
\leq \liminf_{n \to \infty} C_{1-s}^\dagger(f_{X_n}(A^n)|E^nX_n|P_{AE}^n \times P_{X_n})
\]

\[
\leq \limsup_{n \to \infty} C_{1-s}^\dagger(f_{X_n}(A^n)|E^nX_n|P_{AE}^n \times P_{X_n})
\]

\[
\leq -\frac{1}{s} \log \int_{-\infty}^{\infty} \Phi \left( -\frac{L + x}{\sqrt{V_2(A|E|P_{AE})}} \right) \right\}
\]

(59)

This result is proved in Section VII-C. We remark that the converse parts (lower bounds) to (53)–(54) hold for all \( s \geq 0 \) (and not only being upper bounded by 1) owing to the data processing inequalities in (23)–(24).
Fig. 5. Illustration of the second-order asymptotics in Case (B) given by the right-hand-side of (57) for the discrete memoryless multiple source $P_{AE} = [0.7 \ 0.1; 0.1 \ 0.1]$. It is easy to see that the integral there is non-negative.

Observe that in Theorem 3 (Case (A) for instance), the number of compressed symbols $M_n$ satisfies

$$\log M_n = nH_{1+s}(A|E)P_{AE} + \sqrt{n}L,$$

or,

$$\log M_n = nH^\uparrow_{1+s}(A|E)P_{AE} + \sqrt{n}L.$$  \hspace{1cm} (60)

The leading conditional Rényi entropy terms scaling in $n$ are known as the first-order terms, while the terms scaling as $\sqrt{n}$ are known as the second-order terms. The coefficient $L$ is known as the second-order coding rate [16], [43], [44] and the second-order asymptotic characterizations depend on $L$. Note that even though $L$ is termed as the second-order coding rate, it may be negative. Observe that the conditional variances appear in (57)–(59), which suggests that we evaluate the one-shot bounds using the central limit theorem among other techniques. We have tight results (equalities) for Cases (A) and (B) but unfortunately not for Case (C) where the Rényi parameter $\alpha = 1 - s$ for $s \in (0, 1]$. However, in the limit of the second-order coding rate $L$ being large (either in the positive or negative direction), we can assert that one of the terms in the maxima in the lower bounds of (58) and (59) dominates and matches the upper bound and hence, we have a tight result up to the term in $L^2$ (Theorem 4). We now comment specifically on each of the cases.

1) For Case (A), the second-order asymptotic behaviors of $C_{1+s}$ and $C^\uparrow_{1+s}$ when they are normalized by $\frac{1}{\sqrt{n}}$ are linear in $L$.

2) The same is true for Case (B) for large positive $L$ because with $V := V(A|E)P_{AE}$,

$$\lim_{L \to \infty} \frac{1}{L} \int_{-\infty}^{L/\sqrt{V}} \frac{L - \sqrt{V}x}{\sqrt{2\pi}} e^{-x^2/2} \, dx = \lim_{L \to \infty} \left\{ \Phi \left( \frac{L}{\sqrt{V}} \right) - \frac{\sqrt{V}}{L} \int_{-\infty}^{L/\sqrt{V}} xe^{-x^2/2} \sqrt{2\pi} \, dx \right\} = 1.$$  \hspace{1cm} (62)

In contrast, when $L \to -\infty$ in Case (B), the limit is zero. The second-order asymptotics in Case (B) in (57) is shown in Fig. 5 and is obtained via numerical integration to approximate the integral. The limit in (57) is monotonically increasing in $L$. This is intuitive because as $L$ increases, there is potentially more leakage to $E^n$ and less uniformity on the (larger) support $\{1, \ldots, e^{nH(A|E)P_{AE} + \sqrt{n}L} \}$.

3) For Case (C) there is no normalization by $\frac{1}{\sqrt{n}}$ and we only have bounds. However, for large $|L|$, we will see from Theorem 4 that the second-order asymptotic behavior is quadratic in $L$. The bounds on the second-order asymptotics in the two parts (conditional Rényi entropy and its Gallager version) of Case (C) in (58) and (59) are shown in Figs. 6 and 7 respectively.

We conclude that in the second-order asymptotic regime where the number of compressed symbols satisfies (60)–(61), there are distinct differences between the three regimes of the Rényi parameter $\alpha \in [0, 1)$, $\alpha = 1$ and $\alpha \in (1, 2]$. 

Since for Case (C) we only have bounds, we now examine the behavior of the bounds in the limit of large $|L|$ for which we can show tight results up to the quadratic terms.

**Theorem 4 (Large Second-Order Rates).** For Case (C) in Theorem 3 (Rényi parameter $\alpha = 1 - s$ where $s \in (0, 1]$), we have the following asymptotic results as $L \to \infty$:

\[
\liminf_{n \to \infty} C_{1-s}(f_{X_n}(A^n)|E^nX_n|P_{AE}^n \times P_{X_n}) = \frac{L^2}{2sV(A|E|P_{AE})} + O(\log L),
\]

(63)

\[
\liminf_{n \to \infty} C_{1-s}^\dagger(f_{X_n}(A^n)|E^nX_n|P_{AE}^n \times P_{X_n}) = \frac{1-s}{2s} \cdot \frac{L^2}{V_1(A|E|P_{AE}) + V_2(A|E|P_{AE})(1-s)} + O(\log L).
\]

(64)

Furthermore, we have the following asymptotic results as $L \to -\infty$:

\[
\liminf_{n \to \infty} \log f_{X_n} C_{1-s}(f_{X_n}(A^n)|E^nX_n|P_{AE}^n \times P_{X_n}) = -\frac{L^2}{2V(A|E|P_{AE})} + O(\log |L|)
\]

(65)

\[
\liminf_{n \to \infty} \log f_{X_n} C_{1-s}^\dagger(f_{X_n}(A^n)|E^nX_n|P_{AE}^n \times P_{X_n}) = -\frac{L^2}{2V(A|E|P_{AE})} + O(\log |L|)
\]

(66)

The results in Theorem 4 are somewhat analogous and similar those in the study of the moderate-deviations asymptotics in information theory [45]–[49]. Note the difference between the results in (63)–(64) ($L \to \infty$) versus (65)–(66) ($L \to -\infty$). The former pair of results resembles the equivocation results presented in Section III since the effective rate is $(L/\sqrt{n})$-higher than the conditional Rényi entropy and there is no logarithm preceding $C_{1-s}$ and $C_{1-s}^\dagger$. The latter pair of results resembles the exponent results of Section IV since the effective rate is $(|L|/\sqrt{n})$-lower than the conditional Rényi entropy and there is a logarithm preceding $C_{1-s}$ and $C_{1-s}^\dagger$. So the results presented in Theorem 4 are natural in view of Theorems 1 and 2.
Proof of Theorem 4: For ease of notation, let

\[ A := -\frac{1}{s} \log \left( \frac{2^{1-s}}{s^{-s}} \right) \left( 1 - s \right) - \frac{1}{s} \log \Phi \left( - \frac{L}{\sqrt{V(A|E|P_{AE})}} \right) \]  

\[ B := -\frac{1-s}{s} \log \Phi \left( - \frac{L}{\sqrt{V(A|E|P_{AE})}} \right). \]  

be the first and second terms in the maximum in the lower bound in (58). When \( L \to \infty \), the term \( A \) attains the maximum (since \( B \) with the additional factor of \( 1-s \in [0,1] \) is less positive). Also see the right plot of Fig. 6. Thus, in this limiting regime, the lower bound matches the upper bound in (58) up to a constant term. In other words,

\[ \lim_{n \to \infty} \inf_{f_{X_n}} C_{1-s}(f_{X_n}(A^n)|E^n X_n|P_{AE}^n \times P_{X_n}) = -\frac{1}{s} \log \Phi \left( - \frac{L}{\sqrt{V(A|E|P_{AE})}} \right) + O(1) \]  

(69)

where \( O(1) \) denotes a term bounded in \( L \) (but dependent on \( s \)). Now by employing the asymptotic equality

\[ \Phi(-t) = 1 - \Phi(t) \sim \frac{e^{-t^2/2}}{\sqrt{2\pi t}}, \quad \text{as } t \to \infty, \]  

(70)

we obtain from (69) that

\[ \lim_{n \to \infty} \inf_{f_{X_n}} C_{1-s}(f_{X_n}(A^n)|E^n X_n|P_{AE}^n \times P_{X_n}) = \frac{L^2}{2sV(A|E|P_{AE})} + O(\log L), \]  

(71)

which proves (63). Now, instead if \( L \to -\infty \), we find that the term \( B \) attains the maximum in the lower bound in (58) due to the constant negative offset in the definition of \( A \). Also see the left plot of Fig. 6. In this case, taking
the logarithm, we have

\[
\lim_{n \to \infty} \log \inf_{f_{X_n}} C_{1-s}(f_{X_n}(A^n)|E^nX_n|P_{AE}^n \times P_{X_n}) \\
= \log \left[ -\log \left( 1 - \Phi \left( \frac{L}{\sqrt{V(A|E|P_{AE})}} \right) \right) \right] + O(1) \\
= \log \left[ \Phi \left( \frac{L}{\sqrt{V(A|E|P_{AE})}} \right) \right] + O(\log |L|) \\
= -\frac{L^2}{2V(A|E|P_{AE})} + O(\log |L|)
\]

(72)

(73)

(74)

where in (72), \( \log \left( \frac{1-s}{s} \right) \leq O(1) \leq \log(\frac{1}{s}) \) for \(-L\) large enough, in (73), we used the fact that \( \log(1-t) = -t + O(t^2) \) when \( t \downarrow 0 \), and finally in (74), we used (70). This proves (65).

Again for ease of notation, let

\[
D := -\frac{1}{s} \log \left( 2^{s+1-s} \frac{1-s}{s} (1-s) \right) - \frac{1-s}{s} \log \int_{-\infty}^{\infty} \Phi \left( -\frac{L+x}{\sqrt{V_2(A|E|P_{AE})}} \right) \frac{i\pi e^{-x^2/(2V_1)}}{\sqrt{2\pi V_1(A|E|P_{AE})}} \, dx
\]

(75)

\[
E := -\frac{1-s}{s} \log \Phi \left( \frac{L}{\sqrt{V(A|E|P_{AE})}} \right). \\
\]

(76)

be the first and second terms in the maximum in the lower bound in (59). When \( L \to \infty \), the term \( D \) dominates because \( V_2 \leq V \) and thus the integrands in \( D \) are smaller than \( \Phi \left( \frac{L}{\sqrt{V(A|E|P_{AE})}} \right) \) in \( E \). See right plot of Fig. 7. We can then find the \( x \) that dominates the integral in \( D \). We denote this by \( x^* \). Since \( L \) is large, by (70),

\[
\log \left[ \Phi \left( \frac{L+x}{\sqrt{V_2}} \right) \right] - \left[ -\frac{1}{2(1-s)} \left( \frac{L+x}{\sqrt{V_2}} \right)^2 - \frac{x^2}{2V_1} \right] \to 0, \quad \text{as} \ L \to \infty.
\]

(77)

Differentiating the quadratic, we obtain

\[
x^* = -\frac{LV_1(A|E|P_{AE})}{V_1(A|E|P_{AE}) + (1-s)V_2(A|E|P_{AE})}.
\]

(78)

The exponential term \( e^{-x^2/(2V_1(A|E|P_{AE}))} \) controls the behavior of the integral in \( D \). Substituting \( x^* \) into this exponential term yields (64). In the other direction, when \( L \to -\infty \), the term \( E \) attains the maximum. See left plot of Fig. 7. By a similar calculation as in (72)–(74), we obtain the lower bound to (66). Carrying out the calculations,

\[
\begin{align*}
\lim_{n \to \infty} \log \inf_{f_{X_n}} C_{1-s}(f_{X_n}(A^n)|E^nX_n|P_{AE}^n \times P_{X_n}) & \geq \log \left[ -\frac{1-s}{s} \log \Phi \left( -\frac{L}{\sqrt{V(A|E|P_{AE})}} \right) \right] \\
& = \log \left[ -\frac{1-s}{s} \left( -\Phi \left( \frac{L}{\sqrt{V(A|E|P_{AE})}} \right) \right) \right] + O(\log |L|) \\
& = \log \left[ \Phi \left( \frac{L}{\sqrt{V(A|E|P_{AE})}} \right) \right] + O(\log |L|) \\
& = -\frac{L^2}{2V(A|E|P_{AE})} + O(\log |L|).
\end{align*}
\]

(79)

(80)

(81)

(82)

To show that the upper bound in (59) matches the lower bound given by \( E \) in (82), we find that

\[
\left[ 1 - \Phi \left( \frac{L+x}{\sqrt{V_2(A|E|P_{AE})}} \right) \right]^{\frac{1}{1-s}} \geq 1 - \frac{1}{1-s} \Phi \left( \frac{L+x}{\sqrt{V_2(A|E|P_{AE})}} \right).
\]

(83)
which implies that the upper bound in (59) simplifies to

\[
- \frac{1 - s}{s} \log \int_{-\infty}^{\infty} \Phi \left( - \frac{L + x}{\sqrt{2 V_1(A|E|P_{AE})}} \right) \frac{1}{\sqrt{2 \pi V_1(A|E|P_{AE})}} \, dx \\
\leq - \frac{1 - s}{s} \log \int_{-\infty}^{\infty} \left[ 1 - \frac{1}{1 - s} \Phi \left( \frac{L + x}{\sqrt{2 V_1(A|E|P_{AE})}} \right) \right] e^{-x^2/(2 V_1(A|E|P_{AE}))} \, dx \\
= - \frac{1 - s}{s} \log \left[ 1 - \frac{1}{1 - s} \Phi \left( \frac{L}{\sqrt{V_1(A|E|P_{AE})}} \right) \right] \\
= \frac{1}{s} \Phi \left( \frac{L}{\sqrt{V_1(A|E|P_{AE})}} \right) + O \left( e^{-L^4/(4 V_1(A|E|P_{AE})^2)} \right)
\]

where (85) follows because the convolution of two independent zero-mean Gaussians is a Gaussian where the variances add and we also note that \( V = V_1 + V_2 \) per (50). This argument was also used in the second-order analysis of channels with state in [50]. Now taking the logarithm and the limit as \( L \to -\infty \), we match the lower bound in (82) and complete the proof of (66).

VI. ONE-SHOT BOUNDS

To prove Theorems 1, 2 and 3, we leverage the following one-shot (i.e., blocklength \( n \) equal to 1) bounds. The proofs of these one-shot bounds are rather technical and hence we provide them in the appendices.

A. One-Shot Bounds for the Direct Parts

For the direct parts of the equivocation results, we evaluate the following one-shot bounds. The first two bounds in (87) and (88) can be considered as generalizations of the bounds by Hayashi in [17] where \( \epsilon = 1 \).

Lemma 1. For an ensemble of \( \epsilon \)-almost universal hash functions \( f_X : \mathcal{A} \to \mathcal{M} = \{1, \ldots, M\} \), we have for \( s \in [0, 1] \),

\[
e^{-s c_{1,+}(f_X(A)|EX|PAE \times PX)} \leq \epsilon^s + M^s e^{-s H_{1,+}(A|E|P_{AE})}
\]

(87)

\[
e^{-s c_{1,-}(f_X(A)|EX|PAE \times PX)} \leq \epsilon^{1-s} + M^{1-s} e^{-s H_{1,+}(A|E|P_{AE})}
\]

(88)

In the other direction with \( s \in [0, 1] \),

\[
e^{-s c_{1,-}(f_X(A)|EX|PAE \times PX)} \geq 2^{-s} \sum_{(a,e): P_{AE}(a|e) \geq \frac{1}{M}} P_{AE}(a,e) P_{A|E}(a|e) e^{-s M} - 2^{-s} \sum_{(a,e): P_{AE}(a|e) < \frac{1}{M}} P_{AE}(a,e) e^{-s} \]

(89)

\[
\geq \frac{1}{2 M^{1-s}} \sum_{e} P_e \left( \sum_{a: P_{AE}(a|e) \geq \frac{1}{M}} P_{A|E}(a|e)^{-1-s} \right)^{1-s} + (2 \epsilon)^{-1-s} \sum_{e} P_e \left( \sum_{a: P_{AE}(a|e) < \frac{1}{M}} P_{A|E}(a|e)^{1-s} \right)^{1-s}
\]

(90)

For the direct parts of the exponents results, we evaluate the following one-shot bound.

Lemma 2. For an ensemble of universal hash functions \( f_X : \mathcal{A} \to \mathcal{M} = \{1, \ldots, M\} \), we have for any \( s \in [0, 1] \),

\[
e^{1-s c_{1,+}(f_X(A)|EX|PAE \times PX)} \leq 1 + \frac{1}{1 + s} M^s e^{-s H_{1,+}(A|E|P_{AE})}
\]

(91)

For the direct parts of the second-order results, we evaluate the following one-shot bound.

Lemma 3. For an ensemble of an \( \epsilon \)-almost universal hash functions \( f_X : \mathcal{A} \to \mathcal{M} = \{1, \ldots, M\} \), we have for any \( s \in [0, 1] \) and \( c > 0 \),

\[
e^{-s c_{1,-}(f_X(A)|EX|PAE \times PX)} \geq P_{AE} \left\{ (a,e) : P_{A|E}(a|e) \leq \frac{c}{M} \right\} \left( \frac{1}{c + \epsilon} \right)^s
\]

(92)

\[
e^{-1-s c_{1,-}(f_X(A)|EX|PAE \times PX)} \geq \left( \frac{1}{c + \epsilon} \right)^{1-s} \sum_{e} P_e \left( \sum_{a: P_{AE}(a|e) \leq \frac{1}{M}} P_{A|E}(a|e)^{1-s} \right)^{1-s}
\]

(93)
B. One-Shot Bounds for the Converse Parts

For the converse parts of the equivocation results, we evaluate the following one-shot bounds.

**Lemma 4.** Fix \( c > 1 \) and \( s \geq 0 \). Any hash function \( f : \mathcal{A} \to \mathcal{M} = \{1, \ldots, M\} \) satisfies

\[
e^{-sC_{1-s}(f(A)|E|P_{AE})} \leq 2c^{-s} \sum_{e} P_{E}(e) \sum_{a : P_{A|E}(a|e) \geq \frac{c}{M}} P_{A|E}(a|e)^{1-s}M^{-s} + 2 \frac{1}{1-s} s^{1-s} (1-s)P_{AE}\left\{ (a, e) : P_{A|E}(a|e) \leq \frac{c}{M} \right\}
\]

(94)

For the Gallager-type counterpart,

\[
e^{-\frac{1}{1-s}C_{1-s}(f(A)|E|P_{AE})} \leq 2c^{-s} \sum_{e} P_{E}(e) \left( \sum_{a : P_{A|E}(a|e) \geq \frac{c}{M}} P_{A|E}(a|e)^{1-s}M^{-s} \right)^{\frac{1}{1-s}} + \left( 2 \frac{1}{1-s} s^{1-s} (1-s)P_{A|E=e}\left\{ a : P_{A|E}(a|e) < \frac{c}{M} \right\} \right)^{\frac{1}{1-s}}
\]

(95)

For the converse parts of the exponents results, we evaluate the following one-shot bounds.

**Lemma 5.** Fix \( c \geq 1 \) and \( s \in (-\infty, 1] \). Any hash function \( f : \mathcal{A} \to \mathcal{M} = \{1, \ldots, M\} \) satisfies

\[
e^{-sC_{1-s}(f(A)|E|P_{AE})} \leq \sum_{(a,e) : P_{A|E}(a|e) \geq \frac{c}{M}} P_{E}(e)P_{A|E}(a|e)^{1-s}M^{-s} + \sum_{e} P_{E}(e)P_{A|E=e}\left\{ a : P_{A|E}(a|e) < \frac{c}{M} \right\}^{1-s}
\]

(96)

\[
\leq P_{A,E}\left\{ (a, e) : P_{A|E}(a|e) \geq \frac{c}{M} \right\}c^{-s} + P_{A,E}\left\{ (a, e) : P_{A|E}(a|e) < \frac{c}{M} \right\}^{1-s}.
\]

(97)

For the Gallager-type counterpart, for \( s \in [-1, 1) \),

\[
e^{-\frac{1}{1-s}C_{1-s}(f(A)|E|P_{AE})} \leq \sum_{e} P_{E}(e) \left[ P_{A|E=e}\left\{ a : P_{A|E}(a|e) \geq \frac{c}{M} \right\}c^{-s} + P_{A|E=e}\left\{ a : P_{A|E}(a|e) < \frac{c}{M} \right\}^{1-s} \right]^{\frac{1}{1-s}}
\]

(98)

For the converse parts of the second-order results, we evaluate the following one-shot bounds.

**Lemma 6.** Fix \( c > 0 \) and \( s \in [0, \infty) \). Any hash function \( f : \mathcal{A} \to \mathcal{M} = \{1, \ldots, M\} \) satisfies

\[
e^{-sC_{1-s}(f(A)|E|P_{AE})} \leq 2c^{-s}P_{AE}\left\{ (a, e) : P_{A|E}(a|e) \geq \frac{c}{M} \right\} + 2 \frac{1}{1-s} s^{1-s}P_{AE}\left\{ (a, e) : P_{A|E}(a|e) \leq \frac{c}{M} \right\}
\]

(99)

In the other direction, we have

\[
e^{sC_{1+s}(f(A)|E|P_{AE})} \geq \sum_{(a,e) : P_{A|E}(a|e) \geq \frac{c}{M}} P_{E}(e)P_{A|E}(a|e)^{1+s}M^{s} + \sum_{e} P_{E}(e)P_{A|E=e}\left\{ (a, e) : P_{A|E}(a|e) < \frac{c}{M} \right\}^{1+s}
\]

(100)

\[
\geq P_{AE}\left\{ (a, e) : P_{A|E}(a|e) \geq \frac{c}{M} \right\}c^{s} + P_{AE}\left\{ (a, e) : P_{A|E}(a|e) < \frac{c}{M} \right\}^{1+s}.
\]

(101)
For the Gallager-type counterpart, we have

\[
e^{-\frac{s}{1-s}C_{1-s}^{i}(f(A)|E|P_{AE})} \leq \sum_{e} P_{E}(e) \left[ \sum_{a: P_{A|E}(a|e) \geq \frac{c}{M}} P_{A|E}(a|e)^{1-s} M^{-s} \right. \\
+ P_{A|E=e}\left\{ (a, e) : P_{AE}(a|e) < \frac{c}{M} \right\} \left. \right]^\frac{1}{1-s}, \tag{102} \]

\[
e^{-\frac{s}{1-s}C_{1-s}^{i}(f(A)|E|P_{AE})} \leq 2^{\frac{s}{1-s}} \sum_{e} P_{E}(e) \left[ 2e^{-s}P_{A|E=e}\left\{ a : P_{A|E}(a|e) \geq \frac{c}{M} \right\} \right]^\frac{1}{1-s} \\
+ \left( 2^{\frac{s}{1-s}} s^{\frac{1}{1-s}} (1-s)P_{A|E=e}\left\{ a : P_{A|E}(a|e) < \frac{c}{M} \right\} \right)^\frac{1}{1-s}. \tag{103} \]

VII. PROOFS OF THE ASYMPTOTIC RESULTS

In this section, we prove the asymptotic results in Theorems 1, 2, and 3.

**Notation:** Throughout, we let \( a^{n} = (a_{1}, a_{2}, \ldots, a_{n}) \in A^{n} \) and \( e^{n} = (e_{1}, e_{2}, \ldots, e_{n}) \in E^{n} \) denote deterministic length-\( n \) strings. We also let \( A^{n} = (A_{1}, A_{2}, \ldots, A_{n}) \) and \( E^{n} = (E_{1}, E_{2}, \ldots, E_{n}) \) denote random vectors of length \( n \). Finally, we adopt the exponential equality notation: \( a_{n} \dot{=} b_{n} \) if and only if \( \frac{1}{n} \log \frac{a_{n}}{b_{n}} \to 0 \).

**A. Proof of Theorem 1**

1) Direct Parts: **Proof:** We first prove the direct parts (upper bounds). The bound in \((87)\) and Jensen’s inequality (for the concave function \( t \mapsto \log t \)) imply that

\[
C_{1+s}(f_{X_{n}}(A^{n})|E^{n}X_{n}|P_{AE}^{n} \times P_{X_{n}}^{n})
\]

\[
= E_{X_{n}} \left[ \frac{1}{s} \log \left[ e^{-sH_{1+s}(f_{X_{n}}(A^{n})|E^{n}|P_{AE}^{n})} M_{n}^{s} \right] \right] \tag{104} \]

\[
\leq \frac{1}{s} \log E_{X_{n}} \left[ e^{-sH_{1+s}(f_{X_{n}}(A^{n})|E^{n}|P_{AE}^{n})} M_{n}^{s} \right] \tag{105} \]

\[
\leq \frac{1}{s} \log \left( e^{s} + M_{n}^{s} e^{-sH_{1+s}(A^{n}|E^{n}|P_{AE}^{n})} \right) \tag{106} \]

\[
= \frac{1}{s} \log \left( e^{s} + M_{n}^{s} e^{-sH_{1+s}(A|E|P_{AE})} \right) \tag{107} \]

For \( \epsilon \) being a constant, this achieves the upper bound of \((34)\) upon normalizing by \( n \) and taking the \( \lim \sup \).

Similarly, using Jensen’s inequality and \((88)\), we obtain

\[
C_{1+s}(f_{X_{n}}(A^{n})|E^{n}X_{n}|P_{AE}^{n} \times P_{X_{n}}^{n})
\]

\[
= E_{X_{n}} \left[ \frac{1 + s}{s} \log \left[ e^{-\frac{s}{1+s}H_{1+s}(f_{X_{n}}(A^{n})|E^{n}|P_{AE}^{n})} M_{n}^{\frac{1}{1+s}} \right] \right] \tag{108} \]

\[
\leq \frac{1 + s}{s} \log E_{X_{n}} \left[ e^{-\frac{s}{1+s}H_{1+s}(f_{X_{n}}(A^{n})|E^{n}|P_{AE}^{n})} M_{n}^{\frac{1}{1+s}} \right] \tag{109} \]

\[
\leq \frac{1 + s}{s} \log \left( e^{\frac{s}{1+s}} + M_{n}^{\frac{1}{1+s}} e^{-\frac{s}{1+s}H_{1+s}(A^{n}|E^{n}|P_{AE}^{n})} \right) \tag{1010} \]

\[
= \frac{1 + s}{s} \log \left( e^{\frac{s}{1+s}} + M_{n}^{\frac{1}{1+s}} e^{-sH_{1+s}(A|E|P_{AE})} \right) \tag{111} \]

This leads to the upper bound of \((35)\) for constant \( \epsilon \) upon normalizing by \( n \) and taking the \( \lim \sup \).

To obtain \((36)\), we employ Cramer’s theorem \([39]\) on the sequence of random variables \( -\log P_{A|E}^{n}(A^{n}|E^{n}) = \sum_{i=1}^{n} -\log P_{A|E}(A_{i}|E_{i}) \) under the product joint distribution \( P_{AE}^{n} \). It is easy to see by using exponential tail bounds
that
\[
\lim_{n \to \infty} - \frac{1}{n} \log \sum_{(a^n, e^n) : P_{AE}(a^n, e^n) \leq e^{-Rn}} P_{AE}^n(a^n, e^n) = \lim_{n \to \infty} - \frac{1}{n} \log \Pr \left( \frac{1}{n} \sum_{i=1}^{n} \log P_{A|E}(A_i|E_i) \leq -R \right) = \max_{t \geq 0} t(R - H_{1-t}(A|E|P_{AE})).
\]
(113)

Note that the cumulant generating function of the random variable $-\log P_{A|E}(A|E)$ under the joint distribution $P_{AE}$ can be expressed in terms of the conditional Rényi entropy as
\[
\tau(t) := \log \mathbb{E}_{P_{AE}} \left[ e^{t(-\log P_{A|E}(A|E))} \right] = tH_{1-t}(A|E|P_{AE}),
\]
(114)
explaining the presence of this term in (113). We again apply (a generalized version of) Cramer’s theorem to the sequence of random variables $\log P_{A|E}(A^n|E^n)$ under the sub-distribution (non-negative product measure) $P_{AE}^n(a^n, e^n)(P_{-s} A|E)^n(a^n|e^n)$ and event \{(a^n, e^n) : \log P_{A|E}(a^n|e^n) < -nR\}. Note that the cumulant generating function in this case is
\[
\tau_s(t) := \log \sum_{a, e} P_{AE}(a, e)P_{-s} A|E(a|e) \exp \left( t \log P_{A|E}(a|e) \right)
\]
(115)
and we also have that
\[
\tau'_s(0) = -\hat{R}_s
\]
(116)
where \(\hat{R}_s\) is defined in (38). Thus,
\[
\lim_{n \to \infty} - \frac{1}{n} \log \sum_{(a^n, e^n) : P_{A|E}(a^n|e^n) \geq e^{-Rn}} P_{AE}^n(a^n, e^n) P_{A|E}(a^n|e^n)^{-s} = \max_{t \geq 0} \left\{ -tR - (s-t)H_{1-(s-t)}(A|E|P_{AE}) \right\}.
\]
(118)
For the case where $R \geq \hat{R}_s$, the constraint in the optimization above is active, i.e., $t^* = 0$. Conversely, when $R \leq \hat{R}_s$, the constraint is inactive. Thus, we obtain
\[
\lim_{n \to \infty} - \frac{1}{n} \log \sum_{(a^n, e^n) : P_{A|E}(a^n|e^n) \geq e^{-Rn}} P_{AE}^n(a^n, e^n) P_{A|E}(a^n|e^n)^{-s} e^{-snR} = \begin{cases} 
\max_{t \in [0,1]} t \left( R - H_{1-t}(A|E|P_{AE}) \right) & \text{if } R \geq \hat{R}_s \\
0 & \text{if } R \leq \hat{R}_s 
\end{cases}
\]
(119)
where the second clause follows by the identification $t' = s - t$. Now with these preparations, we can employ the one-shot bound in (89) with $\epsilon = 1$ to prove the direct part of (36) as follows: Since (119) is not greater than (113), the former dominates in the exponent and we obtain
\[
\limsup_{n \to \infty} \frac{1}{n} C_{1-s}(f_{X_n}(A^n)|E^n X_n|P_{AE}^n \times P_{X_n}) = -1 \liminf_{n \to \infty} \frac{1}{n} \log \left[ e^{-sC_{1-s}(f_{X_n}(A^n)|E^n X_n|P_{AE}^n \times P_{X_n})} \right] \leq -1 \liminf_{n \to \infty} \frac{1}{n} \log \left[ 2^{-s} \sum_{(a^n, e^n) : P_{A|E}(a^n|e^n) \leq e^{-Rn}} P_{AE}^n(a^n, e^n) \right. \\
+ 2^{-s} \sum_{(a^n, e^n) : P_{A|E}(a^n|e^n) \geq e^{-Rn}} P_{AE}^n(a^n, e^n) P_{A|E}(a^n|e^n)^{-s} e^{-snR} \right],
\]
(121)
where (121) follows from (89). Now assembling (113) and (119) in (121), we obtain the upper bound to (36).
The upper bound of (37) proceeds in an analogous manner but we need to instead employ the Gärtner-Ellis theorem [39] (instead of Cramer’s theorem) and evaluate (90). Doing so to the sequence of random variables

$$- \log P^n_{A|E}(A^n|e^n) = \sum_{i=1}^n - \log P_{A|E}(A_i|e_i)$$

with $e^n$ of fixed type [14] and $A^n$ with the memoryless distribution $P^n_{A|E}(\cdot|e^n)$, we obtain

$$\lim_{n \to \infty} - \frac{1}{n} \log \sum_{e^n} P^n_{E}(e^n) \left( \sum_{a^n : P^n_{A|E}(a^n|e^n) < e^{-nR}} P^n_{A|E}(a^n|e^n) \right)^{\frac{1}{1-s}} = \max_{t \geq 0} \frac{t}{1-s} \left( R - H_{1-t|1-s}(A|E|P_{AE}) \right)$$

where $H_{1-t|1-s}(A|E|P_{AE})$ is the two-parameter conditional Rényi entropy defined in (20). To show (122) in more detail consider $e^n \in \mathcal{T}_Q = \{e^n \in \mathcal{E}^n : \text{type}(e^n) = Q\}$. Then we have that

$$P_{A^n|E^n=e^n}(\cdot) = \exp \left( -n \max_{t \geq 0} \left[ tR - \mathbb{E}_Q \log \sum_a P_{A|E}^{1-t}(a|E) \right] \right),$$

where

$$\mathbb{E}_Q \log \sum_a P_{A|E}^{1-t}(a|E) = \sum_e Q(e) \log \sum_a P_{A|E}^{1-t}(a|e).$$

Let $\mathcal{P}_n(\mathcal{E})$ be the set of $n$-types with alphabet $\mathcal{E}$. Splitting the sum on the left-hand-side in (122) into the polynomially many $n$-types on $\mathcal{E}$, we obtain

$$\sum_{e^n \in \mathcal{E}^n} P^n_{E}(e^n) \left( \sum_{a^n : P^n_{A|E}(a^n|e^n) < e^{-nR}} P^n_{A|E}(a^n|e^n) \right)^{\frac{1}{1-s}} = \max_{Q \in \mathcal{P}_n(\mathcal{E})} \left[ tR - \mathbb{E}_Q \log \sum_a P_{A|E}^{1-t}(a|E) \right]$$

(125) follows from the fact that $P^n_{E}(\mathcal{T}_Q) = \exp(-nD(Q||P_E))$ [14, Ch. 2], the swapping of min and max in (128) follows from the fact that the objective function is convex in $Q$ and concave in $t$. Now by straightforward calculus, the optimizing distribution for fixed $t$ is

$$Q^*(e) = \frac{P_E(e) (\sum_a P_{A|E}^{1-t}(a|e))^{\frac{1}{1-s}}}{Z}$$

(129) where the normalizing constant (partition function)

$$Z := \sum_e P_E(e) (\sum_a P_{A|E}^{1-t}(a|e))^{\frac{1}{1-s}}.$$

(130) Plugging this into (128) we obtain

$$\sum_{e^n \in \mathcal{E}^n} P^n_{E}(e^n) \left( \sum_{a^n : P^n_{A|E}(a^n|e^n) < e^{-nR}} P^n_{A|E}(a^n|e^n) \right)^{\frac{1}{1-s}} = \exp \left( -n \max_{t \geq 0} \left[ tR - \frac{1-s}{t} Z \right] \right)$$

(131) which then yields (122). Note that we have to use the Gärtner-Ellis theorem (and not Cramer’s theorem) because the collection of random variables $- \log P_{A|E}(A_i|e_i), i = 1, \ldots, n$ is independent but not identically distributed.
Again, by applying the Gärtner-Ellis theorem [39] to the sequence of random variables $\log P_{A|E}^{n}(A^n|e^n) = \sum_{i=1}^{n} \log P_{A|E}(A_i|e_i)$ with non-negative measure $P_{A|E}^{n}(\cdot|e^n)^{1-s}$, we have

$$\sum_{a^n: P_{A|E}(a^n|e^n) \geq e^{-nR}} P_{A|E}(a^n|e^n)^{1-s} = \exp \left( -n \max_{t \geq 0} \left[ -tR - \mathbb{E}_Q \log \sum_{a} P_{A|E}^{1-s-t}(a|E) \right] \right)$$

(132)

where $Q$ is the type of $e^n$ and $\mathbb{E}_Q \log \sum_{a} P_{A|E}^{1-s-t}(a|E)$ is defined in (124). So by using a type partitioning argument of sequences $e^n$ similarly to (125)–(131), we obtain

$$\sum_{e^n} P_{E}(e^n) \left( \sum_{a^n: P_{A|E}(a^n|e^n) \geq e^{-nR}} P_{A|E}(a^n|e^n)^{1-s} \right)^{\frac{1}{n}} \leq \exp \left( -n \max_{t \geq 0} \left[ -\frac{tR}{1-s} + \log \sum_{e} P_{E}(e) \left( \sum_{a} P_{A|E}^{1-s-t}(a|e) \right)^{\frac{1}{n}} \right] \right).$$

(133)

Consequently, considering the two different cases similarly to (119), we obtain

$$\lim_{n \to \infty} \frac{1}{n} \log \left[ e^{-\frac{t}{1-s}nR} \sum_{e^n} P_{E}(e^n) \left( \sum_{a^n: P_{A|E}(a^n|e^n) \geq e^{-nR}} P_{A|E}(a^n|e^n)^{1-s} \right)^{\frac{1}{n}} \right]$$

$$= \begin{cases} \frac{s}{1-s} (R - H_{1-s}(A|E|P_{AE})) & \text{if } R \geq \frac{d}{dt} tH_{1-t|1-s}(A|E|P_{AE}) |_{t=s} \\ \max_{t \in [0,1]} \frac{s}{1-s} (R - H_{1-t|1-s}(A|E|P_{AE})) & \text{if } R \leq \frac{d}{dt} tH_{1-t|1-s}(A|E|P_{AE}) |_{t=s} \end{cases}.$$ 

(134)

By inspecting (134) and (122), we observe that the former is not greater than the latter. Thus, the latter dominates the exponential behavior and using (90) with $\epsilon = 1$, we obtain

$$\limsup_{n \to \infty} \frac{1}{n} \log \left[ \frac{1}{n} \sum_{e^n} P_{E}(e^n) \left( \sum_{a^n: P_{A|E}(a^n|e^n) \geq e^{-nR}} P_{A|E}(a^n|e^n)^{1-s} \right)^{\frac{1}{n}} \right]$$

$$= \begin{cases} \frac{s}{1-s} (R - H_{1-s}(A|E|P_{AE})) & \text{if } R \geq \frac{d}{dt} tH_{1-t|1-s}(A|E|P_{AE}) |_{t=s} \\ \max_{t \in [0,1]} \frac{s}{1-s} (R - H_{1-t|1-s}(A|E|P_{AE})) & \text{if } R \leq \frac{d}{dt} tH_{1-t|1-s}(A|E|P_{AE}) |_{t=s} \end{cases}.$$ 

(135)

(136)

Now note that

$$\frac{d}{dt} tH_{1-t|1-s}(A|E|P_{AE}) |_{t=s} = \frac{d}{dt} tH_{1-t}(A|E|P_{AE}) |_{t=s} = \hat{R}_s^t$$

(138)

due to continuity and the equalities in (21) and (39). This completes the justification of the upper bound of (37).

2) Converse Parts: Proof: For the converse, we do not consider the common randomness $X_n$ (i.e., $X_n = \emptyset$) since the bound must hold for all (not just $\epsilon$-almost universal) hash functions $f_{X_n}$. This statement applies to the proofs of all converse bounds in the sequel.

The lower bounds to (34) and (35) can be easily obtained by using the data processing inequalities for Rényi conditional entropies and their Gallager-type counterparts in (22)–(24).

Now for (36), we note that when $\hat{R} \geq \hat{R}_s$, we have

$$s(R - H_{1-s}(A|E|P_{AE}) \leq \max_{t \in [0,1]} t(R - H_{1-t}(A|E|P_{AE}))$$

(139)
and when $R \leq \hat{R}_s$, the opposite inequality holds. So by rearranging (94), we obtain the bound
\[
e^{-sC_{1-s}(f(A^n)|E^n|P_{AE}\times P_{X_n})} \leq 2e^{-s} \sum_{e^n} P_{E^n}(e^n) \sum_{a^n:P_{AE}(a^n|e^n) \geq \frac{c}{M_n}} P_{A|E}^n(a^n|e^n)^{1-s} M_n^{-s} + 2 \left(2^{s(1-s)-1}(1-s)P_{A|E}^n(a^n|e^n) \left\{a^n : P_{A|E}^n(a^n|e^n) < \frac{c}{M_n}\right\}\right)^{\frac{1}{1-s}}
\]
\[
\leq 2e^{-s} e^{\hat{H}_{1-s}(A^n|E^n|P_{AE})} M_n^{-\frac{s}{1-s}} + 2s(1-s)^{\frac{1}{1-s}} \sum_{e^n} P_{E^n}(e^n) e^{\frac{1}{1-s} \hat{H}_{1-s}(A^n|E^n|P_{AE})} \left(\frac{M_n}{c}\right)^{-\frac{1}{1-s}}
\]
\[
= 2e^{-s} e^{\frac{1}{1-s} \hat{H}_{1-s}(A^n|E^n|P_{AE})} M_n^{-\frac{s}{1-s}} + 2s(1-s)^{\frac{1}{1-s}} e^{\frac{1}{1-s} \hat{H}_{1-s}(A^n|E^n|P_{AE})} \left(\frac{M_n}{c}\right)^{-\frac{1}{1-s}}
\]
with $s \geq t \geq 0$. This implies (37) upon taking the logarithm, normalizing by $n$ and taking the limit. The two cases of (37) arise from the two terms in the square parentheses above.

B. Proof of Theorem 2

1) Direct Parts: Proof: For the direct part, we employ (87) with $\epsilon = 1$ and $t \in [s, 1]$. We recall that $M_n = e^n R$. Now we have
\[
C_{1+s}(f_{X_n}(A^n)|E^n X_n|P_{AE}\times P_{X_n}) \leq C_{1+t}(f_{X_n}(A^n)|E^n X_n|P_{AE}\times P_{X_n}) \leq \frac{1}{t} \log \left(1 + M_t e^{-tH_{1+t}(A^n|E^n|P_{AE})}\right) \leq \frac{1}{t} M_t^t e^{-ntH_{1+t}(A|E|P_{AE})}.
\]
Taking the logarithm and optimizing over $t \in [s, 1]$, we obtain the lower bound to (43). Similarly, applying (91) to the case $t \in [s, 1]$, we obtain
\[
C_{1+s}(f_{X_n}(A^n)|E^n X_n|P_{AE}\times P_{X_n}) \leq C_{1+t}(f_{X_n}(A^n)|E^n X_n|P_{AE}\times P_{X_n}) \leq \frac{1}{1+t} \log \left(1 + \frac{1}{1+t} M_t^t e^{-tH_{1+t}(A^n|E^n|P_{AE})}\right) \leq \frac{1}{t} M_t^t e^{-ntH_{1+t}(A|E|P_{AE})}.
\]
which implies the lower bound to (46) upon optimizing over $t \in [s, 1]$.

For the $-s$ versions in (44) and (47), we simply note that
\[
C_{1+s} \geq C_{1-s} \quad \text{and} \quad C_{1+s}^\dagger \geq C_{1-s}^\dagger.
\]
for any $s, s' \in [0, 1]$ because as mentioned in Section II-A (after (13) and (18) respectively), $H_{1+s}$ and $H^1_{1+s}$ are monotonically decreasing in $s$. Thus, we have

\[
\liminf_{n \to \infty} \frac{-1}{n} \log C_{1-s'}(f_{X_n}(A^n)|E^n X_n|P^n AE \times P_{X_n}) \geq \liminf_{n \to \infty} \frac{-1}{n} \log C_{1-s}(f_{X_n}(A^n)|E^n X_n|P^n AE \times P_{X_n}) \geq \liminf_{n \to \infty} \frac{-1}{n} \log C_{1-s'}(f_{X_n}(A^n)|E^n X_n|P^n AE \times P_{X_n}) \geq \liminf_{n \to \infty} \frac{-1}{n} \log C_{1-s}(f_{X_n}(A^n)|E^n X_n|P^n AE \times P_{X_n}).
\] (153)

\[
\liminf_{n \to \infty} \frac{-1}{n} \log C_{1-s'}^t(f_{X_n}(A^n)|E^n X_n|P^n AE \times P_{X_n}) \geq \liminf_{n \to \infty} \frac{-1}{n} \log C_{1-s}(f_{X_n}(A^n)|E^n X_n|P^n AE \times P_{X_n}).
\] (154)

Combining these statements with the bounds derived in (145)–(150) completes the proof of (44) and (47).

2) Converse Parts: Proof: For the converse, we first show the upper bound to (43). Choose a constant $c_0$ satisfying $c_0 > 1 + s$. Let

\[
T_s := \frac{d}{ds} s H_{1+s}(A|E|P_{AE}).
\] (155)

Now assume that $R \leq T_s$. We claim that

\[
\liminf_{n \to \infty} \frac{-1}{n} \log \left[ P^n_{AE}\left\{ (a^n, e^n) : P^n_{AE}(a^n|e^n) \geq c_0 e^{-nT_s} \right\} c_0^s e^{-sT_s} e^{snR} \right] = s H_{1+s}(A|E|P_{AE}) - sR.
\] (156)

This is justified as follows. We know from Cramer’s theorem [39] that

\[
\Pr \left( P^n_{AE}(A^n|E^n) \geq c_0 e^{-nT_s} \right) = \exp \left( -n \max_{t \geq 0} \{ tH_{1+t}(A|E|P_{AE}) - tT_s \} \right).
\] (157)

Differentiating the objective function with respect to $t \geq 0$ yields

\[
\frac{d}{dt} tH_{1+t}(A|E|P_{AE}) = T_s
\] (158)

so the $t$ that satisfies the above equation is $t = s$ by the definition of $T_s$ in (155). As a result,

\[
\Pr \left( P^n_{AE}(A^n|E^n) \geq c_0 e^{-nT_s} \right) = \exp \left( -n \left\{ s H_{1+s}(A|E|P_{AE}) - sT_s \right\} \right).
\] (159)

Plugging this into the left-hand-side of (156) yields the claim. The one-shot bound in (97) (replacing $-s$ by $+s$) implies that

\[
e^{sC_{1+s}(f(A^n)|E^n|P^n_{AE})} \geq P^n_{AE}\left\{ (a^n, e^n) : P^n_{AE}(a^n|e^n) \geq c_0 e^{-nT_s} \right\} c_0^s e^{-sT_s} e^{snR}
\]

\[
+ 1 - (1 + s) P^n_{AE}\left\{ (a^n, e^n) : P^n_{AE}(a^n|e^n) \geq c_0 e^{-nT_s} \right\} = 1 + \left( c_0^s e^{-snT_s} e^{snR} - 1 - s \right) P^n_{AE}\left\{ (a^n, e^n) : P^n_{AE}(a^n|e^n) \geq c_0 e^{-nT_s} \right\}
\]

(160)

\[
1 + \left( c_0^s e^{-snT_s} e^{snR} - 1 - s \right) P^n_{AE}\left\{ (a^n, e^n) : P^n_{AE}(a^n|e^n) \geq c_0 e^{-nT_s} \right\}
\]

(161)

Hence, taking the logarithm of (161), normalizing by $s$ and using (156), we obtain the upper bound to (43). For the other case $R \geq T_s$, we have

\[
\max_{t \geq s} \{ tH_{1+t}(A|E|P_{AE}) - tr \} = \max_{t \geq 0} \{ tH_{1+t}(A|E|P_{AE}) - tr \}
\] (162)

This is also reflected in Fig. 4. Thus, (97) implies that

\[
e^{sC_{1+s}(f(A^n)|E^n|P^n_{AE})} \geq P^n_{AE}\left\{ (a^n, e^n) : P^n_{AE}(a^n|e^n) \geq c_0 e^{-nR} \right\} c_0^s
\]

\[
+ \left( 1 - P^n_{AE}\left\{ (a^n, e^n) : P^n_{AE}(a^n|e^n) \geq c_0 e^{-nsR} \right\} \right)^{1+s}
\]

\[
\geq 1 + \left( c_0^s - 1 - s \right) P^n_{AE}\left\{ (a^n, e^n) : P^n_{AE}(a^n|e^n) \geq c_0 e^{-nR} \right\}
\]

(163)

(164)
Hence,

\[ C_{1+s}(f(A^n)|E^n|P_{AE}^n) \]
\[ = \frac{1}{s} \log (e^{sC_{1+s}(f(A^n)|E^n|P_{AE}^n)}) \]
\[ \geq \frac{1}{s} \log \left( 1 + \left( e^s - 1 - s \right) P_{AE}^n \left\{ (a^n, e^n) : P_{AE}^n(a^n|e^n) \geq c_0 e^{-nT_s} \right\} \right) \]
\[ = \frac{e^s - 1 - s}{s} P_{AE}^n \left\{ (a^n, e^n) : P_{AE}^n(a^n|e^n) \geq c_0 e^{-nT_s} \right\} + o(1) \]

(165) (166) (167)

By combining (156) and (162) we see that for \( R \geq T_s \), we also obtain the upper bound to (43).

The proof of the upper bound to (46) is similar and we present the details here. Again choose the constant \( c_0 \) such that \( c_0^s > 1 + s \). Assume that \( R \leq T_s \). Then the one-shot bound in (98) (replacing \(-s\) by \( +s\)) implies that

\[ e^{1+s} C_{1+s}(f(A^n)|E^n|P_{AE}^n) \]
\[ \geq \sum_{e^n} P_E^n(e^n) \left( P_{A^n|E^n=e^n} \left\{ a^n : P_{A|E}^n(a^n|e^n) \geq c_0 e^{-nT_s} \right\} e_0^s e^{-snT_s} e^{snR} \right. \]
\[ + \left( 1 - P_{A^n|E^n=e^n} \left\{ a^n : P_{A|E}^n(a^n|e^n) \geq c_0 e^{-nT_s} \right\} \right) \]
\[ \geq \sum_{e^n} P_{E}^n(e^n) \left( P_{A^n|E^n=e^n} \left\{ a^n : P_{A|E}^n(a^n|e^n) \geq c_0 e^{-nT_s} \right\} e_0^s e^{-snT_s} e^{snR} \right. \]
\[ + \left( 1 - (1 + s) P_{A^n|E^n=e^n} \left\{ a^n : P_{A|E}^n(a^n|e^n) \geq c_0 e^{-nT_s} \right\} \right) \]
\[ = \sum_{e^n} P_E^n(e^n) \left( 1 + \left( e_0^s e^{-snT_s} e^{snR} - (1 + s) \right) P_{A^n|E^n=e^n} \left\{ a^n : P_{A|E}^n(a^n|e^n) \geq c_0 e^{-nT_s} \right\} \right) \]
\[ = \sum_{e^n} P_{E}^n(e^n) \left( 1 + \frac{e_0^s e^{-snT_s} e^{snR} - (1 + s)}{1 + s} P_{A^n|E^n=e^n} \left\{ a^n : P_{A|E}^n(a^n|e^n) \geq c_0 e^{-nT_s} \right\} \right) \]
\[ = 1 + \frac{e_0^s e^{-snT_s} e^{snR} - (1 + s)}{1 + s} P_{E}^n(e^n) P_{A^n|E^n=e^n} \left\{ a^n : P_{A|E}^n(a^n|e^n) \geq c_0 e^{-nT_s} \right\} , \]

(168) (169) (170) (171) (172)

where (171) follows from the fact that \( \log[(1 + a)^t] = t[a + O(a^2)] \) for \( a \downarrow 0 \). Hence,

\[ C_{1+s}^+(f(A^n)|E^n|P_{AE}^n) \]
\[ = \frac{1}{s} \log \left( \frac{1}{e^{1+s} C_{1+s}(f(A^n)|E^n|P_{AE}^n) + P_{AE}^n \times P_{X^n}^n} \right) \]
\[ \geq \frac{1}{s} \log \left( 1 + \left( e_0^s e^{-snT_s} e^{snR} - (1 + s) \right) \sum_{e^n} P_E^n(e^n) P_{A^n|E^n=e^n} \left\{ a^n : P_{A|E}^n(a^n|e^n) \geq c_0 e^{-nT_s} \right\} \right) \]
\[ = \frac{e_0^s e^{-snT_s} e^{snR} - (1 + s)}{s} \sum_{e^n} P_{E}^n(e^n) P_{A^n|E^n=e^n} \left\{ a^n : P_{A|E}^n(a^n|e^n) \geq c_0 e^{-nT_s} \right\} , \]
\[ = \frac{e_0^s e^{-snT_s} e^{snR} - (1 + s)}{s} P_{AE}^n \left\{ (a^n, e^n) : P_{A|E}^n(a^n|e^n) \geq c_0 e^{-nT_s} \right\} , \]

(173) (174) (175) (176)

where (176) follows from \( \log(1 + a) = a + O(a^2) \) as \( a \downarrow 0 \) and the fact that the summation in (174) vanishes as \( n \) grows. Combining (156) and (176) yields the upper bound to (46) for \( R \leq T_s \). A similar calculation for the case \( R \geq T_s \) also yields the the same upper bound to (46).

We choose the constant \( c \) such that \( (1 - s) > c^{-s} \). We apply Cramer’s Theorem [39] to the sequence of random variables \( \log P_{A|E}^n(A^n|E^n) \). Then,

\[ \lim_{n \to \infty} -\frac{1}{n} \log P_{AE}^n \left\{ (a^n, e^n) : P_{A|E}^n(a^n|e^n) \geq \frac{c}{e^{nR}} \right\} = \max_{t \geq 0} \{ t H_{1+t}(A|E|P_{AE}) - tR \} . \]

(177)
The one-shot bound in (94) implies that

\[ e^{-sC_{1-s}(f(A^n)|E^n|P_{A|E}^n)} \]

\[ \leq P_{A|E}^n \left\{ (a^n, e^n) : P_{A|E}^n(a^n|e^n) \geq \frac{c}{M} \right\} c^{-s} + P_{A|E}^n \left\{ (a^n, e^n) : P_{A|E}^n(a^n|e^n) < \frac{c}{M} \right\} \left(1 - \frac{c}{M} \right)^{1-s} \]

\[ \leq 1 - ((1-s) - c^{-s}) \left(1 - \frac{c}{M} \right) \sum_{e^n} P_{A^n|E^n=e^n}(a^n : P_{A|E}^n(a^n|e^n) \geq \frac{c}{M}) \]  

because \((1-x)^{1-s} \leq 1 - (1-s)x\).

Thus,

\[ C_{1-s}(f(A^n)|E^n|P_{A|E}^n) \]

\[ = -\frac{1}{s} \log \left[ e^{-sC_{1-s}(f(A^n)|E^n|P_{A|E}^n)} \right] \]

\[ \geq -\frac{1}{s} \log \left\{ (a^n, e^n) : P_{A|E}^n(a^n|e^n) \geq \frac{c}{M} \right\} \]  

\[ \geq \frac{1}{s} \log \left\{ (a^n, e^n) : P_{A|E}^n(a^n|e^n) \geq \frac{c}{M} \right\} . \]

Combining (177) and (183), we have the upper bound to (44).

When

\[ \sum_{e^n} P_{A^n|E^n=e^n}(a^n : P_{A|E}^n(a^n|e^n) \geq \frac{c}{M}) \]

is exponentially small, (98) implies that

\[ e^{-\frac{1}{s}C_{1-s}(f(A^n)|E^n|P_{A|E}^n)} \]

\[ \leq \sum_{e^n} P_{A^n|E^n=e^n}(a^n : P_{A|E}^n(a^n|e^n) \geq \frac{c}{M}) c^{-s} \]

\[ + \sum_{e^n} P_{A^n|E^n=e^n}(a^n : P_{A|E}^n(a^n|e^n) < \frac{c}{M}) \left(1 - \frac{c}{M} \right)^{1-s} \]

\[ = \sum_{e^n} P_{A^n|E^n=e^n}(c^{-s} \sum_{a^n} P_{A^n|E^n=e^n}(a^n : P_{A|E}^n(a^n|e^n) \geq \frac{c}{M}) \]

\[ + \left(1 - \frac{c}{M} \right) \sum_{a^n} P_{A^n|E^n=e^n}(a^n : P_{A|E}^n(a^n|e^n) \geq \frac{c}{M}) \right)^{1-s} \]

\[ \leq \sum_{e^n} P_{A^n|E^n=e^n}(1 - ((1-s) - c^{-s}) P_{A^n|E^n=e^n}(a^n : P_{A|E}^n(a^n|e^n) \geq \frac{c}{M}) \]  

\[ \approx \sum_{e^n} P_{A^n|E^n=e^n}(1 - \frac{1-s - c^{-s}}{1-s} P_{A^n|E^n=e^n}(a^n : P_{A|E}^n(a^n|e^n) \geq \frac{c}{M}) \]

\[ = 1 - \frac{1-s - c^{-s}}{1-s} \sum_{e^n} P_{A^n|E^n=e^n}(a^n : P_{A|E}^n(a^n|e^n) \geq \frac{c}{M}) , \]
where (188) follows from the same reasoning as (171). Thus,

\[
C_{1-s}(f(A^n)|E^n|P^n_{AE}) = -\frac{1-s}{s} \log \left[ e^{-\frac{1-s}{s} C_{1-s}(f(A^n)|E^n|P^n_{AE})} \right] \tag{190}
\]

\[
\geq -\frac{1-s}{s} \log \left[ 1 - \frac{1-s-c^s}{1-s} \sum_{a^n} P^n_{E}(e^n) P^n_{A^n|E^n=e^n} \left\{ a^n : P^n_{A|E}(a^n|e^n) \geq \frac{c}{M} \right\} \right] \tag{191}
\]

\[
\geq \frac{1-s}{s} \cdot \frac{1-s-c^s}{1-s} \sum_{a^n} P^n_{E}(e^n) P^n_{A^n|E^n=e^n} \left\{ a^n : P^n_{A|E}(a^n|e^n) \geq \frac{c}{M} \right\} \tag{192}
\]

\[
= \frac{1-s-c^s}{s} \sum_{a^n} P^n_{E}(e^n) P^n_{A^n|E^n=e^n} \left\{ a^n : P^n_{A|E}(a^n|e^n) \geq \frac{c}{M} \right\} \tag{193}
\]

Combining (177) and (193), we have the upper bound to (47).

\[\Box\]

C. Proof of Theorem 3

1) Direct Parts: Proof: First, we prove the upper bounds for Case (A) where the Rényi parameter \(\alpha = 1+s\) for \(s \in (0, 1]\). Substituting \(e^{nH_{1+s}(A|E|P_{AE})+\sqrt{nL}}\) into \(M_n\) in the chain of inequalities in (104)–(107), we obtain for the class of \(\epsilon\)-almost universal hash function that

\[
C_{1+s}(f_{X_n}(A^n)|E^n X_n|P^n_{AE} \times P_{X_n}) \leq \frac{1}{s} \log \left( e^s + e^{s\sqrt{nL}} \right) \tag{194}
\]

Set \(\epsilon\) to be a constant (not varying with \(n\)). Normalizing by \(\sqrt{n}\) and taking the \(\lim sup\) as \(n \to \infty\) yields the upper bound to (53).

In an exactly analogous way, the upper bound to (54) can be shown by substituting \(e^{nH_{1+s}(A|E|P_{AE})+\sqrt{nL}}\) into \(M_n\) in the chain of inequalities in (148)–(150).

Substituting \(e^{nH_{1+s}(A|E|P_{AE})+\sqrt{nL}}\) into \(M_n\) in the chain of inequalities in (145)–(147) with \(t = s\), we obtain

\[
C_{1+s}(f_{X_n}(A^n)|E^n X_n|P^n_{AE} \times P_{X_n}) \leq M_n e^{-snH_{1+s}(A|E|P_{AE})} = e^{s\sqrt{nL}} \tag{195}
\]

which implies the upper bound to (55) after we take the logarithm, normalize both sides by \(\sqrt{n}\) and take the \(\lim sup\) as \(n \to \infty\).

In an exactly analogous way, the upper bound to (56) can be shown by substituting \(e^{nH_{1+s}(A|E|P_{AE})+\sqrt{nL}}\) into \(M_n\) in the chain of inequalities in (108)–(111). This completes the proof for the direct part of Case (A) of Theorem 3.

Case (B) follows by partitioning the space \(A^n \times \mathcal{E}^n\) into pairs of sequences of the same joint type [14]. Let \(Q_{AE}\) denote a generic joint type on \(A \times \mathcal{E}\). Let \(U^{(Q_{AE})}\) be the uniform distribution over the type class \(T_{Q_{AE}} \subset A^n \times \mathcal{E}^n\). Let

\[
U_{f(A^n),E^n}^{(Q_{AE})}(i, e^n) := \sum_{a^n : f(a^n) = i} U^{(Q_{AE})}(a^n, e^n) \tag{196}
\]

be the distribution on \(\{1, \ldots, \|f\|\} \times \mathcal{E}^n\) when the hash function \(f\) is applied to the variable \(A^n\) and denote

\[
U_{E^n}^{(Q_{AE})}(e^n) := \sum_{i=1}^{\|f\|} U_{f(A^n),E^n}^{(Q_{AE})}(i, e^n) \tag{197}
\]

as its \(\mathcal{E}^n\)-marginal. Because the probability of pairs of sequences of the same joint type have the same \(P^n_{AE}\) probability, we can write

\[
P_{f(A^n),E^n}(i, e^n) = \sum_{Q_{AE} \in P^n_{A \times \mathcal{E}}} P^n_{AE}(T_{Q_{AE}}) U_{f(A^n),E^n}^{(Q_{AE})}(i, e^n). \tag{198}
\]

By using (198) and the fact that the relative entropy is convex, we have

\[
C_1(f(A^n)|E^n|P^n_{AE}) = D(P_{f(A^n),E^n} \| P_{\text{mix},f(A^n)} \times P_{E^n}) \leq \sum_{Q_{AE} \in P^n_{A \times \mathcal{E}}} P^n_{AE}(T_{Q_{AE}}) D \left( U_{f(A^n),E^n}^{(Q_{AE})} \| P_{\text{mix},f(A^n)} \times U_{E^n}^{(Q_{AE})} \right). \tag{199}
\]
Recall also that $P_{\text{mix}, f(A^n)}$ is the uniform distribution over the alphabet $\{1, \ldots, \|f\|\}$. Now, we analyze the relative entropy term in (200). Let $T_{Q_{AE}}(e^n) := \{a^n : (a^n, e^n) \in T_{Q_{AE}}\}$ be the conditional type class of $Q_{AE}$ given $e^n$, also known as the $Q_{AE}$-shell. By the method of types [14, Ch. 2], we know that for $e^n$ of type $Q_{E}$,

$$\log |T_{Q_{AE}}(e^n)| = nH(A|E|Q_{AE}) + O(\log n).$$

(201)

Furthermore, by Taylor expansion of $H(A|E|Q_{AE})$ around $P_{AE}$ as in the rate redundancy lemma [42], [51], we have

$$H(A|E|Q_{AE}) = H(A|E|P_{AE}) + \sum_{a,e} (Q_{AE}(a,e) - P_{AE}(a,e))h_{A|E}(a|e) + O(\|Q_{AE} - P_{AE}\|^2)$$

(202)

where the conditional entropy density $h_{A|E}(a|e)$ is defined as

$$h_{A|E}(a|e) := \log \frac{1}{P_{A|E}(a|e)}$$

(203)

and $\|Q - P\| = \sum_{z \in Z} |Q(z) - P(z)|$ is the variational distance between $Q$ and $P$. For brevity, we denote the $\sqrt{n}$-scaled version of the second term in (202) as

$$b_n(Q_{AE}) := \sqrt{n} \left( \sum_{a,e} (Q_{AE}(a,e) - P_{AE}(a,e))h_{A|E}(a|e) \right).$$

(204)

Note from the central limit theorem that if $Q_{AE}$ is a random type formed from $n$ independent copies of $P_{AE}$,

$$b_n(Q_{AE}) = \sqrt{n} \left( \frac{1}{n} \sum_{i=1}^{n} h_{A|E}(A_i|E_i) - H(A|E|P_{AE}) \right) \to N(0, V(A|E|P_{AE}))$$

(205)

in distribution. Fix $\delta > 0$. Now upper bounding the relative entropy $D_1$ with the collision relative entropy $D_2$ [5] (recall from Section II-A that $H_{1+s}$ is monotonically decreasing on $(0, \infty)$), applying the universal2 property of the hash function $f$ to the collision relative entropy (see (104)–(107) with $\epsilon = s = 1$ or [5]), and combining the above notations and bounds, we obtain for all $e^n \in T_{Q_{E}}$ and all $n$ large enough (depending on $\delta$) that

$$D \left( U^{(Q_{AE})}_{f(A^n), E^n} \left| P_{\text{mix}, f(A^n)} \times U^{(Q_{AE})}_{E^n} \right. \right) \leq D_2 \left( U^{(Q_{AE})}_{f(A^n), E^n} \left| P_{\text{mix}, f(A^n)} \times U^{(Q_{AE})}_{E^n} \right. \right)$$

$$\quad \leq \log \left( 1 + \frac{M_n}{|T_{Q_{AE}}(e^n)|} \right)$$

$$\quad \leq \log \left( 1 + \exp \left[ \sqrt{n} \left( L - b_n(Q_{AE}) + o(b_n(Q_{AE})) + O(\log n) \right) \right] \right)$$

$$\quad \leq \left\{ \begin{array}{ll}
\sqrt{n} (L - b_n(Q_{AE}) + o(b_n(Q_{AE}))) + O(\log n) & b_n(Q_{AE}) \leq L + \delta \\
\sqrt{n} (L - b_n(Q_{AE})) & b_n(Q_{AE}) > L + \delta
\end{array} \right.$$

(206)

(207)

(208)

(209)

Note that we assumed that $\|f\| = M_n = e^{nH(A|E|P_{AE}) + \sqrt{n}L}$. Thus, by plugging (209) back into (200), we obtain that for all $n$ large enough (depending on $\delta$),

$$\frac{1}{\sqrt{n}} C_1(f(A^n)|E^n|P_{AE}^n) \leq \sum_{Q_{AE} \in P_n(A \times E) : b_n(Q_{AE}) \leq L + \delta} P_{AE}^n(T_{Q_{AE}})(L - (1 - \delta)b_n(Q_{AE})) + O \left( \frac{\log n}{\sqrt{n}} \right).$$

(210)

Let $V := V(A|E|P_{AE})$. By the central limit-type convergence in (205), we obtain

$$\limsup_{n \to \infty} \frac{1}{\sqrt{n}} C_1(f(A^n)|E^n|P_{AE}^n) \leq \int_{-\infty}^{L + \delta} \frac{L - (1 - \delta)b}{\sqrt{2\pi V}} e^{-b^2/(2V)} db$$

(211)

By a change of variables to $x := b/\sqrt{V}$ and taking $\delta \downarrow 0$, we immediately obtain the direct part (upper bound) of Case (B) in (57).
For Case (C), the upper bound to (58) can be obtained by specializing the one-shot bound in (92) with \( \epsilon = 1 \), \( M_n = e^{nH(A|E|P_{AE})+\sqrt{nL}} \) and \( c = e^{-n^{1/4}} \). With these choices, we have

\[
e^{-sC_{1-s}(f_{X_n}(A^n))|E^n|P_{AE}^{n}}
\]

\[
\geq \Pr \left( P_{AE}^{n}(A^n|E^n) \leq \frac{e^{-n^{1/4}}}{e^{nH(A|E|P_{AE})+\sqrt{nL}}} \left( \frac{1}{1+e^{-n^{1/4}}} \right)^{s} \right)
\]

\[
= \Pr \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[ -\log P_{AE}(A_i|E_i) - H(A|E|P_{AE}) \right] \geq L + \frac{1}{\sqrt{n}} \left( \frac{1}{1+e^{-n^{1/4}}} \right)^{s} \right)
\]

(212)

The probability is an information spectrum [38] term with \( n \) i.i.d. terms and

\[
\mathbb{E}_{P_{A_i,E_i}} \left[ -\log P_{AE}(A_i|E_i) \right] = H(A|E|P_{AE}),
\]

(214)

\[
\text{Var}_{P_{A_i,E_i}} \left[ -\log P_{AE}(A_i|E_i) \right] = V(A|E|P_{AE})
\]

(215)

for each \( 1 \leq i \leq n \). So by the central limit theorem, for any \( s > 0 \), the right-hand-side of (213) converges uniformly as follows:

\[
\lim_{n \to \infty} \Pr \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[ -\log P_{AE}(A_i|E_i) - H(A|E|P_{AE}) \right] \geq L + \frac{1}{\sqrt{n}} \right) = \Phi \left( - \frac{L}{\sqrt{V(A|E|P_{AE})}} \right).
\]

(216)

Plugging (216) into (213), taking the logarithm, and normalizing by \(-s\) yields the upper bound to (58).

In a similar way, the upper bound to (59) can be obtained by specializing the one-shot bound in (93) with \( \epsilon = 1 \), \( M_n = e^{nH(A|E|P_{AE})+\sqrt{nL}} \) and \( c = e^{-n^{1/4}} \). The calculation for the specialization is similar to the converse part which is detailed in full in (246)–(249) in the next section. This completes the proof for the direct part of Case (C) of Theorem 3.

2) Converse Parts: Proof: We now prove the lower bounds for Case (A). The first two bounds can be shown using the data processing inequalities in (22)–(24). In particular, the lower bound to (53) can be evaluated as follows:

\[
C_{1+s}(f(A^n)|E^n|P_{AE}^n) = nH_{1+s}(A|E|P_{AE}) + \sqrt{nL} - H_{1+s}(f(A^n)|E^n|P_{AE}^n)
\]

(217)

\[
\geq nH_{1+s}(A|E|P_{AE}) + \sqrt{nL} - H_{1+s}(A^n|E^n|P_{AE}^n)
\]

(218)

\[
= nH_{1+s}(A|E|P_{AE}) + \sqrt{nL} - nH_{1+s}(A|E|P_{AE})
\]

(219)

\[
= \sqrt{nL},
\]

(220)

where (218) follows from (23). The lower bound to (54) follows completely analogously using (24).

The lower bound to (55) can be shown by first relaxing (100) as follows:

\[
e^{sC_{1+s}(f(A)|E|P_{AE})}
\]

\[
\geq \sum_{(a,e):P_{AE}(a|e) \geq \frac{c}{M}} P_{E}(e)P_{AE}(a|e)^{1+s}M^s + \sum_{e} P_{E}(e)P_{AE}=P_{AE}\left\{(a,e) : P_{AE}(a|e) < \frac{c}{M} \right\}^{1+s}
\]

(221)

\[
\geq \sum_{e} P_{E}(e)P_{AE}=P_{AE}\left\{(a,e) : P_{AE}(a|e) < \frac{c}{M} \right\}^{1+s}
\]

(222)

\[
\geq P_{AE}\left\{(a,e) : P_{AE}(a|e) < \frac{c}{M} \right\}^{1+s}
\]

(223)

\[
= \left[ 1 - P_{AE}\left\{(a,e) : P_{AE}(a|e) \geq \frac{c}{M} \right\} \right]^{1+s}
\]

(224)

\[
\geq 1 - (1+s)P_{AE}\left\{(a,e) : P_{AE}(a|e) \geq \frac{c}{M} \right\}
\]

(225)

where (223) uses Jensen’s inequality (for the convex function \( t \mapsto t^{1+s} \)) and (225) uses the inequality \( (1-x)^{1+s} \geq 1 - (1+s)x \) (also due to the convexity of \( t \mapsto t^{1+s} \)). Hence we have for the \( n \)-shot setting

\[
sC_{1+s}(f(A^n)|E^n|P_{AE}^n) \geq \log \left( 1 - (1+s)P_{AE}^n\left\{(a^n,e^n) : P_{AE}^n(a^n|e^n) \geq \frac{c}{M_n} \right\} \right)
\]

(226)
Applying the modified Gärtner-Ellis theorem derived in Hayashi-Tan [52, Appendix A] to the sequence of random variables $-\log P^n_{A^n|E^n} (A^n|E^n)$ with $M_n = e^{nH_{1+} (A|E|P_{AE}) + \sqrt{nL}}$ and $c = 1$, we have

$$\lim_{n \to \infty} \frac{1}{\sqrt{n}} \log P^n_{AE} \left\{ (a^n, e^n) : P^n_{A^n|E^n} (a^n | e^n) \geq \frac{c}{M_n} \right\} = -sL$$

(227)

The modification here is due to the different normalization of $\sqrt{n}$ as opposed to the normalization by $n$ in the usual Gärtner-Ellis theorem in [39]. Also see Remark (a) to Theorem 2.3.6 in [39]. Combining (227) with (226) yields the lower bound to (55). The lower bound to (56) can be proved in a completely analogous way by relaxing the one-shot bound in (102).

For the converse part of Case (B), we use Theorem 8 of [16], which analyzes the second-order asymptotics of intrinsic randomness [38, Ch. 2] [53]. In particular, from the proof of Theorem 8 of [16] (second column page 4634), we deduce that for each $e^n \in E^n$,

$$H(f(A^n)|P^n_{A^n|E^n = e^n})$$

$$\leq \sqrt{n} \int_{-\infty}^{L_n} a dF^n_e (a) + nH(A|E|P_{AE})$$

$$+ P^n_{A^n|E^n = e^n} \left\{ a^n : P^n_{A^n|E^n = e^n} (a^n) \leq \frac{1}{M_n} \right\} \left( \sqrt{nL_n} - \log P^n_{A^n|E^n = e^n} \left\{ a^n : P^n_{A^n|E^n = e^n} (a^n) \leq \frac{1}{M_n} \right\} \right)$$

(228)

where the second-order coding rate at length $n$ is

$$L_n := \frac{1}{\sqrt{n}} \left( \log M_n - nH(A|E|P_{AE}) \right).$$

(229)

and the distribution function $F^n_e (x)$ which is dependent on $e^n$ is defined as

$$F^n_e (x) := P^n_{A^n|E^n = e^n} \left\{ a^n : -\frac{1}{n} \log P^n_{A^n|E^n = e^n} (a^n) \leq H(A|E|P_{AE}) + \frac{x}{\sqrt{n}} \right\}. $$

(230)

In fact, $F^n_e (x)$ only depends $e^n$ through its type. Our next step is to take the expectation of (228) over $e^n$ with distribution $P^n_E$. Let

$$g(e^n) := P^n_{A^n|E^n = e^n} \left\{ a^n : P^n_{A^n|E^n = e^n} (a^n) \leq \frac{1}{M_n} \right\}. $$

(231)

Since $t \mapsto -t \log t$ is concave, by Jensen’s inequality, we have

$$\mathbb{E}[g(E^n)(\gamma - \log g(E^n))] \leq \mathbb{E}[\gamma g(E^n)] - \mathbb{E}[g(E^n)] \log \mathbb{E}[g(E^n)].$$

(232)

Consequently, with $\gamma := \sqrt{n}L_n$,

$$H(f(A^n)|E^n|P^n_{AE})$$

$$\leq \sqrt{n} \int_{-\infty}^{L_n} a dF_n (a) + nH(A|E|P_{AE})$$

$$+ P^n_{AE} \left\{ (a^n, e^n) : P^n_{A^n|E^n} (a^n | e^n) \leq \frac{1}{M_n} \right\} \left( \sqrt{nL_n} - \log P^n_{AE} \left\{ (a^n, e^n) : P^n_{A^n|E^n} (a^n | e^n) \leq \frac{1}{M_n} \right\} \right)$$

(233)

where $F_n$ is the averaged distribution function defined as

$$F_n (x) := \sum_{e^n} P^n_{E^n} (e^n) F^n_e (x) = P^n_{AE} \left\{ (a^n, e^n) : -\frac{1}{n} \log P^n_{A^n|E^n} (a^n | e^n) \leq H(A|E|P_{AE}) + \frac{x}{\sqrt{n}} \right\}. $$

(234)
Thus, by invoking the definition of $L_n$ in (229) and $F_n$ in (234), we obtain the inequality
\[
\frac{1}{\sqrt{n}} \left( H(f(A^n) | E^n | P_{AE}^n) - nH(A | E | P_{AE}) \right) \\
\leq \int_{-\infty}^L a \, dF_n(a) + P_{AE}^n \left\{ (a^n, e^n) : P_{A|E}(a^n | e^n) \leq \frac{1}{M_n} \right\} \left( L_n - \frac{\log P_{AE}^n \left\{ (a^n, e^n) : P_{A|E}(a^n | e^n) \leq \frac{1}{M_n} \right\}}{\sqrt{n}} \right) \\
= \int_{-\infty}^L a \, dF_n(a) + (1 - F_n(L_n)) \left( L_n - \frac{\log(1 - F_n(L_n))}{\sqrt{n}} \right)
\] (235)

By the central limit theorem
\[
F_n(x) \to F(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-y^2/(2V)} \, dy, \quad \forall x \in \mathbb{R}.
\] (237)

Taking the lim sup of (236), and using the central limit result in (237), we obtain
\[
\lim_{n \to \infty} \frac{1}{\sqrt{n}} \left( H(f(A^n) | E^n | P_{AE}^n) - nH(A | E | P_{AE}) \right) \leq \int_{-\infty}^L a \, dF(a) + L(1 - F(L)).
\] (238)

Since, we have the simple relation
\[
C_1(f(A^n) | E^n | P_{AE}^n) = D(P_f(A^n), E^n || P_{mix,f(A^n)} × P_{E^n})
= -H(f(A^n) | E^n | P_{AE}^n) + \log M_n
= -H(f(A^n) | E^n | P_{AE}^n) + nH(A | E | P_{AE}) + \sqrt{n}L,
\] (239) (240) (241)

we immediately obtain the desired lower bound for the second-order asymptotics of $C_1$:
\[
\lim inf_{n \to \infty} \frac{1}{\sqrt{n}} C_1(f(A^n) | E^n | P_{AE}^n) \geq \int_{-\infty}^L (L - a) \, dF(a) = \int_{-\infty}^{L/\sqrt{V}} \frac{L - \sqrt{V}x}{\sqrt{2\pi}} e^{-x^2/2} \, dx.
\] (242)

For Case (C), the first part of the maximum in the lower bound in (58), namely $A$ in (67), follows from (99) and the second part of the maximum, namely $B$ in (68), follows from (97) with the common choice of $c = e^{n^{1/4}}$. In particular, specializing the bound in one-shot bound in (97) with this choice of $c$, we obtain
\[
e^{-sC_{1-\epsilon}(f(A^n) | E^n | P_{AE}^n)} \leq \left(e^{n^{1/4}}\right)^{-s} + \Pr \left( P_{A|E}^n(A^n | E^n) \leq \frac{e^{n^{1/4}}}{e^{nH(A | E | P_{AE}) + \sqrt{n}L}} \right)^{1-s}
\] (243)

where we trivially upper bounded the first probability in the one-shot bound by 1. The first term in (243) goes to zero (since $s > 0$) while the second term is an information spectrum term that asymptotically behaves as
\[
\lim_{n \to \infty} \Pr \left( P_{A|E}^n(A^n | E^n) \leq \frac{e^{n^{1/4}}}{e^{nH(A | E | P_{AE}) + \sqrt{n}L}} \right) = \Phi \left( -\frac{L}{\sqrt{V(A | E | P_{AE})}} \right).
\] (244)

by the central limit theorem and the statistics computed in (214)–(215). Hence, taking the logarithm in (243), and normalizing by $-s$, we obtain the second term in the maximum in the lower bound in (58), namely $B$. In exactly the same way, specializing the bound in (99), we obtain
\[
e^{-sC_{1-\epsilon}(f(A^n) | E^n | P_{AE}^n)} \leq 2(e^{n^{1/4}})^{-s} + 2 \sum_{e^n} P_{E}^n(e^n) \left( \frac{2}{s^{1-s}} \frac{1}{s^{1-s}} \Pr \left( P_{A|E}^n(A^n | E^n) \leq \frac{e^{n^{1/4}}}{e^{nH(A | E | P_{AE}) + \sqrt{n}L}} \right) \right)
\] (245)

Applying the central limit theorem to the probability in the second term recovers $A$ in the lower bound in (58).

The method to obtain the two terms in the maximum in the lower bound in (59) is more complicated than that for (58) because we need to condition on various sequences $e^n \in \mathcal{E}^n$. In particular, to obtain the lower bound $D$ in (75), we evaluate (103) with $c = e^{n^{1/4}}$. We obtain
\[
e^{-\frac{1}{1-s}C_{1-\epsilon}(f(A^n) | E^n | P_{AE}^n)} \leq 2 \sum_{e^n} P_{E}^n(e^n) \left( \frac{2}{s^{1-s}} \frac{1}{s^{1-s}} \Pr \left( P_{A|E}^n(A^n | E^n) \leq \frac{e^{n^{1/4}}}{M_n} \right) \right).
\] (246)
As usual, the first term goes to zero. To compute the probability in the second term, let us denote the type (empirical distribution) \([14]\) of \(e^n\) by \(Q_{en}\) in \(P_{n}(E)\) for the moment. Then we have

\[
P_{A^n|E^n=e^n}\left\{a^n : P^n_{A|E}(a^n|e^n) < \frac{c}{M_n}\right\} - \Phi\left(-\frac{L + H(A|E|P_{AE}) - H(A|E|P_{AE}|Q_{en})}{\sqrt{V_2(A|E|P_{AE}|Q_{en})}}\right) \leq O\left(\frac{1}{\sqrt{n}}\right)
\]

by the Berry-Esseen theorem \([54, \text{Sec. XVI.7}]\), where the conditional entropy given another distribution \(Q_{en}\), denoted as \(H(A|E|P_{AE}|Q_{en})\), was defined in \((10)\), and conditional varentropy another distribution \(Q_{en}\) is defined as

\[
V_2(A|E|P_{AE}|Q_{en}) := \sum_a Q_{en}(e) \sum_a P_{A|E}(a|e) \left[\log \frac{1}{P_{A|E}(a|e)} - H(A|P_{A|E=E})\right]^2.
\]

Note that \(V_2(A|E|P_{AE}|P_E) = V_2(A|E|P_{AE})\) defined in \((51)\). In \((247)\), the remainder term \(O\left(\frac{1}{\sqrt{n}}\right)\) is uniform in \(L\) and \(Q_{en}\). We now plug this into \((246)\) and notice that we are then averaging over all types \(Q_{en}\) (where \(E^n\) has distribution \(P^n_{E}\)). Now, employing a weak (expectation) form of the Berry-Esseen theorem \([55, \text{Thm. 2.2.14}]\) with \(x = H(A|E|P_{AE}) - H(A|E|P_{AE}|Q_{en})\) yields

\[
e^{-\frac{s}{1-s}C_{1-s}(f(A^n)|E^n|P_{AE})} \leq 2^{\frac{1}{1-s}} \left[O\left(n^{-\frac{s}{4(n-\eta)}}\right) + \left(2^{\frac{1}{1-s}} s^{\frac{1}{1-s}} (1 - s)\right)^{\frac{1}{1-s}} \times \ldots \times \int_{-\infty}^{\infty} \Phi\left(-\frac{L + x}{\sqrt{V_2(A|E|P_{AE})}}\right) + O\left(\frac{1}{\sqrt{n}}\right)\right]^{\frac{1}{1-s}} e^{-x^2/(2V_1(A|E|P_{AE}))} d\pi + O\left(\frac{1}{\sqrt{n}}\right).
\]

Now we take the logarithm, divide both sides by \(-\frac{s}{1-s}\), and take the limit as \(n \to \infty\). This yields the lower bound \(D\) in \((75)\). Note that here unlike in the steps leading to \((85)\), we cannot add \(V_1\) and \(V_2\) due the exponentiation of the first term by \(\frac{1}{1-s}\) in the integral.

Using similar techniques, we can obtain the lower bound \(E\) in \((76)\). In particular, evaluate \((98)\) with the same choice of \(c\). Here, in fact, no averaging over \(E^n\) is needed because the first term in \((98)\) vanishes by our choice of \(c = e^{n^{1/4}}\).

This completes the proof of the converse parts of Theorem 3.

\[\square\]

VIII. Conclusion

A. Summary

In this paper, we have derived the fundamental limits of the asymptotic behavior of the equivocation when a hash function \(f\) is applied to the source (Theorem 1). We have also showed that optimal key generation rates change when we use alternative Rényi information measures (Corollary 1). Under these Rényi quantities, we have evaluated the corresponding exponential rates of decay of the security measures (Theorem 2) as well as their second-order coding rates (Theorems 3 and 4). The Rényi information measures generalize the ubiquitous Shannon information measures and may be useful in many settings as described in the Introduction. To establish our asymptotic theorems, we have introduced new families of non-asymptotic achievability and converse bounds on the Rényi information measures and their Gallager counterparts and used various probabilistic limit theorems (such as large deviation theorems and the central limit theorem) to evaluate these bounds when the number of realizations of the joint source tends to infinity.

B. Future Research Directions

In the future, we plan to explore various extensions to the results contained herein.

1) We would like to study security problems such as the remaining uncertainty of a source \(A^n\) when another party observes a compressed version \(f(A^n) \in M := \{1, \ldots, M_n\}\) and another correlated source \(E^n\). Namely, we aim to study the asymptotic behavior of the conditional Rényi entropy \(H_{1+s}(A^n|f(A^n), E^n|P_{AE})\) and its Gallager counterpart \(H_{1+s}^g(A^n|f(A^n), E^n|P_{AE})\).

2) Another set of related problems involve the analyses of the asymptotic behavior of \(H_{1+s}(f(A^n)|E^n|P_{AE})\) and \(H_{1+s}^g(f(A^n)|E^n|P_{AE})\). These represent the uncertainties of an eavesdropper with regard to the message.
index \( f(A^n) \in \mathcal{M} \). The eavesdropper, however, is equipped with correlated observations \( E^n \). We anticipate that some of the techniques developed in the current paper may be useful to perform various calculations.

3) In this paper, we focused primarily on analyzing \( C_{1+s} \) and \( C_{1+s}^+ \) for \( s \in [-1, 1] \). It may be of interest to study the various asymptotic behaviors of \( C_{1+s} \) and \( C_{1+s}^+ \) for general \( s \in \mathbb{R} \) since for example, \( H_{\min} = \lim_{s \to \infty} H_{1+s} \) and \( H_{\min} [6]–[9] \) is a fundamental quantity in cryptography and information-theoretic security as mentioned in Section I-A. Indeed, \( e^{-H_{\min}(A|E|P_{AE})} \) is the best (highest) probability of successfully guessing \( A \) given \( E \). As remarked after Theorems 1 and 3, we already have the converse parts for all \( s \geq 0 \) for the results in (34), (35), (53) and (54). They follow immediately from various information processing inequalities. It would be ideal, though challenging, to complete the story.

4) Lastly, we aim to apply the results and techniques contained in this paper to information-theoretic security problems such as the wiretap channel [12] and secret key agreement [19] as was done by various researchers in [17], [18], [20], [21].

APPENDIX A

PROOF OF LEMMA 1

A. Proof of (87)

Proof: The derivation here is similar to that in [17], [22] for universal-2 hash functions. Throughout, for any function \( f: A \to \mathcal{M} \), we let

\[
 f^{-1}(i) := \{a \in A : f(a) = i\}, \quad \forall i \in \mathcal{M}.
\]

Now, for any \( a \), due to the \( \epsilon \)-almost universal-2 property of \( f_X \), we have

\[
 \mathbb{E}_X \sum_{a' \in f_X^{-1}(f_X(a))} P_{A|E}(a'|e) \leq P_{A|E}(a|e) + \frac{\epsilon}{M} \sum_{a' \neq a} P_{A|E}(a'|e)
\]

(251)

\[
 \leq P_{A|E}(a|e) + \frac{\epsilon}{M}.
\]

(252)

Starting from the definition of the conditional Rényi divergence, we have

\[
 e^{-sH_{1+s}(f_X(A)|EX|P_{AE} \times P_X)}
\]

\[
 = \mathbb{E}_X \sum_{e} P_{E}(e) \sum_{i=1}^{M} \left( \sum_{a \in f_X^{-1}(i)} P_{A|E}(a|e) \right)^{1+s}
\]

(253)

\[
 = \mathbb{E}_X \sum_{e} P_{E}(e) \sum_{a} P_{A|E}(a|e) \left( \sum_{a' \in f_X^{-1}(f_X(a))} P_{A|E}(a'|e) \right)^{s}
\]

(254)

\[
 \leq \sum_{e} P_{E}(e) \sum_{a} P_{A|E}(a|e) \left( \mathbb{E}_X \sum_{a' \in f_X^{-1}(f_X(a))} P_{A|E}(a'|e) \right)^{s}
\]

(255)

\[
 \leq \sum_{e} P_{E}(e) \sum_{a} P_{A|E}(a|e) \left( P_{A|E}(a|e) + \frac{\epsilon}{M} \right)^{s}
\]

(256)

\[
 \leq \sum_{e} P_{E}(e) \sum_{a} P_{A|E}(a|e) \left( P_{A|E}(a|e)^s + \left( \frac{\epsilon}{M} \right)^s \right)
\]

(257)

\[
 = \left( \frac{\epsilon}{M} \right)^s + \sum_{e} P_{E}(e) \sum_{a} P_{A|E}(a|e)^{1+s}
\]

(258)

\[
 = \frac{\epsilon^s}{M^s} + e^{-sH_{1+s}(A|E|P_{AE})},
\]

(259)

where (255), we used the concavity of \( t \mapsto t^s \) for \( s \in [0, 1] \), in (256) we used the fact that \( f_X \) is a \( \epsilon \)-almost universal-2 hash function, and in (257) we used the inequality \( \sum_i a_i \leq \sum_i a_i^s \) for \( s \in [0, 1] \) [13, Problem 4.15(f)]. By (30), we have

\[
 C_{1+s}(f_X(A)|EX|P_{AE} \times P_X) = \log M - H_{1+s}(f_X(A)|EX|P_{AE} \times P_X).
\]

(260)

Uniting (259) and (260) proves (87) as desired.
B. Proof of (88)

Proof: Along exactly the same lines, we also have
\[
e^{-\frac{1}{1+s}H_{1+s}^{\uparrow}(f_{X}(A)|EX|P_{AE}\times P_{X})}
\]
\[
= \mathbb{E}_{X} \sum_{e} P_{E}(e) \left( \sum_{i=1}^{M} \left( \sum_{a \in f_{X}^{-1}(i)} P_{A|E}(a|e) \right)^{1+s} \right)^{\frac{1}{1+s}}
\]
\[
= \mathbb{E}_{X} \sum_{e} P_{E}(e) \left( \sum_{a} P_{A|E}(a|e) \left( \sum_{a' \in f_{X}^{-1}(f_{X}(a))} P_{A|E}(a'|e) \right)^{s} \right)^{\frac{1}{1+s}}
\]
\[
\leq \sum_{e} P_{E}(e) \left( \sum_{a} P_{A|E}(a|e) \left( \mathbb{E}_{X} \sum_{a' \in f_{X}^{-1}(f_{X}(a))} P_{A|E}(a'|e) \right)^{s} \right)^{\frac{1}{1+s}}
\]
\[
\leq \sum_{e} P_{E}(e) \left( \sum_{a} P_{A|E}(a|e) \left( P_{A|E}(a|e) + \frac{\epsilon}{M} \right)^{s} \right)^{\frac{1}{1+s}}
\]
\[
\leq \sum_{e} P_{E}(e) \left( \sum_{a} P_{A|E}(a|e) \left( P_{A|E}(a|e)^{s} + \left( \frac{\epsilon}{M} \right)^{s} \right) \right)^{\frac{1}{1+s}}
\]
\[
= \sum_{e} P_{E}(e) \left( \left( \frac{\epsilon}{M} \right)^{s} + \sum_{a} P_{A|E}(a|e)^{1+s} \right)^{\frac{1}{1+s}}
\]
\[
\leq \sum_{e} P_{E}(e) \left( \left( \frac{\epsilon}{M} \right)^{\frac{s}{1+s}} + \left( \sum_{a} P_{A|E}(a|e)^{1+s} \right)^{\frac{1}{1+s}} \right)
\]
\[
= \left( \frac{\epsilon}{M} \right)^{\frac{s}{1+s}} + \sum_{e} P_{E}(e) \left( \sum_{a} P_{A|E}(a|e)^{1+s} \right)^{\frac{1}{1+s}}
\]
\[
= \frac{\epsilon^{\frac{s}{1+s}}}{M^{\frac{s}{1+s}}} + e^{-\frac{1}{1+s}H_{1+s}^{\uparrow}(A|E|P_{AE})}.
\]
Combining this with the relation between $H_{1+s}^{\uparrow}$ and $C_{1+s}^{\uparrow}$ in (32), we obtain (88). □

C. Proof of (89)

Proof: For any $a$, we have
\[
\mathbb{E}_{X} \sum_{a' \in f_{X}^{-1}(f_{X}(a))} P_{A|E}(a'|e)
\]
\[
\leq P_{A|E}(a|e) + \frac{\epsilon}{M} \sum_{a' \neq a} P_{A|E}(a'|e)
\]
\[
\leq P_{A|E}(a|e) + \frac{\epsilon}{M}
\]
\[
\leq 2 \max \left\{ P_{A|E}(a|e), \frac{\epsilon}{M} \right\}.
\]
When $P_{A|E}(a|e) \leq \frac{\epsilon}{M}$, we have
\[
\mathbb{E}_{X} \sum_{a' \in f_{X}^{-1}(f_{X}(a))} P_{A|E}(a'|e) \leq P_{A|E}(a|e) + \frac{\epsilon}{M} \leq \frac{\epsilon + \epsilon}{M}.
\]

The relations in (272) and (273) will turn out to be useful in the proofs of the direct one-shot bounds.
Indeed, starting from (272), we have

\[ e^{-sC_{1-s}(f_X(A))EX|P_{AE}\times P_X} \]

\[ = \frac{1}{M^s} e^{sH_{1-s}(f_X(A))EX|P_{AE}\times P_X} \]

\[ \geq \frac{1}{M^s} \sum_e P_E(e) \sum_a P_{A|E}(a|e) \left( \mathbb{E}_X \sum_{a' \in f_X^{-1}(f_X(a))} P_{A|E}(a'|e) \right)^{-s} \]

\[ \geq \frac{1}{M^s} \sum_e P_E(e) \sum_a P_{A|E}(a|e) \left( P_{A|E}(a|e) + \frac{e}{M} \right)^{-s} \]

\[ = \sum_{a,e} P_{AE}(a,e) (MP_{A|E}(a|e) + e)^{-s} \]

\[ \geq \sum_{a,e} P_{AE}(a,e) \left( 2 \max \{ MP_{A|E}(a|e), e \} \right)^{-s} \]

\[ \geq 2^{-s} \sum_{a,e} P_{AE}(a,e) \min \{ P_{A|E}(a|e)^{-s}M^{-s}, e^{-s} \} \]

\[ = 2^{-s} \sum_{a,e; P_{A|E}(a|e) \geq \epsilon M^{-1}} P_{AE}(a,e) P_{A|E}(a|e)^{-s}M^{-s} \]

\[ + 2^{-s} \sum_{a,e; P_{A|E}(a|e) < \epsilon M^{-1}} P_{AE}(a,e) e^{-s}. \]

We obtain (89).

\[ \square \]

D. Proof of (90)

**Proof:** Using (272) and the convexity of \( a \mapsto a^{1-s} \) we have

\[ e^{\frac{s}{1-s}H_{1-s}(f_X(A))EX|P_{AE}\times P_X} \]

\[ = \mathbb{E}_X \sum_e P_E(e) \left( \sum_{i=1}^M \left( \sum_{a \in f_X^{-1}(i)} P_{A|E}(a|e) \right)^{1-s} \right)^{\frac{1}{1-s}} \]

\[ = \mathbb{E}_X \sum_e P_E(e) \left( \sum_a P_{A|E}(a|e) \left( \mathbb{E}_X \sum_{a' \in f_X^{-1}(f_X(a))} P_{A|E}(a'|e) \right)^{-s} \right)^{\frac{1}{1-s}} \]

\[ \geq \sum_e P_E(e) \left( \sum_a P_{A|E}(a|e) \left( \mathbb{E}_X \sum_{a' \in f_X^{-1}(f_X(a))} P_{A|E}(a'|e) \right)^{-s} \right)^{\frac{1}{1-s}} \]

\[ \geq \sum_e P_E(e) \left( \sum_a P_{A|E}(a|e) \left( P_{A|E}(a|e) + \frac{e}{M} \right)^{-s} \right)^{\frac{1}{1-s}} \]

\[ \geq \sum_e P_E(e) \left( \sum_a P_{A|E}(a|e) \left( 2 \max \{ P_{A|E}(a|e), \frac{e}{M} \} \right)^{-s} \right)^{\frac{1}{1-s}} \]

\[ = 2^{-\frac{s}{1-s}} \sum_e P_E(e) \left( \sum_a P_{A|E}(a|e) \min \{ P_{A|E}(a|e)^{-s}, \frac{e^{-s}}{M^{-s}} \} \right)^{\frac{1}{1-s}} \]

\[ = 2^{-\frac{s}{1-s}} \sum_e P_E(e) \left( \sum_{a: P_{A|E}(a|e) \geq \frac{e}{M}} P_{A|E}(a|e)^{1-s} + \frac{e^{-s}}{M^{-s}} \sum_{a: P_{A|E}(a|e) < \frac{e}{M}} P_{A|E}(a|e) \right)^{\frac{1}{1-s}} \]

\[ \geq 2^{-\frac{s}{1-s}} \sum_e P_E(e) \left( \left( \sum_{a: P_{A|E}(a|e) \geq \frac{e}{M}} P_{A|E}(a|e) \right)^{\frac{1}{1-s}} + \left( \frac{e^{-s}}{M^{-s}} \sum_{a: P_{A|E}(a|e) < \frac{e}{M}} P_{A|E}(a|e)^{1-s} \right)^{\frac{1}{1-s}} \right) \]
\[
\geq 2^{-\frac{1-\epsilon}{M}} \sum_e P_E(e) \left( \left( \sum_{a:|E(a)| \geq \frac{1}{M}} P_A |E(a|e) \right)^{1+s} + \left( \frac{e^{-s}}{M^{1-s}} \sum_{a:|E(a|e) < \frac{1}{M}} P_A |E(a|e) \right)^{1+s} \right)
\]
\[= 2^{-\frac{1-\epsilon}{M}} \sum_e P_E(e) \left( \sum_{a:|E(a|e) \geq \frac{1}{M}} P_A |E(a|e) \right)^{1+s}
\]
\[+ 2^{-\frac{1-\epsilon}{M}} \sum_e P_E(e) \left( \sum_{a:|E(a|e) < \frac{1}{M}} P_A |E(a|e) \right)^{1+s}.
\]

Thus we obtain (90).

\[\square\]

**APPENDIX B**

**PROOF OF LEMMA 2**

A. Proof of (91)

*Proof:* Since \((1 + x)^{\frac{1}{1+s}} \leq 1 + \frac{1}{1+s} x\), and \(x \mapsto x^{\frac{1}{1+s}}\) is concave for \(s \in [0,1]\), we have

\[M^{\frac{1}{1+s}} e^{\frac{1}{1+s} H^{1+s}_{1,+}(f_X(A)|EX|PAE \times PX)}
\]
\[= M^{\frac{1}{1+s}} E_X \sum_e P_E(e) \left( \sum_{a=1}^{M} \left( \sum_{a \in f_X^{-1}(i)} P_A |E(a|e) \right)^{1+s} \right)^{\frac{1}{1+s}}
\]
\[= M^{\frac{1}{1+s}} E_X \sum_e P_E(e) \left( \sum_{a} P_A |E(a|e) \left( \sum_{a' \in f_X^{-1}(f_X(a))} P_A |E(a'|e) s \right)^{\frac{1}{1+s}} \right)
\]
\[\leq M^{\frac{1}{1+s}} \sum_e P_E(e) \left( \sum_{a} P_A |E(a|e) \left( E_X \sum_{a' \in f_X^{-1}(f_X(a))} P_A |E(a'|e) s \right)^{\frac{1}{1+s}} \right)
\]
\[\leq M^{\frac{1}{1+s}} \sum_e P_E(e) \left( \sum_{a} P_A |E(a|e) \left( P_A |E(a|e) s + \frac{1}{M} \right) \right)^{\frac{1}{1+s}}
\]
\[\leq M^{\frac{1}{1+s}} \sum_e P_E(e) \left( \sum_{a} P_A |E(a|e) \left( P_A |E(a|e) s + \frac{1}{M} \right)^{\frac{1}{1+s}} \right)
\]
\[= M^{\frac{1}{1+s}} \sum_e P_E(e) \left( \frac{1}{M} + \sum_{a} P_A |E(a|e) \right)^{\frac{1}{1+s}}
\]
\[= \sum_e P_E(e) \left( 1 + M^s \sum_{a} P_A |E(a|e) \right)^{\frac{1}{1+s}}
\]
\[\leq \sum_e P_E(e) \left( 1 + \frac{1}{1+s} M^s \sum_{a} P_A |E(a|e) \right)
\]
\[= 1 + \frac{1}{1+s} M^s \sum_e P_E(e) \sum_{a} P_A |E(a|e) \right)
\]
\[= 1 + \frac{1}{1+s} M^s e^{-sH_{1,+}(A|E|PAE)}
\]

Using (260), we obtain (91).
APPENDIX C
PROOF OF LEMMA 3

A. Proof of (92)

Proof: Using (273), we have
\[
e^{-sC_{1-\epsilon}(f_X(A)|EX|PAE \times PX)}
\]
\[
= \frac{1}{M^s} e^{sH_{1-\epsilon}(f_X(A)|EX|PAE \times PX)}
\]  
\[
= \frac{1}{M^s} \mathbb{E}_X \sum_e P_E(e) \sum_{i=1}^{M} \left( \sum_{a \in f_X^{-1}(i)} P_{A|E}(a|e) \right)^{1-s}
\]  
\[
= \frac{1}{M^s} \mathbb{E}_X \sum_e P_E(e) \sum_a P_{A|E}(a|e) \left( \sum_{a' \in f_X^{-1}(f_X(a))} P_{A|E}(a'|e) \right)^{-s}
\]  
\[
\geq \frac{1}{M^s} \sum_e P_E(e) \sum_{a: P_{A|E}(a|e) \leq \frac{1}{M}} P_{A|E}(a|e) \left( \mathbb{E}_X \sum_{a' \in f_X^{-1}(f_X(a))} P_{A|E}(a'|e) \right)^{-s}
\]  
\[
\geq \frac{1}{M^s} \sum_e P_E(e) \left( c + \epsilon \right)^{M - s}
\]  
\[
= \mathbb{P}_{A|E} \left\{ (a, e) : P_{A|E}(a|e) \leq \frac{c}{M} \right\} \left( \frac{1}{c + \epsilon} \right)^s.
\]

We obtain (92) as desired.

B. Proof of (93)

Proof: Using (273), we have
\[
e^{-sH_{1-\epsilon}(f_X(A)|EX|PAE \times PX)}
\]
\[
= \mathbb{E}_X \sum_e P_E(e) \left( \sum_{i=1}^{M} \left( \sum_{a \in f_X^{-1}(i)} P_{A|E}(a|e) \right)^{1-s} \right)^{\frac{1}{1-s}}
\]  
\[
= \mathbb{E}_X \sum_e P_E(e) \left( \sum_a P_{A|E}(a|e) \left( \sum_{a' \in f_X^{-1}(f_X(a))} P_{A|E}(a'|e) \right)^{-s} \right)^{\frac{1}{1-s}}
\]  
\[
\geq \sum_e P_E(e) \left( \sum_{a: P_{A|E}(a|e) \leq \frac{1}{M}} P_{A|E}(a|e) \left( \mathbb{E}_X \sum_{a' \in f_X^{-1}(f_X(a))} P_{A|E}(a'|e) \right)^{-s} \right)^{\frac{1}{1-s}}
\]  
\[
\geq \sum_e P_E(e) \left( c + \epsilon \right)^{M - s}
\]  
\[
= \left( \frac{c + \epsilon}{M} \right)^{-\frac{s}{M}} \sum_e P_E(e) \left( \sum_{a: P_{A|E}(a|e) \leq \frac{1}{M}} P_{A|E}(a|e) \right)^{\frac{1}{1-s}}.
\]

By combining with (32), we obtain (93).
APPENDIX D
PROOF OF LEMMA 4

A. Proof of (94)

Proof: Define the function
\[ g(x, y) := x + y - 2x^{1-s}y^s. \]
Then, we can show that
\[ \min_y g(x, y) = x(1 - 2^{-\frac{1}{1-s}}s^{\frac{1}{1-s}}(1 - s)), \]
which is attained when \( y = x(2s)^{-\frac{1}{1-s}} \). Next, consider
\[ 1 - e^{-sC_{1-s}(f(A)|E|P_{AE})} \]
\[ = \sum_e P_E(e) \sum_a g\left(P_{A|E}(a|e), \frac{1}{M}\right) \]
\[ = \sum_e P_E(e) \sum_{a:P_{A|E}(a|e) \geq \frac{c}{M}} g\left(P_{A|E}(a|e), \frac{1}{M}\right) + \sum_{a:P_{A|E}(a|e) < \frac{c}{M}} g\left(P_{A|E}(a|e), \frac{1}{M}\right) \]
\[ \geq \sum_e P_E(e) \sum_{a:P_{A|E}(a|e) \geq \frac{c}{M}} \left(P_{A|E}(a|e) - 2P_{A|E}(a|e)^{1-s}M^s\right) \]
\[ + \sum_{a:P_{A|E}(a|e) < \frac{c}{M}} P_{A|E}(a|e)(1 - 2^{-\frac{1}{1-s}}s^{\frac{1}{1-s}}(1 - s)) \]
\[ = 1 - 2e^{-s} \sum_e P_E(e) \sum_{a:P_{A|E}(a|e) \geq \frac{c}{M}} P_{A|E}(a|e)^{1-s}M^s - 2s^{-\frac{1}{1-s}}s^{\frac{1}{1-s}}(1 - s) \sum_{a:P_{A|E}(a|e) < \frac{c}{M}} P_{A|E}(a|e) \]
\[ = 1 - 2e^{-s} \sum_e P_E(e) \sum_{a:P_{A|E}(a|e) \geq \frac{c}{M}} P_{A|E}(a|e)^{1-s}M^s - 2s^{-\frac{1}{1-s}}s^{\frac{1}{1-s}}(1 - s)P_{AE}\{ (a, e) : P_{A|E}(a|e) < \frac{c}{M} \}, \]
which implies (94).

B. Proof of (95)

We first state a useful and easy lemma:

Lemma 7. Let \( x, y \geq 0 \) and \( t \geq 1 \). Then we have
\[ (x + y)^t \leq 2^{t-1}(x^t + y^t). \]
Proof: It is clear that \( a \mapsto a^t \) is convex for \( a \geq 0 \). Thus,
\[ (x + y)^t = 2^t\left(\frac{x}{2} + \frac{y}{2}\right)^t \leq 2^t\left(\frac{x^t}{2} + \frac{y^t}{2}\right) = 2^{t-1}(x^t + y^t), \]
which proves the claim.

Proof of (95): Using the previously proved bound in (94) with \( |E| = 1 \), we obtain
\[ e^{-sC_{1-s}(f(A)|P_{A|E}=e)} \leq 2e^{-s} \sum_{a:P_{A|E}(a|e) \geq \frac{c}{M}} P_{A|E}(a|e)^{1-s}M^{-s} \]
\[ + 2s^{-\frac{1}{1-s}}s^{\frac{1}{1-s}}(1 - s)P_{AE}\{ (a, e) : P_{A|E}(a|e) \leq \frac{c}{M} \}, \]
\[ = 2e^{-s} \sum_{a:P_{A|E}(a|e) \geq \frac{c}{M}} P_{A|E}(a|e)^{1-s}M^{-s} + 2s^{-\frac{1}{1-s}}s^{\frac{1}{1-s}}(1 - s)P_{AE}\{ (a, e) : P_{A|E}(a|e) \leq \frac{c}{M} \}, \]
\[ = 2e^{-s} \sum_{a:P_{A|E}(a|e) \geq \frac{c}{M}} P_{A|E}(a|e)^{1-s}M^{-s} + 2s^{-\frac{1}{1-s}}s^{\frac{1}{1-s}}(1 - s)P_{AE}\{ (a, e) : P_{A|E}(a|e) \leq \frac{c}{M} \}, \]
which implies (95).
Taking average over \( P_E \) and using the bound in (323), we have
\[
e^{-\frac{1}{1-s} C_{1-s} (f(A)|E|P_AE)}
\] 
\[
= \sum_e P_E(e) \left( e^{-s C_{1-s} (f(A)|P_A|E=e)} \right)^{\frac{1}{1-s}}
\] 
\[
\leq \sum_e P_E(e) \left[ 2c^{-s} \sum_{a : P_A|E(a|e) \geq \frac{c}{M}} P_{A|E}(a|e) \right]^{\frac{1}{1-s} - s}
\] 
\[
+ 2^{\frac{1}{1-s}} s^{\frac{1}{1-s}} (1 - s) P_A e \left\{ (a, e) : P_{A|E}(a|e) \leq \frac{c}{M} \right\} \right]^{\frac{1}{1-s}}
\] 
where in the last step, we applied Lemma 7 with \( t = \frac{1}{1-s} \geq 1 \) to the term in parentheses in (325). Thus we obtain (95).

\[\Box\]

**APPENDIX E**

**PROOF OF LEMMA 5**

These inequalities can be shown by the information processing inequality as follows.

### A. Proofs of (96) and (97)

**Proof:** Depending on \( e \in \mathcal{E} \), we choose the function \( f_e : \mathcal{A} \to \mathcal{M} \) as \( \arg \max_f e^{-sD_{1-s}(P_{f(A)|E=e}||P_{\text{mix},A})} \). Then, when \( P_{A|E=e}(a) \geq \frac{1}{M} \), we have \( f_e^{-1}(f_e(a)) = 1 \). Now, we divide the set \( \mathcal{M} \) into two parts as follows. \( \mathcal{M}_1 := f_e(\{a : P_{A|E=e}(a) \geq \frac{1}{M} \}) \) and \( \mathcal{M}_2 := \mathcal{M}_1^c \). Next, we define the map \( g : \mathcal{M} \to \mathcal{M}_1 \cup \{0\} \) as
\[
g(i) := \begin{cases} i & \text{when } i \in \mathcal{M}_1 \\ 0 & \text{when } i \in \mathcal{M}_2 \end{cases}
\] 
(327)

Due to the information processing inequality for Rényi divergence in (9), we obtain
\[
e^{-sD_{1-s}(P_{f_e(A)|E=e}||P_{\text{mix},A})}
\] 
\[
\leq e^{-sD_{1-s}(P_{f_e(A)|E=e}||P_{\text{mix},A})}
\] 
\[
= \sum_{a : P_{A|E=e}(a) \geq \frac{1}{M}} P_E(e) P_{A|E}(a|e) \right]^{\frac{1}{1-s}}
\] 
\[
+ P_{A|E=e} \right\{ a : P_{A|E}(a|e) \leq \frac{c}{M} \right\} \right]^{\frac{1}{1-s}}
\] 
(329)

Taking the average over \( P_E(e) \), we obtain (96).

Further, we have
\[
\sum_{a : P_{A|E=e}(a) \geq \frac{1}{M}} P_E(e) P_{A|E}(a|e) \right]^{\frac{1}{1-s}}
\] 
\[
\leq \sum_{a : P_{A|E=e}(a) \geq \frac{1}{M}} P_E(e) P_{A|E}(a|e) \right]^{\frac{1}{1-s}}
\] 
\[
+ P_{A|E=e} \right\{ a : P_{A|E}(a|e) \geq \frac{c}{M} \right\} \right]^{\frac{1}{1-s}}
\] 
(330)

Taking the average over \( P_E(e) \), we obtain (97).
B. Proof of (98)

**Proof:** Here, we employ the following expression of $C_{1-s}^s(A|E|P_{AE})$:

$$e^{-\frac{1}{1-s}C_{1-s}^s(A|E|P_{AE})} = \frac{1}{|A|^{1-s}} \sum_{e} P_E(e) \left( \sum_a P_{A|E}(a|e)^{1-s} \right)^{\frac{1}{1-s}}. \quad (333)$$

To minimize $e^{-\frac{1}{1-s}C_{1-s}^s(f(A)|E|P_{AE})}$, it is enough to minimize $\sum_{i\in\mathcal{M}}(\sum_{a\in f^{-1}(i)} P_{A|E=e}(a))^{1-s}$ for each $e$.

Fortunately, the discussion in the proof in Appendix E-A (and, in particular, the bound (96)) shows that this value is upper bounded by

$$\sum_{a: P_{A|E=e}(a) \geq \frac{c}{M}} P_E(e) P_{A|E}(a|e)^{1-s} + P_{A|E=e}\left\{ a : P_{A|E}(a|e) < \frac{c}{M} \right\}^{\frac{1}{1-s}} M^s. \quad (334)$$

Thus,

$$e^{-\frac{1}{1-s}C_{1-s}^s(f(A)|E|P_{AE})} \leq \frac{1}{M^{1-s}} \sum_{e} P_E(e) \left( P_{A|E=e}\left\{ a : P_{A|E=e}(a) \geq \frac{c}{M} \right\} c^{-s} M^s + P_{A|E=e}\left\{ a : P_{A|E}(a|e) < \frac{c}{M} \right\}^{\frac{1}{1-s}} M^s \right)^{\frac{1}{1-s}} \quad (335)$$

$$= \sum_{e} P_E(e) \left( P_{A|E=e}\left\{ a : P_{A|E=e}(a) \geq \frac{c}{M} \right\} c^{-s} + P_{A|E=e}\left\{ a : P_{A|E}(a|e) < \frac{c}{M} \right\}^{\frac{1}{1-s}} \right)^{\frac{1}{1-s}}. \quad (337)$$

Hence, we obtain (98) as desired.

\section*{APPENDIX F}
\addcontentsline{toc}{section}{APPENDIX F}
\section*{PROOF OF LEMMA 6}
\addcontentsline{toc}{subsection}{PROOF OF LEMMA 6}

A. Proof of (99)

**Proof:** The proof of this bound is very similar to the proof of (94) which is presented in Appendix D-A. In particular, starting from (318), consider instead the following bounds

$$1 - e^{-sC_{1-s}(f(A)|E|P_{AE})} \geq \sum_{e} P_E(e)\left( \sum_{a: P_{A|E}(a|e) \geq \frac{c}{M}} P_{A|E}(a|e) - 2P_{A|E}(a|e)c^{-s} \right) + \sum_{a: P_{A|E}(a|e) < \frac{c}{M}} P_{A|E}(a|e) (1 - 2\frac{1}{1-s} s^{\frac{1}{1-s}} (1 - s)) \quad (338)$$

$$\geq (1 - 2c^{-s}) P_{AE}\{ (a, e) : P_{A|E}(a|e) \geq \frac{c}{M} \} + (1 - 2\frac{1}{1-s} s^{\frac{1}{1-s}} (1 - s)) P_{AE}\{ (a, e) : P_{A|E}(a|e) < \frac{c}{M} \} \quad (339)$$

$$= 1 - 2c^{-s} P_{AE}\{ (a, e) : P_{A|E}(a|e) \geq \frac{c}{M} \} - 2\frac{1}{1-s} s^{\frac{1}{1-s}} (1 - s) P_{AE}\{ (a, e) : P_{A|E}(a|e) < \frac{c}{M} \}. \quad (340)$$

This completes the proof of (99).

B. Proofs of (100) and (101)

**Proof:** The proofs of these bounds are very similar to those of (96) and (97) in Appendix E-A and thus are omitted.

C. Proofs of (102) and (103)

**Proof:** The proof of (102) is very similar to the proof of (98) in Appendix E-B and is thus omitted. Finally, the proof of (103) is very similar to the proof of (95) in Appendix D-B and is thus also omitted.
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